



# Analytic Spin and Pseudospin Solutions to the Dirac Equation for the Manning-Rosen Plus Hellmann Potential and Yukawa-Like Tensor Interaction

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**Abstract:** The Dirac equation was solved for the Manning-Rosen plus Hellmann potential including a Yukawa-like tensor potential with arbitrary spin-orbit coupling quantum number  $\kappa$ . In the framework of the spin and pseudospin (pspin) symmetry, the energy eigenvalue equation and the corresponding eigenfunctions were obtained in closed form by using the Nikiforov-Uvarov method. Also Special cases of the potential have been considered and their energy eigen values as well as their corresponding eigen functions were obtained for both relativistic and non-relativistic scope.

**Keywords:** Dirac Equation, Manning-Rosen Potential, Hellmann Potential, Spin and Pseudospin Symmetry, Nikiforov-Uvarov Method

## 1. Introduction

Relativistic symmetries of the Dirac Hamiltonian were discovered many years ago. However, these symmetries have recently been recognized empirically in nuclear and hadronic spectroscopies [1]. Within the framework of the Dirac equation, the pseudospin (pspin) symmetry is used to feature deformed nuclei and superdeformation to establish an effective shell-model [2, 3]. The spin symmetry is relevant for mesons [4] and occurs when the difference of the scalar  $S(r)$  and vector  $V(r)$  potentials is constant, i.e.  $\Delta(r) = Cs$ , and the pspin symmetry occurs when the sum of the scalar and vector potentials is constant, i.e.  $\Delta(r) = Cps$  [5]. Numerous scientists have investigated the Dirac equation by using a variety of potentials and different methods, such as the spin symmetry in the antinucleon spectrum and tensor type Coulomb potential with spin-orbit number  $k$  in a state of spin symmetry and p-spin symmetry [6], bound states of the Dirac equation with position-dependent mass for the Eckart

potential [7], the exact solution of Klein-Gordon with the Poschl-Teller double-ring-shaped Coulomb potential[8], the exact solution of the Dirac equation for the Coulomb potential plus NAD potential by using the Nikorov-Uvarovmethod [9], Deng-Fan potential and the Coulomb potential tensor using the asymptotic iteration method (AIM) [10], Poschl-Teller potential plus the Manning Rosen radial section with the hypergeometry method [11], the solution of Klein-Gordon equation for Hulthen non-central potential inradial part with Romanovski polynomial [12] and the solution of the Schrodinger equation with the Hulthen plus Manning-Rosen potential [13], the Scarf potential with the new tensor coupling potential for spin and pseudospin symmetries using Romanovski polynomials [14], for the q-deformed hyperbolic Poschl-Teller potential and the trigonometric Scarf II non central potential by using AIM [15], eigensolutions of the deformed Woods-Saxon potential via AIM [16], approximate solutions of the Klein Gordon equation with an improved Manning Rosen potential in D-

dimensions using SUSYQM [17], and eigen spectra of the Dirac equation for a deformed Woods-Saxon potential via the similarity transformation [18].

The aim of this work is to solve the Dirac equation for the Manning-Rosen plus Hellmann (MRH) potential in the presence of spin and pspin symmetries and by including a Yukawa-like tensor potential. The MRH potential takes the following form:

$$V(r) = - \left[ \frac{C e^{-\alpha r} + D e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r} \quad (1)$$

Where  $\alpha$  is the screening parameter,  $C$ ,  $D$  and  $V_0, V_1$  are the depths of the potential.

This paper is organized as follows. In section 2, we briefly introduce the Dirac equation with scalar and vector potentials with arbitrary spin-orbit coupling quantum number  $\kappa$  including tensor interaction under spin and pspin symmetry limits. The Nikiforov-Uvarov (NU) method was presented in section 3, the energy eigenvalue equations and corresponding eigenfunctions are obtained in section 4. In section 5, we discussed some special cases of the potential, and finally, our conclusion was given in section 6.

## 2. The Dirac Equation with Tensor Coupling Potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential  $S(r)$ , a repulsive vector potential  $V(r)$  and a tensor potential  $U(r)$  (in units  $\hbar = c = 1$ ) is

$$[\vec{\alpha} \cdot \vec{p} + \beta(M + S(r)) - i\beta \vec{\alpha} \cdot \vec{r} U(r)] \psi(\vec{r}) = [E - V(r)] \psi(\vec{r}) \quad (2)$$

Where  $E$  the relativistic binding energy of the system is,  $p = -i\vec{\nabla}$  is the three-dimensional momentum operator and  $M$  is the mass of the fermionic particle.  $\vec{\alpha}$  and  $\beta$  are the  $4 \times 4$  usual Dirac matrices given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (3)$$

Where  $I$  is the  $2 \times 2$  unitary matrix and  $\vec{\sigma}$  are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

The eigenvalues of the spin-orbit coupling operator are  $\kappa = (j + \frac{1}{2}) > 0$  and  $\kappa = -(j + \frac{1}{2}) < 0$  for unaligned spin  $j = l - \frac{1}{2}$  and aligned spin  $j = l + \frac{1}{2}$ , respectively. The set  $(H^2, K, J^2, J_z)$  can be taken as the complete set of

conservative quantities with  $\vec{J}$  being the total angular momentum operator and  $K = (\vec{\sigma} \cdot \vec{L} + 1)$  is the spin-orbit where  $\vec{L}$  is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum  $j$ , the spin-orbit quantum number  $\kappa$  and the radial quantum number  $n$ . Hence, they can be written as follows:

$$\psi_{n,\kappa}(\vec{r}) = \begin{pmatrix} f_{n,\kappa}(\vec{r}) \\ g_{n,\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{n,\kappa}(r) & Y_{jm}^l(\theta, \varphi) \\ iG_{n,\kappa}(r) & Y_{jm}^l(\theta, \varphi) \end{pmatrix} \quad (5)$$

where  $f_{n,\kappa}(\vec{r})$  is the upper (large) component and  $g_{n,\kappa}(\vec{r})$  is the lower (small) component of the Dirac spinors.  $Y_{jm}^l(\theta, \varphi)$  and  $Y_{jm}^l(\theta, \varphi)$  are spin and pspin spherical harmonics, respectively, and  $m$  is the projection of the angular momentum on the  $z$ -axis. Substituting equation (5) into equation (2) and making use of the following relations

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}) \quad (6a)$$

$$(\vec{\sigma} \cdot \vec{P}) = \vec{\sigma} \cdot \hat{r} \left( \hat{r} \cdot \vec{P} + i \frac{\vec{\sigma} \cdot \vec{L}}{r} \right) \quad (6b)$$

Together with the properties;

$$(\vec{\sigma} \cdot \vec{L}) Y_{jm}^l(\theta, \varphi) = (\kappa - 1) Y_{jm}^l(\theta, \varphi),$$

$$(\vec{\sigma} \cdot \vec{L}) Y_{jm}^l(\theta, \varphi) = -(\kappa - 1) Y_{jm}^l(\theta, \varphi), \quad (7)$$

$$(\vec{\sigma} \cdot \hat{r}) Y_{jm}^l(\theta, \varphi) = -Y_{jm}^l(\theta, \varphi),$$

$$(\vec{\sigma} \cdot \hat{r}) Y_{jm}^l(\theta, \varphi) = -Y_{jm}^l(\theta, \varphi),$$

One obtains two coupled differential equations whose solutions are the upper and lower radial wave functions  $F_{n,\kappa}(r)$  and  $G_{n,\kappa}(r)$  as

$$\left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n,\kappa}(r) = (M + E_{n\kappa} - \Delta(r)) G_{n,\kappa}(r) \quad (8a)$$

$$\left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n,\kappa}(r) = (M - E_{n\kappa} + \Sigma(r)) F_{n,\kappa}(r) \quad (8b)$$

where

$$\Delta(r) = V(r) - S(r) \quad (9a)$$

$$\Sigma(r) = V(r) + S(r) \quad (9b)$$

After eliminating  $F_{n,\kappa}(r)$  and  $G_{n,\kappa}(r)$  in equations (8), the following two Schrodinger-like differential equations for the upper and lower radial spinor components were obtained:

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r} U(r) - \frac{dU(r)}{dr} - U^2(r) \right] F_{n,\kappa}(r) + \frac{\frac{d\Delta(r)}{dr}}{M + E_{n\kappa} - \Delta(r)} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n,\kappa}(r) =$$

$$[(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))] F_{n,\kappa}(r) \quad (10)$$

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r} U(r) + \frac{dU(r)}{dr} - U^2(r) \right] G_{n,\kappa}(r) + \frac{\frac{d\Sigma(r)}{dr}}{M - E_{n\kappa} + \Sigma(r)} \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n,\kappa}(r) =$$

$$[(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]G_{n,\kappa}(r) \quad (11)$$

respectively, where  $\kappa(\kappa - 1) = \hat{l}(\hat{l} + 1)$  and  $\kappa(\kappa + 1) = l(l + 1)$ .

The quantum number  $\kappa$  is related to the quantum numbers for spin symmetry  $l$  and pspin symmetry  $\hat{l}$  as

$$\kappa = \begin{cases} -(l + 1) = -(j + \frac{1}{2})(s_{1/2}, p_{3/2}, etc) \\ j = l + \frac{1}{2}, alignedspin(\kappa < 0), \\ +l = +(j + \frac{1}{2})(p_{1/2}, d_{3/2}, etc) \\ j = l - \frac{1}{2}, unalignedspin(\kappa > 0), \end{cases} \quad (12)$$

and the quasi degenerate doublet structure can be expressed in terms of a pspin angular momentum  $\hat{s} = 1/2$  and pseudo-orbital angular momentum  $\hat{l}$ , which is defined as

$$\kappa = \begin{cases} -\hat{l} = -(j + \frac{1}{2})(s_{1/2}, p_{3/2}, etc) \\ j = \hat{l} - \frac{1}{2}, alignedspin(\kappa < 0), \\ +(\hat{l} + 1) = +(j + \frac{1}{2})(d_{3/2}, f_{5/2}, etc) \\ j = \hat{l} + \frac{1}{2}, unalignedspin(\kappa > 0), \end{cases} \quad (13)$$

where  $\kappa = \pm 1, \pm 2, \dots$ . For example,  $(1s_{1/2}, 0d_{3/2})$  and  $(0p_{3/2}, 0f_{5/2})$  can be considered as pspin doublets

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa H e^{-\alpha r}}{r^2} - \frac{H e^{-\alpha r}}{r^2} - \frac{\alpha H e^{-\alpha r}}{r} - \frac{H^2 e^{-2\alpha r}}{r^2} \right] F_{n,\kappa}(r) = \left[ \gamma \left( - \left[ \frac{C e^{-\alpha r} + D e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r} \right) + \beta^2 \right] F_{n,\kappa}(r) \quad (16a)$$

where  $\kappa = l$  and  $\kappa = -l - 1$  for  $\kappa < 0$  and  $\kappa > 0$ , respectively. Also,

$$\gamma = (M + E_{n\kappa} - C_s) \text{ and } \beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_s). \quad (16b)$$

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H e^{-\alpha r}}{r^2} + \frac{H e^{-\alpha r}}{r^2} + \frac{\alpha H e^{-\alpha r}}{r} - \frac{H^2 e^{-2\alpha r}}{r^2} \right] G_{n,\kappa}(r) = \left[ \tilde{\gamma} \left( - \left[ \frac{C e^{-\alpha r} + D e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r} \right) + \tilde{\beta}^2 \right] G_{n,\kappa}(r) \quad (17a)$$

where  $\kappa = -\tilde{l}$  and  $\kappa = \tilde{l} + 1$  for  $\kappa < 0$  and  $\kappa > 0$ , respectively. Also,

$$\tilde{\gamma} = (E_{n\kappa} - M - C_{ps}) \text{ and } \tilde{\beta}^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) \quad (17b)$$

to obtain the analytic solution, we use an approximation for the centrifugal term as [21]

$$\frac{1}{r^2} = \frac{\alpha^2}{(1 - e^{-\alpha r})^2} \quad (18)$$

Finally, for the solutions to equations (16) and (17) with the above approximation, we will employ the NU method, which is briefly introduced in the following section

### 3. The Nikiforov-Uvarov Method

The NU method is based on the solutions of a generalized

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)} \Psi'(s) + \frac{1}{s^2(1 - c_3 s)^2} [-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3] \Psi(s) = 0 \quad (20)$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1 - c_3 s), \bar{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \quad (21)$$

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions.

#### 2.1. Spin Symmetry Limit

In the spin symmetry limit,  $\frac{d\Delta(r)}{dr} = 0$  or  $\Delta(r) = C_s =$  constant, with  $\Sigma(r)$  taking as the MRH potential eq. (1) and the Yukawa-like tensor potential. i.e

$$\Sigma(r) = V(r) = - \left[ \frac{C e^{-\alpha r} + D e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0}{r} - \frac{V_1 e^{-\alpha r}}{r} - \frac{V_2 e^{-2\alpha r}}{r^2} \quad (14)$$

$$U(r) = -\frac{H}{r} e^{-\alpha r} \quad (15)$$

Under this symmetry, equation (10) is recast in the simple form

#### 2.2. Pseudospin Symmetry Limit

Genocchio [19], showed that there is a connection between pspin symmetry and near equality of the time component of a vector potential and the scalar potential,  $V(r) \approx -S(r)$ . After that, Meng et al [20] derived that if  $\frac{d\Sigma(r)}{dr} = 0$  or  $\Sigma(r) = C_{ps} =$  constant, then pspin symmetry is exact in the Dirac equation. Here, we are taking  $\Delta(r)$  as the MRH potential eq. (1) and the tensor potential as the Yukawa-like potential. Thus, equation (11) is recast in the simple form

They satisfy the following sets of equation respectively

$$c_2 n - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n - 1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (22)$$

$$(c_2 - c_3)n + c_3n^2 - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (23)$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l} S^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1)} (1 - 2c_3 s) \quad (24)$$

Where

$$\begin{aligned} [c_4 = \frac{1}{2}(1 - c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5^2 + \epsilon_1, c_7 = 2c_4c_5 - \epsilon_2, c_8 = c_4^2 + \epsilon_3, \\ c_9 = c_3c_7 + c_3^2c_8 + c_6, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \\ c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8})] \end{aligned} \quad (25)$$

and  $P_n$  is the orthogonal polynomials.

## 4. Solutions to the Dirac Equation

Here, the Dirac equation with the MRH potential and tensor potential was solved by using the NU method.

### 4.1. The spin Symmetric Case

To obtain the solution to equation (16), by using the transformation  $s = e^{-\alpha r}$ , it can be written as follows:

$$\frac{d^2 F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2} \left[ -\kappa(\kappa + 1) - 2\kappa Hs - 2Hs + Hs^2 - H^2s^2 + \frac{\gamma}{\alpha^2} (Cs + Ds^2 + V_0\alpha(1-s) - V_1\alpha s(1-s)) - \frac{\beta^2}{\alpha^2} (1-s)^2 \right] F_{n,\kappa}(s) = 0 \quad (26)$$

Eq. (26) is further simplified as

$$\begin{aligned} \frac{d^2 F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2} \left[ -\left( \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha} V_1 + H^2 - H \right) s^2 + \left( \frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_1 - \frac{\gamma}{\alpha} V_0 + \frac{\gamma}{\alpha^2} C - 2\kappa H - 2H \right) s - \right. \\ \left. \left( \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1) \right) \right] F_{n,\kappa}(s) = 0 \end{aligned} \quad (27)$$

Comparing eq. (27) with eq. (20), we obtain

$$\begin{aligned} [c_1 = 1, \epsilon_1 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha} V_1 + H^2 - H \\ c_2 = 1, \epsilon_2 = \frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_1 - \frac{\gamma}{\alpha} V_0 + \frac{\gamma}{\alpha^2} C - 2\kappa H - 2H] \\ c_3 = 1, \epsilon_3 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1) \end{aligned} \quad (28)$$

and from eq. (25), we further obtain

$$\begin{aligned} [c_4 = 0, c_5 = -\frac{1}{2}, \\ c_6 = \frac{1}{4} + \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha} V_1 + H^2 - H, c_7 = -\left( \frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_1 - \frac{\gamma}{\alpha} V_0 + \frac{\gamma}{\alpha^2} C - 2\kappa H - 2H \right), \\ c_8 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1), c_9 = \left( \eta_\kappa - \frac{1}{2} \right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C - \gamma V_2, \end{aligned}$$

Where  $\eta_\kappa = \kappa + H + 1$ ,

$$\begin{aligned}
c_{10} &= 1 + 2 \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)}, \\
c_{11} &= 2 + 2 \left( \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C} + \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)} \right) \\
c_{12} &= \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)}, \\
c_{13} &= -\frac{1}{2} - \left( \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C} + \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)} \right) \quad (29)
\end{aligned}$$

In addition, the energy eigenvalue equation can be obtained by using eq. (23) as follows:

$$\left( n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C} + \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)} \right)^2 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha} V_1 + H^2 - H \quad (30)$$

By substituting the explicit forms of  $\gamma$  and  $\beta^2$  after equation (16) into equation (30), one can readily obtain the closed form for the energy formula.

$$\begin{aligned}
&\left( n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2} (M + E_{n\kappa} - C_s) - \frac{C}{\alpha^2} (M + E_{n\kappa} - C_s) + \sqrt{\frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa} - C_s)) - \frac{V_0}{\alpha} (M + E_{n\kappa} - C_s) + \eta_\kappa(\eta_\kappa - 1)}} \right)^2 = \\
&\frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa} - C_s)) - \frac{D}{\alpha^2} (M + E_{n\kappa} - C_s) - \frac{V_1}{\alpha} (M + E_{n\kappa} - C_s) + H^2 - H \quad (31)
\end{aligned}$$

On the other hand, to find the corresponding wave functions, referring to equation (29) and eq. (24), we obtain the upper component of the Dirac spinor from eq. 24 as

$$F_{n,\kappa}(s) = B_{n,\kappa} s \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)} (1 - s)^{\frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C}} P_n \left( 2 \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha} V_0 + \kappa(\kappa + 1)}, 2 \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2} D - \frac{\gamma}{\alpha^2} C} \right) (1 - 2s) \quad (32)$$

Where  $B_{n,\kappa}$  is the normalization constant. The lower component of the Dirac spinor can be calculated from equation (8a)

$$G_{n,\kappa}(r) = \frac{1}{(M + E_{n\kappa} - C_s)} \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n,\kappa}(r) \quad (33)$$

where  $E_{n\kappa} \neq -M + C_s$ .

#### 4.2. The Pseudospin Symmetric Case

To avoid repetition in the solution of equation (17), we follow the same procedures explained in section 4.1 and hence obtain the following energy eigenvalue equation:

$$\left( n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 - \frac{\tilde{\gamma}}{\alpha^2} D - \frac{\tilde{\gamma}}{\alpha^2} C} + \sqrt{\frac{\tilde{\beta}^2}{\alpha^2} - \frac{\tilde{\gamma}}{\alpha} V_0 + \kappa(\kappa - 1)} \right)^2 = \frac{\tilde{\beta}^2}{\alpha^2} - \frac{\tilde{\gamma}}{\alpha^2} D - \frac{\tilde{\gamma}}{\alpha} V_1 + H^2 + H \quad (34)$$

By substituting the explicit forms of  $\tilde{\gamma}$  and  $\tilde{\beta}^2$  after equation (17b) into equation (34), one can readily obtain the closed form for the energy formula as

$$\begin{aligned}
&\left( n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2} (E_{n\kappa} - M - C_{ps}) - \frac{C}{\alpha^2} (E_{n\kappa} - M - C_{ps}) + \sqrt{\frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})) - \frac{V_0}{\alpha} (E_{n\kappa} - M - C_{ps}) + \kappa(\kappa - 1)}} \right)^2 = \\
&\frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa} + C_s)) - \frac{D}{\alpha^2} (E_{n\kappa} - M - C_{ps}) - \frac{V_1}{\alpha} (E_{n\kappa} - M - C_{ps}) + H^2 + H \quad (35)
\end{aligned}$$

and the corresponding wave functions for the upper Dirac spinor as

$$G_{n,\kappa}(r) = \tilde{B}_{n,\kappa} S^{\sqrt{\frac{\tilde{B}^2}{\alpha^2} - \tilde{V}_0 + \kappa(\kappa-1)}} (1-s)^{\frac{1}{2} + \sqrt{(\Lambda_\kappa - \frac{1}{2})^2 - \frac{\tilde{V}}{\alpha^2} D - \frac{\tilde{V}}{\alpha^2} C}} P_n \left( 2 \sqrt{\frac{\tilde{B}^2}{\alpha^2} - \frac{\tilde{V}}{\alpha} V_0 + \kappa(\kappa-1)}, 2 \sqrt{(\Lambda_\kappa - \frac{1}{2})^2 - \frac{\tilde{V}}{\alpha^2} D - \frac{\tilde{V}}{\alpha^2} C} \right) (1-2s) \quad (36)$$

where  $\Lambda_\kappa = \kappa + H$  and  $\tilde{B}_{n,\kappa}$  is the normalization constant. Finally, the Upper-spinor component of the Dirac equation can be obtained via equation (8b) as

$$F_{n,\kappa}(r) = \frac{1}{(M - E_{n\kappa} + C_{ps})} \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n,\kappa}(r) \quad (37)$$

where  $E_{n\kappa} \neq M + C_{ps}$ .

In this section, some special cases of the energy eigenvalues were studied and given by Equations (31) and (35) for the spin and pseudospin symmetries, respectively.

Case 1: If one sets  $C_s = 0, C_{ps} = 0, V_0 = V_1 = 0$  in eq. (31) and eq. (35), the energy equation of Manning-Rosen potential for spin and pseudospin symmetric Dirac theory respectively was obtained:

## 5. Discussion

$$\left( n + \frac{1}{2} + \sqrt{\left( \eta_\kappa - \frac{1}{2} \right)^2 - \frac{D}{\alpha^2} (M + E_{n\kappa}) - \frac{C}{\alpha^2} (M + E_{n\kappa})} + \sqrt{\frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) + \kappa(\kappa + 1)} \right)^2 = \frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) - \frac{D}{\alpha^2} (M + E_{n\kappa}) + H^2 - H \quad (38)$$

and

$$\left( n + \frac{1}{2} + \sqrt{\left( \Lambda_\kappa - \frac{1}{2} \right)^2 - \frac{D}{\alpha^2} (E_{n\kappa} - M) - \frac{C}{\alpha^2} (E_{n\kappa} - M)} + \sqrt{\frac{1}{\alpha^2} (M + E_{n\kappa})(M - E_{n\kappa}) + \kappa(\kappa - 1)} \right)^2 = \frac{1}{\alpha^2} (M + E_{n\kappa})(M - E_{n\kappa}) - \frac{D}{\alpha^2} (E_{n\kappa} - M) + H^2 + H \quad (39)$$

Case 2: If one sets  $C_s = 0, C_{ps} = 0, C = D = 0$  in eq. (31) and eq. (35), the energy equation of the Hellmann potential for spin and pseudospin symmetric Dirac theory were obtained respectively:

$$\left( n + \eta_\kappa + \sqrt{\frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) - \frac{V_0}{\alpha} (M + E_{n\kappa}) + \kappa(\kappa + 1)} \right)^2 = \frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) - \frac{V_1}{\alpha} (M + E_{n\kappa}) + H^2 - H \quad (40)$$

and

$$\left( n + \Lambda_\kappa + \sqrt{\frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) - \frac{V_0}{\alpha} (E_{n\kappa} - M) + \kappa(\kappa - 1)} \right)^2 = \frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) - \frac{V_1}{\alpha} (E_{n\kappa} - M) + H^2 + H \quad (41)$$

Case 3: If one sets  $C_s = 0, C_{ps} = 0, V_1 = 0, C = 0, D = 0$ , in eq. (31) and eq. (35), the energy equation of coulomb potential for spin and pseudospin symmetric Dirac theory were obtained respectively:

$$\left( n + \eta_\kappa + \sqrt{\frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) - \frac{V_0}{\alpha} (M + E_{n\kappa}) + \kappa(\kappa + 1)} \right)^2 = \frac{1}{\alpha^2} ((M - E_{n\kappa})(M + E_{n\kappa})) + H^2 - H \quad (42)$$

and

$$\left( n + \Lambda_\kappa + \sqrt{\frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) - \frac{V_0}{\alpha} (E_{n\kappa} - M) + \kappa(\kappa - 1)} \right)^2 = \frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) + H^2 + H \quad (43)$$

Case 4: If one sets  $C_s = 0, C_{ps} = 0, V_0 = 0, C = 0, D = 0, V_1 = -V_1$  in eq. (31) and eq. (35), the energy equation of the yukawa potential for pseudospin and spin symmetric Dirac theory were obtained respectively, as

$$\left( n + \Lambda_\kappa + \sqrt{\frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) + \kappa(\kappa - 1)} \right)^2 = \frac{1}{\alpha^2} ((M + E_{n\kappa})(M - E_{n\kappa})) + \frac{V_1}{\alpha} (E_{n\kappa} - M) + H^2 + H \quad (44)$$

and

$$\left(n + \eta_\kappa + \sqrt{\frac{1}{\alpha^2}((M - E_{n\kappa})(M + E_{n\kappa})) + \kappa(\kappa + 1)}\right)^2 = \frac{1}{\alpha^2}((M - E_{n\kappa})(M + E_{n\kappa})) + \frac{V_1}{\alpha}(M + E_{n\kappa}) + H^2 - H \quad (45)$$

Case 5: If one sets  $C_s = 0, C_{ps} = 0, V_0 = V_1 = 0, C = 0, D = 0$ , in eq. (31) and eq. (35), the energy equation of the inversely quadratic yukawa potential for pseudospin and spin symmetric Dirac theory were obtained respectively, as

$$\left(n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 - V_2(E_{n\kappa} - M)} + \sqrt{\frac{1}{\alpha^2}((M + E_{n\kappa})(M - E_{n\kappa})) + \kappa(\kappa - 1)}\right)^2 = \frac{1}{\alpha^2}((M + E_{n\kappa})(M - E_{n\kappa})) - V_2(E_{n\kappa} - M) + H^2 + H \quad (46)$$

and

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - V_2(M + E_{n\kappa})} + \sqrt{\frac{1}{\alpha^2}((M - E_{n\kappa})(M + E_{n\kappa})) + \kappa(\kappa + 1)}\right)^2 = \frac{1}{\alpha^2}((M - E_{n\kappa})(M + E_{n\kappa})) - V_2(M + E_{n\kappa}) + H^2 - H \quad (47)$$

Case 5: Here, the relativistic limit of the energy eigenvalues and wavefunctions of our solutions were discussed. If we take  $C_s = 0, H = 0, \kappa \rightarrow l$  and put  $S(r) = V(r) = \Sigma(r)$ , the nonrelativistic limit of energy equation 31 and wave function 32 under the following appropriate transformations  $M + E_{n\kappa} \rightarrow \frac{2\mu}{\hbar^2}$ , and  $M - E_{n\kappa} \rightarrow -E_{nl}$  becomes

$$E_{nl} = -\frac{\alpha^2 \hbar^2}{2\mu} \left\{ \left[ \frac{2l(l+1) - \frac{2\mu C}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\alpha \hbar^2} + \frac{2\mu V_1}{\alpha \hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1) \sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2 \hbar^2} - \frac{2\mu D}{\alpha^2 \hbar^2}} \right]^2 + \frac{2\mu V_0}{\alpha \hbar^2} - l(l+1) \right\} \quad (48)$$

and the associated wave functions  $F_{n\kappa}(s) \rightarrow R_{n,l}(s)$  are

$$R_{n,l}(s) = N_{n,l} s^{U/2} (1-s)^{(V-1)/2} P_n^{(U,V)}(1-2s), \quad (49)$$

$$\text{where } U = 2\sqrt{\frac{2\mu E_{nl}}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\alpha \hbar^2}} + l(l+1) \text{ and } V = 2\sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2 \hbar^2} - \frac{2\mu D}{\alpha^2 \hbar^2}} \quad (50)$$

Case 7: If  $V_0 = V_1 = 0$  in eq. (45), we obtain the energy equation of Manning-Rosen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2 \hbar^2}{2\mu} \left\{ \left[ \frac{2l(l+1) - \frac{2\mu C}{\alpha^2 \hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1) \sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2 \hbar^2} - \frac{2\mu D}{\alpha^2 \hbar^2}} \right]^2 - l(l+1) \right\} \quad (51)$$

Case 8: If  $C = D = 0$  in eq. (45), we obtain the energy equation of the Hellmann potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2 \hbar^2}{2\mu} \left\{ \left[ \frac{2l(l+1) - \frac{2\mu V_0}{\alpha \hbar^2} + \frac{2\mu V_1}{\alpha \hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1) \sqrt{\left(l + \frac{1}{2}\right)^2}} \right]^2 + \frac{2\mu V_0}{\alpha \hbar^2} - l(l+1) \right\} \quad (52)$$

Case 9: If  $V_1 = 0, C = D = 0$  in eq. (45), we obtain the energy equation of the coulomb potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2 \hbar^2}{2\mu} \left\{ \left[ \frac{2l(l+1) - \frac{2\mu V_0}{\alpha \hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1) \sqrt{\left(l + \frac{1}{2}\right)^2}} \right]^2 + \frac{2\mu V_0}{\alpha \hbar^2} - l(l+1) \right\} \quad (53)$$

Case 10: If one  $V_0 = 0, V_1 = -V_1, C = D = 0$  in eq. (45), we obtain the energy equation of the Yukawa potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2 \hbar^2}{2\mu} \left\{ \left[ \frac{2l(l+1) - \frac{2\mu V_1}{\alpha \hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2}} \right]^2 - l(l+1) \right\} \quad (54)$$

## 6. Conclusion

The Dirac and Klein-Gordon wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. In recent years, a lot of effort has been put into solving these relativistic wave equations for various potentials by using different methods. In this paper, under spin and pspin symmetry limits, we have obtained the Analytic spin and pseudospin solutions to the Dirac equation for the Manning-Rosen plus Hellmann potential and Yukawa-like tensor interaction using the conventional Nikiforov-Uvarov method.

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