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Eco-Epidemiological Modelling and Analysis of Prey-Predator Population

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Abstract: In this paper, prey-predator model of five Compartments are constructed with treatment is given to infected prey and infected predator. We took predation incidence rates as functional response type II and disease transmission incidence rates follow simple kinetic mass action function. The positivity, boundedness, and existence of the solution of the model are established and checked. Equilibrium points of the models are identified and Local stability analysis of Trivial Equilibrium point, Axial Equilibrium point, and Disease-free Equilibrium points are performed with the Method of Variation Matrix and Routh Hourwith Criterion. It is found that the Trivial equilibrium point E_o is always unstable, and Axial equilibrium point E_A is locally asymptotically stable if $\beta k - (t_1+d_2) < 0$, $qp_1k - d_3(s+k) < 0$, & $qp_3k - (t_2+d_4)(s+k) < 0$ conditions hold true. Global Stability analysis of endemic equilibrium point of the model has been proved by Considering appropriate Liapunove function. In this study, the basic reproduction number of infected prey is obtained to be the following general formula $R_{01}=[(qp_1-d_3)^2 k\beta d_3s^2]/[(qp_1-d_3)\{(qp_1-d_3)^2ks(t_1+d_2)+rsqp_2(kqp_1-kd_3-d_3s)\}]$ and the basic reproduction number of infected predator population is computed and results are written as the general formula of the form as $R_{02}=[(qp_1-d_3)(qp_3 d_3)k+\alpha rsq(kqp_1-kd_3-d_3s)]/[(qp_1-d_3)^2(t_2+d_4)k]$. If the basic reproduction number is greater than one, then the disease will persist in prey-predator system. If the basic reproduction number is one, then the disease is stable, and if basic reproduction number less than one, then the disease is dies out from the prey-predator system. Finally, simulations are done with the help of DEDiscover software to clarify results.

Keywords: Eco-Epidemiology, Prey-Predator, Stability, Variation Matrix, Reproduction Number, Simulation

1. Introduction

Mathematical Modeling of prey-predator systems of interaction of species have a long history since original remarkable work was done by Lotka-Volterra Model in 1920 [1, 3, 5, 6], and SIR model Compartment of systems of population is another vital area of research after pioneering work of Kermack and Mckendrick [1-3, 5-10]. Anderson and May where the first who combined these two modeling systems, while Chattopadhyay and Arino were the first who used the term "eco-epidemiology" for such models [3, 5, 7]. The dynamics of disease in prey-predator systems now become an interesting area of research due to the fact that prey-predator interaction is rich and complex in nature [4, 6, 7, 11-13]. Several mathematical models have been proposed and studied on prey-predator systems [1-7, 9-12]. Many studies focused on the study of disease in a prey only [1-5, 7, 12], other researchers were interested in the study of disease within the predator population only [14], and there are also some studies on diseases in both prey and predators [6, 9, 11] In this paper, we proposed and studied infectious disease on both prey and predator interaction of species with treatment given to infected prey and infected predator.

2. Model Formulation and Assumptions

In this paper, the prey-predator population divided into five compartments. let us denote X(t)-Susceptible prey, $W(t)$ - infected prey, $Y(t)$ -Susceptible predator, $Z(t)$ - infected predator, $H(t)$ – both infected prey and infected predator population under treatment. In the absence of infectious disease, the susceptible prey population grows logistically with intrinsic

growth rate r and environmental carrying capacity k and only susceptible prey can reproduce. In the presence of infectious disease, susceptible predator become infected predator when they come into contact with infected predator, susceptible prey become infected prey when they come into contact with infected prey and the contact process assumed to follow bilinear functional with convolution rate α, β respectively. The predation functional response of predator towards the prey assumed to follow a different holling type II functional response form with p_1, p_2 respective predation coefficients of $X(t), W(t)$ due to susceptible predator, and p_3, p_4 respective predation coefficient of $X(t), Y(t)$ due to infected predator. suppose Consumed prey converted into predator with efficiency q and also half saturated constant s. It is also assumed that Infected prey $W(t)$ and infected predator $Z(t)$ can only recover through treatment, and treated at treatment rate of t_1, t_2 respectively. The prey-predator population $H(t), W(t), Y(t)$, and $Z(t)$ suffer from infectious disease with death rate d_1, d_2, d_3, and d_4 respectively. Moreover, Assume that all variables and parameters used in the model are non negative.

Table 1. *Notations and description of variables.*

Variables	Descriptions
X(t)	Population size of susceptible prey
W(t)	Population size of infectedprey
Y(t)	Population size of susceptible predator
Z(t)	Population size of infectedpredator
$H(t)$	Population size of infected population under treatment

Table 2. *Notations and Description of parameters.*

parameters	Description of parameters
$r,\ k$	Intrinsic growth rate and Carrying capacity of susceptible prey
$\alpha,\ \beta$	disease transmission rates of prey and predator respectively
$t_1,\ t_2$	Treatment rate of infected prey and infected predator respectively
$r_1,\ r_2$	Recovery rate of infected prey and infected predator respectively
$p_i, i = 1,2,3,4$	predation coefficients
$f_i, i = 1,2,3,4$	functional response
$d_i, i = 1,2,3,4$	death rates
$q,\ s$	efficiency of predation, and half-saturation constant respectively

According to the above assumptions, we have the following Model flow diagram

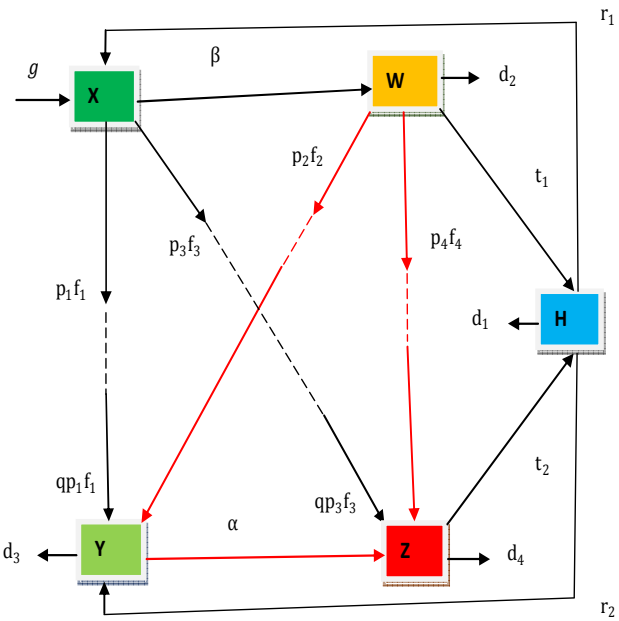


Figure 1. *Flow Diagram.*

From the Model flow diagram in Figure 1 we have the following set of differential equations

$$dX/dt = g + r_1H - \beta XW - p_1f_1 - p_3f_3 \quad (1)$$

$$dW/dt = \beta XW - t_1W - d_2W - p_2f_2 - p_4f_4 \quad (2)$$

$$dY/dt = qp_1f_1 + qp_2f_2 + r_2H - \alpha YZ - d_3Y \quad (3)$$

$$dZ/dt = qp_3f_3 + qp_4f_4 + \alpha YZ - t_2Z - d_4Z \quad (4)$$

$$dH/dt = t_1W + t_2Z - d_1H - r_1H - r_2H \quad (5)$$

with initial conditions $X(0) \geq 0, W(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0, H(0) \geq 0, p_i > 0, i = 1,2,3,4, \& \ 0 < q \leq 1$

Depending on the assumptions of per capita growth of function $g(X, W)$ for susceptible prey, and different type II functional responses $f_i, i = 1,2,3,4$. We have more feasible model (6)-(10) emanated from model(1)-(5) as:

$$dX/dt = rX(1 - [X + W]/k) + r_1H - \beta XW - [p_1XY]/[s + X] - [p_3XZ]/[s + X] = f(X, W, Y, Z, H) \quad (6)$$

$$dW/dt = \beta XW - t_1W - d_2W - [p_2WY]/[s + W] - [p_4WZ]/[s + W] = g(X, W, Y, Z, H) \quad (7)$$

$$dY/dt = [qp_1XY]/[s + X] + [qp_2WY]/[s + W] + r_2H - \alpha YZ - d_3Y = h(X, W, Y, Z, H) \quad (8)$$

$$dZ/dt = [qp_3XZ]/[s + X] + [qp_4WZ]/[s + W] + \alpha YZ - t_2Z - d_4Z = i(X, W, Y, Z, H) \quad (9)$$

$$dH/dt = t_1W + t_2Z - d_1H - r_1H - r_2H = j(X, W, Y, Z, H) \quad (10)$$

with initial conditions $X(0) \geq 0, W(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0, H(0) \geq 0, p_1, p_2, p_3, p_4 > 0 \ \& \ 0 < q \leq 1$

3. Mathematical Analysis of the Model

In this section, positivity, boundedness, and existence of the solution of the model is checked. This mathematical analysis of the model could be considered as primarly results.

Theorem 3.1 [Boundedness] All solutions of Model Equations (6)-(10) are bounded in feasible region $\mathbb{R}^5_+$

Proof: each solutions $X(t), W(t), Y(t), Z(t), H(t)$ of the model is bounded if and only if total population N is bounded.

Let total population of prey-predator $N = X + W + Y + Z + H$. For $\Lambda > 0$ be constant,

$$[dN/dt] + \Lambda N = [dX/dt] + [dW/dt] + [dY/dt] + [dZ/dt] + [dH/dt] + \Lambda N \quad (11)$$

By substitute all model Equations (6)-(10)into (11) and removing all negative terms, we have the following results

$[dN/dt] + \Lambda N \leq$
$rX + [(qp_1XY)/(s + X)] + [(qp_2WY)/(s + W)] + [(qp_4WZ)/(s + W)] + [(qp_3XZ)/(+X)] + \Lambda N = \mu$. Then $dN/dt + \Lambda N \leq \mu$, Solving the differential inequality, yields:

$N(t) \leq [\mu/\Lambda](1 - e^{-\Lambda t}) + N(0)e^{-\Lambda t}$ att $\rightarrow \infty, N \rightarrow [\mu/\Lambda]$. we know that total prey-predator population is non-negative and hence $0 \leq N(t) \leq [\mu/\Lambda]$ So we have invariant feasible region:

$\Omega = \{(X, W, Y, Z, H) \in \mathbb{R}^5_+ : 0 \leq N(t) \leq [\mu/\Lambda]\}$. This prove the theorem and the model is Mathematically well posed

Theorem 3.2[Positivity] All solutions of Model (6)-(10) are non negative.

Proof: To prove theorem 3.2, We have to show that variables X (t), W(t), Y (t), Z (t), H (t) of the Models (6)-(10) are non-negative $\forall t \geq 0$.

1. Positivity of X(t):From the Susceptible prey Model (6), $dX/dt = rX(1 - [(X + W)/k]) + r_1H - \beta XW - [(p_1XY)/(s + X)] - [(p_3XZ)/(s + X)]$ Without loss of generality, After removing all the positive terms from the right hand side of the differential equation, we have the following differential inequality;
$dX/dt \geq -([(rX^2 + rXW)/k] + \beta XW + [(P_1XY + P_3XZ)/(S + X)])$
divide both sides by negative yields:

$-[dX/dt] \leq [(rX^2 + rXW)/k] + \beta XW + [(P_1XY + P_3XZ)/(S + X)]$, But It is also clear that the following inequality holds
$[(rX^2 + rXW)/k] + \beta XW + [(P_1XY + P_3XZ)/(S + X)] \leq rX^2 + rXW + \beta XW + p_1XY + p_3XZ = X(rX + rW + \beta W + p_1Y + p_3Z)$
Assume that $rW + \beta W + p_1Y + p_3Z = C$, Then the differential inequality reduced to $-[dX/dt] \leq X(rX + C)$.This inequality can be arranged for integration by partial fraction as $\int [1/\{X(rX + C)\}]dX \geq \int -d\,t$, integrating results
$\int([1/c]/[X] + [(-r/c)]/[rX + C])dX \geq -\int dt$, Thus $[1/C][(ln|X|)/C] - [1/C]\ln|rX + C| \geq -t + Q$, where Q is integration constant. Using rules of logarithm the inequality can be written as:
$\ln|X/(rX + C)| \geq -Ct + CQ$. Finally solving for X will give as $X(t) \geq [(ACe^{-Ct})/(1 - rAe^{-Ct})]$, where $A = e^{CQ}$. Therefore $X(t) > 0$for$1 - rAe^{-Ct} > 0$. That is $X(t)$ is non-negative for $t > [1/C]\ln(rA)$

2. Positivity of W(t): From infected prey Model (7) $dW/dt = \beta XW - t_1W - d_2W - [(p_2WY)/(s + W)] - [(p_4WZ)/(s + W)]$, Without loss of original generality, after removing the positive term (βXW). we obtain the following differential inequality,
$dW/dt \geq -(t_1W + d_2W + [(p_2WY)/(s + W)] + [(p_4WZ)/(s + W)])$ if
$-\frac{dw}{dt} \leq (t_1W + d_2W + [(p_2WY)/(s + W)] + [(p_4WZ)/(s + W)])$, But it is clear that the inequality $t_1W + d_2W + [(p_2WY)/(s + W)] + [(p_4WZ)/(s + W)] \leq t_1W + d_2W + p_2WY + p_4WZ = (t_1 + d_2 + p_2Y + p_4Z)$ Wholds true. Now Assume that$t_1 + d_2 + p_2Y + p_4Z = C$. Then we have$-[dW/dt] \leq CWt$, Now applying integration yield $\ln|W| \geq -Ct + Q$, where Q is integration constant, Then solving for the variable $W(t)$ gives the equation $W(t) \geq e^{-Ct+Q}$ which is exponential function and positive at all time. Hence $W(t)$ is positive.

3. Positivity of Y(t):From Susceptible predator Model (8) $dY/dt = [(qp_1XY)/(s + X)] + [(qp_2WY)/(s + W)] + r_2H - \alpha YZ - d_3Y$, without loss of original generality, after removing all positive terms
$([(qp_1XY)/(s + X)] + [(qp_2WY)/(s + W)] + r_2H)$
we obtain differential equation;$dY/dt \geq -(\alpha z + d_3)y$, Then applying integration by separable of variable method results, $\ln|y| \geq -(\alpha z + d_3)t + Q$,where Q integration constant and solving for variable $Y(t)$,we obtain the solution $|y| \geq e^{-(\alpha z+d_3)t+Q}$. Therefore $y(t) \geq e^{-(\alpha z+d_3)t+Q}$ is a positive exponential function.. hence y(t) is positive.

4. Positivity of Z(t):From the infected predator Model (9) $dZ/dt = [(qp_2WZ)/(s + W)] + [(qp_3XZ)/(s + W)] + \alpha YZ - t_2Z - d_4Z$
after removing all positive terms;
$([(qp_2WZ)/(s + W)] + [(qp_3XZ)/(s + W)] + \alpha YZ)$,
we obtain the differential inequality;
$dZ/dt \geq -(t_2 + d_4)z$
Applying integration by separable of variable method yield $ln|z| \geq -(t_2 + d_4)t + Q$ where Q integration constant integration by separable of variable method, Then solving for Z will result $z(t) \geq e^{-(t_2+d_4)t+Q}$ whichis exponential function that is positive at all time. hence $Z(t)$ is positive

5. Positivity of H(t):From infected prey and infected predator population under treatment model (10)
$dH/dt = t_1W + t_2Z - d_1H - r_1H - r_2H$, Without loss of generality, after removing all positive terms, we have the differential inequality $dH/dt \geq -(d_1 + r_1 + r_2)H$ iff $dH/H \geq -(d_1 + r_1 + r_2)dt$ which implies that $ln|H| \geq -(d_1 + r_1 + r_2)t + Q$, and solving the variable H provides$|H| \geq e^{-(d_1+r_1+r_2)t+Q}$ is exponential function which is positive at all time. Therefore $(t) > 0$, and hence H(t) is positive. Thus, variables$X(t), W(t), Y(t), Z(t)$ and $H(t)$are all positive quantities and remain in $\mathbb{R}^5_+$for all t.

Theorem 3.3 [Existence] All Solutions of the model (6) -(10) together with the initial conditions$X(0) > 0, W(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0, H(0) \geq 0$ exist in$\mathbb{R}^5_+$ i.e., the model variables $X(t), W(t), Y(t), Z(t)$ and$H(t)$ exist for all t and remain in $\mathbb{R}^5_+$.

Proof: From the system of differential equation (6)-(10) given as have partial derivatives in the following Table 3 According to Derrick and Groosman theorem, let Ω denote the region $\Omega = \{(X, W, Y, Z, H) \in \mathbb{R}^5_+ ; N \leq (\mu/\Lambda)\}$. Then model (2)-(10) have a unique solution if all partial derivatives of the above functions are continuous and bounded in Ω.

Here, The continuity and the Boundedness can be shown as follows:

Table 3. *Partial derivatives.*

For $f(X,\ W,\ Y,\ Z,\ H)$:	For $h(X,\ W,\ Y,\ Z,\ H)$:
$\lvert\partial f/\partial X\rvert = \lvert r - [(2rx + rW)/k] - \beta W - \left[\left(s(p_1Y + p_3Z)\right)/((s + X)^2)\right]\rvert < \infty$	$\lvert\partial h/\partial X\rvert = \lvert(sqp_1Y)/((s + X)^2)\rvert < \infty$
$\lvert\partial f/\partial W\rvert = \lvert - rX/k - \beta X\rvert < \infty$	$\lvert\partial h/\partial W\rvert = \lvert(sqp_2Y)/((s + X)^2)\rvert < \infty$
$\lvert\partial f/\partial Y\rvert = \lvert - (p_1X)/(s + X)\rvert < \infty$	$\lvert\partial h/\partial Y\rvert = \lvert(qp_1X)/(s + X) + (qp_2W)/(s + W)\ + -\alpha Z - d_3\rvert < \infty$
$\lvert\partial f/\partial Z\rvert = \lvert - (p_3X)/(s + X)\rvert < \infty$	$\lvert\partial h/\partial Z\rvert = \lvert -\alpha y - d_3\rvert < \infty$
$\lvert\partial f/\partial H\rvert = \lvert r_1\rvert < \infty$	$\lvert\partial h/\partial H\rvert = r_2 < \infty$
For $g(X,\ W,\ Y,\ Z,\ H)$:	For $i(X,\ W,\ Y,\ Z,\ H)$:
$\lvert\partial g/\partial X\rvert = \lvert\beta W\rvert < \infty$	$\lvert\partial i/\partial X\rvert = \lvert(sqp_3Z)/((s + X)^2)\rvert < \infty$
$\lvert\partial g/\partial W\rvert = \left\lvert\beta X - t_1 - d_2 - \left\{\left(s(p_2Y + p_4Z)\right)/((s + W)^2)\right\}\right\rvert < \infty$	$\lvert\partial i/\partial W\rvert = \lvert(sqp_4Z)/((s + W)^2)\rvert < \infty$
$\lvert\partial g/\partial Y\rvert = \ \lvert - (p_2W)/(s + W)\rvert < \infty$	$\lvert\partial i/\partial Y\rvert = \lvert\alpha Z\rvert < \infty,$
$\lvert\partial g/\partial Z\rvert = \lvert - (p_4W)/(s + w)\rvert < \infty$	$\lvert\partial i/\partial Z\rvert = \lvert(qp_3X)/(s + X) + (qp_4W)/(s + W) - \alpha Y - t_2 - d_4\rvert < \infty$
$\lvert\partial g/\partial H\rvert = 0 < \infty$	$\lvert\partial i/\partial H\rvert = 0 < \infty$
For $j(X,\ W,\ Y,\ Z,\ H)$:	
$\lvert\partial j/\partial X\rvert = 0 < \infty$	
$\lvert\partial j/\partial W\rvert = t_1 < \infty$	
$\lvert\partial j/\partial Y\rvert = 0 < \infty$	
$\lvert\partial j/\partial Z\rvert = \lvert t_2\rvert < \infty$	
$\lvert\partial j/\partial H\rvert = \lvert - d_1 - r_1 - r_2\rvert < \infty$	

Thus, all the partial derivatives of these functions exist, continuous, and bounded in a regionΩ forallpositive values ofmodel variable and model parameter. Hence, by Derrick and Groosman theorem, a solution for the model (6)-(10) exists and unique.

4. Stability Analysis

Stability analysis in the absence of predators in model, That is when y(t) and Z(t) are Zero, model (6)-(10) can be Written as

$$dX/dt = rX(1 - [(X + W)/k]) + r_1H - \beta XW = f(X, W, H) \quad (12)$$

$$dW/dt = \beta XW - t_1W - d_2W\ = g(X, W, H) \quad (13)$$

$$dH/dt = t_1W - d_1H - r_1H = h(X, W, H) \quad (14)$$

The system (12)-(14) has the following equilibrium points: trivial equilibrium points $E_o(0,\ 0,\ 0)$, Axial Equilibrium point $E_A(k,\ 0,\ 0)$ and positive equilibrium point $E_o(X,\ W,\ H)$ where,

$$X = k - [(k\beta)/r] - [\beta/(d_2 + t_1)] + \left[(k\beta r_1t_1)/\left(r(d_1 + r_1)(d_2 + t_1)\right)\right],$$

$$W = \beta/(d_2 + t_1),$$

$H = \beta/[(d_1 + r_1)(d_2 + t_1)]$, with Next Generation Matrix

$$J(X,\ W,\ H) = \begin{pmatrix} r - [(2rX + rW)/k] - \beta W & -[(rX)/k] - \beta X & r_1 \\ \beta W & \beta X - t_1 - d_2 & 0 \\ 0 & t_1 & -d_1 - r_1 \end{pmatrix}$$

Theorem 4.1 The Trivial Equilibrium E_o isa Saddle point With Unstable manifold in X-direction and Stable Manifold in the WY-plane.

Proof: The Jacobian matrix at E_o is given by

$$J(E_o) = \begin{pmatrix} r & 0 & r_1 \\ 0 & -t_1 - d_2 & 0 \\ 0 & t_1 & -d_1 - r_1 \end{pmatrix}$$

to compute eigen values compute the $\det(J(E_o) - \lambda I_3) = 0$

$$\begin{vmatrix} r - \lambda & 0 & r_1 \\ 0 & -t_1 - d_2 - \lambda & 0 \\ 0 & t_1 & -d_1 - r_1 - \lambda \end{vmatrix} = 0$$

Then $(r - \lambda)(-t_1 - d_2 - \lambda)(-d_1 - r_1 - \lambda) = 0$ is the characteristic polynomial. Thus Eigen Values are:$\lambda_1 = r > 0, \lambda_2 = -t_1 - d_2 < 0, \lambda_3 = -d_1 - r_1 < 0$ is a saddle point with unstable manifold in X-direction and stable manifold in WY-plane.

Theorem 4.2 The Axial Equilibrium $E_A$is a saddle point if $\beta k - t_1 - d_2 > 0$and unstable manifold in X- direction if $\beta k - t_1 - d_2 < 0$,then $E_A$stable

Proof: The Jacobian matrix at E_A is given by

$$J(E_A) = \begin{pmatrix} -r & -r - \beta k & r_1 \\ 0 & \beta k - t_1 - d_2 & 0 \\ 0 & t_1 & -d_1 - r_1 \end{pmatrix}$$

to compute eigen values compute the $\det(J(E_o) - \lambda I_3) = 0$

$$\begin{vmatrix} -r - \lambda & -r - \beta k & r_1 \\ 0 & \beta k - t_1 - d_2 - \lambda & 0 \\ 0 & t_1 & -d_1 - r_1 - \lambda \end{vmatrix} = 0$$

Then $(-r - \lambda)(\beta k - t_1 - d_2 - \lambda)(-d_1 - r_1 - \lambda) = 0$ is characteristic polynomial. Thus $\lambda_1 = -r < 0, \lambda_2 = \beta k - t_1 - d_2, \lambda_3 = -d_1 - r_1 < 0$,hence Axial equilibrium point is saddle point if $\beta k - t_1 - d_2 > 0$and stable if $\beta k - t_1 - d_2 < 0$

Stability Analysis in the absence of infectious Disease in the system(2) That is When there is no disease(t),Z(t) and H(t) are all zero and Model(6)-(10) becomes

$$dX/dt = rX(1 - [(X + W)/k]) - [(p_1XY)/(s + X)] = f(X, Y) \quad (15)$$

$$dY/dt = [(qp_1XY)/(s + X)] - d_3Y = g(X, Y) \quad (16)$$

This system Contain trivial $E_o(0, \quad 0)$, Axial $E_A(k, \quad 0)$ and positive $E_o(X, \quad Y)$ Equilibrium points, where

$$X = \left\{\left[r(k-s) + \sqrt{r}\sqrt{k^2r + 2krs + rXs^2 - 4kp_1}\right]/[2r]\right\}$$

$$Y = \left\{(qp_1)/\left(d_3\left(s + \left[\left(r(k-s) + \sqrt{r}\sqrt{k^2r + 2krs + rs^2 - 4kp_1}\right)/(2r)\right]\right)\right)\right\}$$

with Jacobian matrix is given by

$$J(X,Y) = \begin{pmatrix} r - [(2rX)/k] - [(p_1YS)/((s+X)^2)] & [qp_1X]/[s+k] \\ [qp_1YS]/[(s+X)^2] & [qp_1X]/[s+X] - d_3 \end{pmatrix}$$

*Theorem 4.4*The trivial equilibrium $E_o$isa saddle point with unstable manifold in X-direction and stable manifold in Y-direction

Proof: The Jacobian matrix at$E_o$is given by

$$J(E_o) = \begin{pmatrix} r & 0 \\ 0 & -d_3 \end{pmatrix}$$

hence eigen values are:$\lambda_1 = r > 0, \lambda_2 = -d_3 < 0$ which is a saddle point.

Theorem 4.5 The axial equilibrium E_A is stable if$\frac{qp_1k}{s+k} - d_3 < 0$,otherwise unstable

proof: The Jacobian Matrix at E_A is given by

$$J(E_A) = \begin{pmatrix} -r & -[p_1k]/[s+k] \\ 0 & -[p_1k]/[s+k] - d_3 \end{pmatrix}$$

To find eigen values compute $\det(J(E_A) - \lambda I_3) = 0$

$$\begin{vmatrix} -r - \lambda & -[p_1k]/[s+k] \\ 0 & [qp_1k]/[s+k] - d_3 - \lambda \end{vmatrix} = 0$$

Eigen values are: $\lambda_1 = -r < 0, \lambda_2 = [qp_1k]/[s+k] - d_3$

Thus E_A is stable if$\frac{qp_1k}{s+k} - d_3 < 0$andotherwise unstable.

Theorem 4.6 The positive equilibrium E isstable if$\{r - (2rX)/k - [p_1YS]/[(s+X)^2]\} + \{[qp_1k]/[s+k] - d_3\} > 0$and $\{r - (2rX)/k - [p_1YS]/[(s+X)^2]\} * \{[qp_1k]/[s+k] - d_3\} + \{[qp_1{}^2XYS]/[(s+X)^3]\} > 0$

proof: The Jacobian Matrix at E is given by

$$J(X,Y) = \begin{pmatrix} r - [2rX]/k - [p_1YS]/(s+X)^2 & [qp_1X]/[s+k] \\ [qp_1YS]/[(s+X)^2] & [qp_1X]/[s+X] - d_3 \end{pmatrix}$$

Then compute $\det(J(E) - \lambda I_3) = 0$

$$\begin{vmatrix} r - [2rX]/k - [p_1YS]/[(s+X)^2] - \lambda & [qp_1X]/[s+k] \\ [qp_1YS]/[(s+X)^2] & [qp_1X]/[s+X] - d_3 - \lambda \end{vmatrix} = 0$$

Then

$$\left(\underbrace{r - [(2rX)/k] - [(p_1YS)/((s+X)^2)]}_{a} - \lambda\right)\left(\underbrace{[(qp_1X)/(s+X)] - d_3}_{b} - \lambda\right) + \underbrace{[(sqp_1{}^2XY)/((s+X)^3)]}_{c} = 0$$

is characteristic polynomial. Using Routh Hourwith criterion the quadratic polynomial is stable if $a + b > 0, ab + c > 0$ otherwise unstable. Equilibrium points Model(2)-(10) are Steady state points of the form $(X, \quad W, \quad Y, \quad Z, \quad H)$ofmodel (2) that satisfies$dX/dt = dW/dt = dY/dt = dZ/dt = dH/dt = 0$,provided that each variable is non-negative. In Model(2) Five steady state points are identified and listed here: trivial steady state $E_o(0, \quad 0, \quad 0, \quad 0, \quad 0)$, Axial steady state$E_A(k, \quad 0, \quad 0, \quad 0, \quad 0)$,Disease-free steady state $\bar{E}(\bar{X}, \quad 0, \quad \bar{Y}, \quad 0, \quad 0)$ and endemic steady state $E^*(X^*, W^*, Y^*, Z^*, H^*)$. computation of disease free and endemic equilibrium points are presented as follows:

Disease free equilibrium points[DFEP] of model(2)-(10) are steady state Solutions when there is no infectious disease in the population. In the absence of infectious disease in prey-predator system the variables$W(t) = Z(t) = H(t) = 0$ and$dX/dt = dW/dt = dY/dt = dZ/dt = dH/dt = 0$,Then model(2)-(10) become

$$\begin{cases} r\bar{X}(1-[\bar{X}/k])-[(p_1\bar{X}\bar{Y})/(s+X)]=0 \\ [(qp_1\bar{X}\bar{Y})/(s+\bar{X})]-d_3\bar{Y}\ =0 \end{cases}$$

Thus solving $\bar{X}$ and$\bar{Y}$,it is found that

$$\bar{X}=\{(\mathrm{d_3s})/(\mathrm{qp_1}-\mathrm{d_3})\}\&\bar{Y}=\left\{\left(\mathrm{rsq(kqp_1}-\mathrm{kd_3}-\mathrm{d_3s})\right)/((\mathrm{qp_1}-\mathrm{d_3})^2\mathrm{k})\right\},\text{and}$$

Hence disease-free equilibrium point(DFEP) of Model (2)is given by

$$\bar{\mathrm{E}}=\{\bar{\mathrm{X}},\quad 0,\quad \bar{\mathrm{Y}},\quad 0,\quad 0\}=\left\{(\mathrm{d_3s})/(\mathrm{qp_1}-\mathrm{d_3}),\quad 0,\quad \left(\mathrm{rsq(kqp_1}-\mathrm{kd_3}-\mathrm{d_3s})\right)/((\mathrm{qp_1}-\mathrm{d_3})^2\mathrm{k}),\quad 0,\quad 0\right\}$$

The endemic equilibrium point[EEP]is positive equilibrium point $\mathrm{E^*(X^*,W^*,Y^*,Z^*,H^*)}$ obtained by solving modelequation (2) as $\mathrm{dX/dt=dW/dt=dY/dt=dZ/dt=dH/dt=0}$ for which all variables are non zero

$$\begin{cases} \mathrm{rX^*(1-[(X^*+W^*)/k])+r_1H^*-\beta X^*W^*-[(p_1X^*Y^*+p_3X^*Z^*)/(s+X^*)]=0} \\ \mathrm{\beta X^*W^*-t_1W^*-d_2W^*-[(p_2W^*Y^*+p_4W^*Z^*)/(s+W^*)]\ =0} \\ \mathrm{[(qp_1X^*Y^*)/(s+X^*)]+[(qp_2W^*Y^*)/(s+W^*)]+r_2H^*-\alpha Y^*Z^*-d_3Y^*\ =0} \\ \mathrm{[(qp_3X^*Z^*)/(s+X^*)]+[(qp_4W^*Z^*)/(s+W^*)]+\alpha Y^*Z^*-(t_2+d_4)Z^*\ =0} \\ \mathrm{t_1W^*+t_2Z^*-(d_1+r_1+r_2)H^*\ =0} \end{cases}$$

Then solving for the variables $\mathrm{X^*,W^*,Y^*,\ Z^*,\&}H^*$, the endemic equilibrium points of the model exists, and a simplified result obtained as:

$$X^*=\{-r\beta d_1+[2kr-rs-2k\beta]d_1d_2-r\beta r_1+2krd_2r_1\}$$

$$\mathrm{W^*=\left\{\left[\beta-s(d_2+t_1)-\sqrt{4(s\beta-p_2-p_4)(t_1+d_2)+(s[t_1+d_2]-\beta)^2}\right]/[2(t_1+d_2)]\right\}}$$

$$Y^*=\left\{-\alpha+\left[\left(\beta-s(d_2+t_1)\right)rr_2\right]/[2kr_1d_3(t_1+d_2)]\right\}$$

$$Z^*=\left\{\left(\alpha+[qp_4]/\left[s+\left(\beta-(t_1+d_2)+\sqrt{4(s\beta-p_2-p_4)(t_1+d_2)+(s[t_1+d_2]-\beta)^2}\right)/\left(2(t_1+d_2)\right)\right]\right)/(t_2+d_4)\right\}$$

$$\mathrm{H^*=\{(\beta-r+[r\beta-rsd_2-rst_1]/[2k(t_1+d_2)])/r_1\}}$$

To study the Stability analysis of equilibrium points of model (6)-(10), it is better to linearize mode (6)-(10) using Variation matrix. Then the Variation Matrix of these functions (6)-(10)is given by

$$\mathrm{V(X,W,Y,Z,H)}=\begin{pmatrix} \mathrm{f_X} & \mathrm{f_W} & \mathrm{f_Y} & \mathrm{f_Z} & \mathrm{f_H} \\ \mathrm{g_X} & \mathrm{g_W} & \mathrm{g_Y} & \mathrm{g_Z} & \mathrm{g_H} \\ \mathrm{h_X} & \mathrm{h_W} & \mathrm{h_Y} & \mathrm{h_Z} & \mathrm{h_H} \\ \mathrm{i_X} & \mathrm{i_W} & \mathrm{i_Y} & \mathrm{i_Z} & \mathrm{i_H} \\ \mathrm{j_X} & \mathrm{j_W} & \mathrm{j_Y} & \mathrm{j_Z} & \mathrm{j_H} \end{pmatrix}$$

Where each element of the matrix represent partial derivatives of functions (6)-(10) with respect to model variables, and Computations of each element of the variation matrix given as:

$$\mathrm{V(X,W,Y,Z,H)}=\begin{pmatrix} \mathrm{f_X} & -(\mathrm{rX})/\mathrm{k}-\beta\mathrm{X} & -(\mathrm{p_1X})/(\mathrm{s}+\mathrm{X}) & -(\mathrm{p_3X})/(\mathrm{s}+\mathrm{X}) & \mathrm{r_1} \\ \beta\mathrm{W} & \mathrm{g_W} & -(\mathrm{p_2W})/(\mathrm{s}+\mathrm{W}) & -(\mathrm{p_4W})/(\mathrm{s}+\mathrm{w}) & 0 \\ [\mathrm{sqp_1Y}]/[(\mathrm{s}+\mathrm{X})^2] & [\mathrm{sqp_2Y}]/[(\mathrm{s}+\mathrm{X})^2] & \mathrm{h_Y} & -\alpha\mathrm{y}-\mathrm{d_3} & \mathrm{r_2} \\ [\mathrm{sqp_3Z}]/[(\mathrm{s}+\mathrm{X})^2] & [\mathrm{sqp_4Z}]/[(\mathrm{s}+\mathrm{W})^2] & \alpha\mathrm{Z} & \mathrm{i_z} & 0 \\ 0 & \mathrm{t_1} & 0 & \mathrm{t_2} & -\mathrm{d_1}-\mathrm{r_1}-\mathrm{r_2} \end{pmatrix} \quad (17)$$

where

$$\mathrm{f_X=r-[(2rx)/k]-[(rW)/k]-\beta W-\{[s(p_1Y+p_3Z)]/[(s+X)^2]\},}$$

$$\mathrm{g_W=\beta X-t_1-d_2-\{[s(p_2Y+p_4Z)]/[(s+W)^2]\},}$$

$$\mathrm{h_Y=\{(qp_1X)/(s+X)\}+\{[qp_2W]/[s+W]\}-\alpha Z-d_3,}$$

$$\mathrm{i_z=\{[qp_3X]/[s+X]\}+\{[qp_4W]/[s+W]\}-\alpha Y-t_2-d_4,}$$

Theorem 4.7[TEP] Trivial equilibrium point $\mathrm{E_o(0,\quad 0,\quad 0,\quad 0,\quad 0)}$ is always locally asymptotically unstable.

Proof: Consider the Variation Matrix (17) at E_o

$$V(E_o) = \begin{pmatrix} r & 0 & 0 & 0 & r_1 \\ 0 & -t_1 - d_2 & 0 & 0 & 0 \\ 0 & 0 & -d_3 & 0 & r_2 \\ 0 & 0 & 0 & -t_2 - d_4 & 0 \\ 0 & t_1 & 0 & t_2 & -d_1 - r_1 - r_2 \end{pmatrix}$$

Eigen value of variation matrix can be computed from the characteristic polynomial $\det(V(E_o) - \lambda I_5) = 0$

$$\begin{vmatrix} r - \lambda & 0 & 0 & 0 & r_1 \\ 0 & -t_1 - d_2 - \lambda & 0 & 0 & 0 \\ 0 & 0 & -d_3 - \lambda & 0 & r_2 \\ 0 & 0 & 0 & -t_2 - d_4 - \lambda & 0 \\ 0 & a_1 & 0 & a_2 & -d_1 - r_1 - r_2 - \lambda \end{vmatrix} = 0$$

$\Rightarrow (r - \lambda)(-t_1 - d_2 - \lambda)(-d_3 - \lambda)(-t_2 - d_4 - \lambda)(-d_1 - r_1 - r_2 - \lambda) = 0$ is the characteristic polynomial.

The eigen values are:

$\lambda_1 = r > 0, \lambda_2 = -t_1 - d_2 < 0, \lambda_3 = -d_3 < 0, \lambda_4 = -t_2 - d_4 < 0, \lambda_5 = -d_1 - r_1 - r_2 < 0$, Thus the trivial equilibrium point is a saddle point with locally asymptotically unstable manifold in X-direction, and locally asymptotically stable manifold in W, Y, Z, H directions.

Theorem 4.8[AEP] Axial Equilibrium Point $E_A(k, 0,0,0,0)$ exists and always locally asymptotically stable in model(2)if and only if model parameters satisfy the conditions: $\beta k - t_1 - d_2 < 0, [(qp_1 k)/(s + k)] - d_3 < 0, \&[(qp_3 k)/(s + k)] - t_2 - d_4 < 0$. otherwise E_A is locally asymptotically unstable.

Proof: Consider the Variation matrix (17) at $J(E_A)$

$$V(E_A) = \begin{pmatrix} -r & -r - \beta k & -[(p_1 k)/(s + k)] & -[(p_3 k)/(s + k)] & r_1 \\ 0 & \beta k - t_1 - d_2 & 0 & 0 & 0 \\ 0 & 0 & [(qp_1 k)/(s + k)] - d_3 & d_3 & r_2 \\ 0 & 0 & 0 & [(qp_3 k)/(s + k)] - t_2 - d_4 & 0 \\ 0 & t_1 & 0 & t_2 & -d_1 - r_1 - r_2 \end{pmatrix}$$

Then it is possible to find eigen value from characteristic matrix as $Det(V(E_A) - \lambda I_5)$

$$\begin{vmatrix} -r - \lambda & -r - \beta k & -[(p_1 k)/(s + k)] & -[(p_3 k)/(s + k)] & r_1 \\ 0 & \beta k - t_1 - d_2 - \lambda & 0 & 0 & 0 \\ 0 & 0 & [(qp_1 k)/(s + k)] - d_3 - \lambda & d_3 & r_2 \\ 0 & 0 & 0 & [(qp_3 k)/(s + k)] - t_2 - d_4 - \lambda & 0 \\ 0 & t_1 & 0 & t_2 & -d_1 - r_1 - r_2 - \lambda \end{vmatrix}$$

$$(-r - \lambda)(\beta k - t_1 - d_2 - \lambda)([(qp_1 k)/(s + k)] - d_3 - \lambda)([(qp_3 k)/(s + k)] - t_2 - d_4 - \lambda)(-d_1 - r_1 - r_2 - \lambda)$$

is characteristic polynomial. Then Eigen values are roots of this polynomial

$$\lambda_1 = -r < 0, \lambda_2 = -d_1 - r_1 - r_2 < 0, \lambda_3 = \beta k - t_1 - d_2, \lambda_4 = [(qp_1 k)/(s + k)] - d_3, \lambda_5 = [(qp_3 k)/(s + k)] - t_2 - d_4$$

The Axial equilibrium point E_A is locally asymptotically stable, if

$$\beta k - t_1 - d_2 < 0, [(qp_1 k)/(s + k)] - d_3 < 0, \&[(qp_3 k)/(s + k)] - t_2 - d_4 < 0$$

otherwise E_A is locally asymptotically unstable.

Theorem 4.9 [DFEP] The disease- free equilibrium point

$$\bar{E}(\bar{X}, \quad 0, \quad \bar{Y}, \quad 0, \quad 0) = \{[(d_3 s)/(qp_1 - d_3)], \quad 0, \quad [(rsq(kqp_1 - kd_3 - d_3 s))/((qp_1 - d_3)^2 k)], \quad 0, \quad 0\}$$

exists and always locally asymptotically stable if and only if the model parameter satisfy conditions:

$$\beta\bar{X} - t_1 - d_2 - [(sp_2\bar{Y})/(s^2)] \leq 0, [(qp_3\bar{X})/(s + X)] - \alpha\bar{Y} - t_2 - d_4 \leq 0 \text{ and } \{r - [(2r\bar{X})/k] - [(sp_1\bar{Y})/((s + \bar{X})^2)]\} + \{[(qp_1\bar{X})/(s + \bar{X})] - d_3\} > 0$$

$$\&\{r - [(2r\bar{X})/k] - [(sp_1\bar{Y})/((s + \bar{X})^2)]\} * \{[(qp_1\bar{X})/(s + \bar{X})] - d_3\} + [(sqp_1{}^2\bar{X}^2)/((s + \bar{X})^3)] > 0$$

Proof: Consider the Variation matrix (17) at disease-free equilibrium point $\bar{E}(\bar{X}, 0, \bar{Y}, 0,0)$such that $V(\bar{E}) =$

$$\begin{pmatrix} r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)] & -[(r\bar{X})/k]-\beta\bar{X} & -[(p_1\bar{X})/(s+\bar{X})] & -[(p_3\bar{X})/(s+\bar{X})] & r_1 \\ 0 & \beta\bar{X}-t_1-d_2-\frac{sp_2\bar{Y}}{s^2} & 0 & 0 & 0 \\ [(sp_1\bar{Y})/((s+\bar{X})^2)] & [(sqp_2\bar{Y})/((s+\bar{X})^2)] & [(qp_1\bar{X})/(s+\bar{X})]-d_3 & -\alpha\bar{Y}-d_3 & r_2 \\ 0 & 0 & 0 & [(qp_3\bar{X})/(s+\bar{X})]-\alpha\bar{Y}-t_2-d_4 & 0 \\ 0 & t_1 & 0 & t_2 & -d_1-r_1-r_2 \end{pmatrix}$$

Then it is possible to find the determinant of the variation matrix as$Det(\,V(\bar{E})-\lambda I_5)$

$$\begin{vmatrix} r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)]-\lambda & -[(r\bar{X})/k]-\beta\bar{X} & -[(p_1\bar{X})/(s+\bar{X})] & -[(p_3\bar{X})/(s+\bar{X})] & r_1 \\ 0 & \beta\bar{X}-t_1-d_2-\frac{sp_2\bar{Y}}{s^2}-\lambda & 0 & 0 & 0 \\ [(sp_1\bar{Y})/((s+\bar{X})^2)] & [(sqp_2\bar{Y})/((s+\bar{X})^2)] & [(qp_1\bar{X})/(s+\bar{X})]-d_3-\lambda & -\alpha\bar{Y}-d_3 & r_2 \\ 0 & 0 & 0 & [(qp_3\bar{X})/(s+\bar{X})]-\alpha\bar{Y}-t_2-d_4-\lambda & 0 \\ 0 & t_1 & 0 & t_2 & -d_1-r_1-r_2-\lambda \end{vmatrix}$$

$$(\beta\bar{X}-t_1-d_2-[(sp_2\bar{Y})/(s^2)]-\lambda)*([(qp_3\bar{X})/(s+\bar{X})]-\alpha\bar{Y}-t_2-d_4-\lambda)*(-d_1-r_1-r_2-\lambda)*$$

$$\{(r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)]-\lambda)*([(qp_1\bar{X})/(s+\bar{X})]-d_3-\lambda)+[(sqp_1{}^2\bar{X}^2)/((s+\bar{X})^3)]\}=0$$

Eigen values are:

$$\lambda_1=\beta\bar{X}-t_1-d_2-[(sp_2\bar{Y})/(s^2)],\lambda_2=[(qp_3\bar{X})/(s+\bar{X})]-\alpha\bar{Y}-t_2-d_4,\lambda_3=-d_1-r_1-r_2$$

and the remaining eigen values can be obtained from the roots of quadratic equation:

$$\left(\underbrace{r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)]}_{a}-\lambda\right)\left(\underbrace{[(qp_1\bar{X})/(s+\bar{X})]-d_3}_{b}-\lambda\right)+\underbrace{[(sq\,p_1{}^2\bar{X}^2)/((s+\bar{X})^3)]}_{c}=0$$

It is known that a quadratic equation$(a-\lambda)(b-\lambda)+c=0$ is locally asymptotically stable iff$a+b>0\;\&\;ab+c>0$,Using such Routh Hurwitz criterion, the disease free equilibrium point$\bar{E}(\bar{X},0,\bar{Y},0,0)$is locallyasymptotically stable if$\{r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)]\}+\{[(qp_1\bar{X})/(s+\bar{X})]-d_3\}>0$,

$$\&\{r-[(2r\bar{X})/k]-[(sp_1\bar{Y})/((s+\bar{X})^2)]\}*\{[(qp_1\bar{X})/(s+\bar{X})]-d_3\}+(sq\,p_1{}^2\bar{X}^2)/((s+\bar{X})^3)>0$$

$\lambda_1=\beta\bar{X}-t_1-d_2-[(sp_2\bar{Y})/(s^2)]\le 0,\&\lambda_2=[(qp_3\bar{X})/(s+\bar{X})]-\alpha\bar{Y}-t_2-d_4\le 0$ where, $\bar{X}=\{(d_3s)/(qp_1-d_3)\}\&\bar{Y}=\left[\left(rsq(kqp_1-kd_3-d_3s)\right)/((qp_1-d_3)^2k)\right]$,Otherwise the disease free equilibrium point is asymptotically unstable. Now let see again, the Global stability analysis of model(2) around the endemic equilibrium point or positive equilibrium point $E^*(X^*,W^*,Y^*,Z^*,H^*)$ which showsco-existence. For that let us state following theorem and prove by taking appropriate Liapunove function L.

Theorem 4.10[Global stability] Endemic Equilibrium point $E^*(X^*,W^*,Y^*,Z^*,H^*)$ exists and globally asymptotically stable.

Proof: Define appropriate Liapunove function

$L(X,\ W,\ Y,\ Z,\ H)=\{[(X-X^*)^2]/2\}+\{[\alpha_1(W-W^*)^2]/[2]\}+\{[\alpha_2(Y-Y^*)^2]/[2]\}+\{[\alpha_3(Z-Z^*)^2]/[2]\}+\{[(H-H^*)^2]/[2]\}$ .where $\alpha_1,\alpha_2,\alpha_3,\alpha_4>0$ are chosen properly such that, $dL/dt=0\ \forall(X^*,W^*,Y^*,Z^*,H^*)\in\mathbb{R}_+^5$ and $dL/dt\le 0\ \forall(X,W,Y,Z,H)\in\mathbb{R}_+^5$.This implies$E^*$of the system is Liapunove stable and$dL/dt<0,\forall(X,W,Y,Z,H)\in\mathbb{R}_+^5$near E^*. This implies E^* is globally asymptotically stable point. Now differentiate the liapunove function L with respect oftimet as:

$$dL/dt=(X-X^*)[dX/dt]+\alpha_1(W-W^*)[dW/dt]+\alpha_2(Y-Y^*)[dY/dt]+\alpha_3(Z-Z^*)[dZ/dt]+\alpha_4(H-H^*)[dH/dt] \qquad (18)$$

Now substitute the model (6)-(10) into (18), we have the following equation

$$\begin{aligned} dL/dt=&(X-X^*)[rX(1-[X+W]/k)+r_1H-\beta XW-[p_1XY]/[s+X]-[p_3XZ]/[s+X]]\\ &+\alpha_1(W-W^*)[\beta XW-t_1W-d_2W-[p_2WY]/[s+W]-[p_4WZ]/[s+W]]\\ &+\alpha_2(Y-Y^*)[[qp_1XY]/[s+X]+[qp_2WY]/[s+W]+r_2H-\alpha YZ-d_3Y]\\ &+\alpha_3(Z-Z^*)[[qp_3XZ]/[s+X]+[qp_4WZ]/[s+W]+\alpha YZ-t_2Z-d_4Z]\\ &+\alpha_4(H-H^*)[t_1W+t_2Z-d_1H-r_1H-r_2H] \end{aligned}$$

Take out X,W,Y,Z,H from each bracket and write a change as follows

$dL/dt=(X-X^*)(X-X^*)\left[r(1-[(X+W)/k])+[(r_1H)/X]-\beta W-[(p_1Y)/(s+X)]-[(p_3Z)/(s+X)]\right]+$
$\alpha_1(W-W^*)(W-W^*)\left[\beta X-t_1-d_2-[(p_2Y)/(s+W)]-[(p_4Z)/(s+W)]\right]+\alpha_2(Y-Y^*)(Y-Y^*)\left[[(qp_1X)/(s+X)]+\right.$

$[(qp_2W)/(s+W)] + [(r_2H)/Y] - \alpha Z - d_3\big] + \alpha_3(Z-Z^*)(Z-Z^*)\big[[(qp_3X)/(s+X)] + [(qp_4W)/(s+W)] + \alpha Y - t_2 - d_4\big] + \alpha_4(H-H^*)(H-H^*)[[(t_1W)/H] + [(t_2Z)/H] - d_1 - r_1 - r_2]$, By rearranging, it is obtain that:

$dL/dt = -(X-X^*)^2\big[-r(1-[(X+W)/k]) - [(r_1H)/X] + \beta W + [(p_1Y)/(s+X)] + [(p_3Z)/(s+X)]\big] - \alpha_1(I-I^*)^2\big[-\beta X + t_1 + d_2 + [(p_2Y)/(s+W)] + [(p_4Z)/(s+W)]\big] - \alpha_2(Y-Y^*)^2[-[(qp_2W)/(s+W)] - [(r_2H)/Y] + \alpha Z + d_3] - \alpha_3(Z-Z^*)^2[-[(qp_3X)/(s+X)] - [(qp_4W)/(s+W)] - \alpha Y + t_2 + d_4] - \alpha_4(H-H^*)^2[-[(t_1W)/H] - [(t_2Z)/H] + d_1 + r_1 + r_2]$ .Thus it is possible to set$\alpha_1, \alpha_2, \alpha_3, \alpha_4$such that$dL/dt \le 0$ and endemic equilibrium point E^* is globally stable point.

5. Basic Reproduction Number

The basic reproduction number denoted by R_0 and defined as the expected number of people getting secondary infection among the whole susceptible population. This number shows a potential for spread of disease within a given population. When $R_0 < 1$ each infected individual produces on average less than one new infected individual so that the disease is expected to die out. On the other hand if$R_0 > 1$, then each individual produces more than one new infected individual so that the disease is expected to continue spreading in the population

Theorem 5.1 The basic reproduction number for infected prey at Disease free equilibrium point(DFEP)

$$\bar{E}(\bar{X},0,\bar{Y},0,0) = ([d_3s]/[qp_1 - d_3]\,,\quad 0,\quad [rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\,,\quad 0,\quad 0) \text{is given by}$$

$$R_{01} = [(qp_1 - d_3)^2k\beta d_3s^2]/[(qp_1 - d_3)\{(qp_1 - d_3)^2ks(t_1 + d_2) + rsqp_2(kqp_1 - kd_3 - d_3s)\}]$$

Proof: Consider infected prey equation in (7):

$dW/dt = \beta XW - t_1W - d_2W - [p_2WY]/[s+W] - [p_4WZ]/[s+W] = (\beta X - [t_1 + d_2 + [p_2Y]/[s+W] + [p_4Z]/[s+W]])\,W$. Now let us define functions F and V, $F = \beta X, V = t_1 + d_2 + [p_2Y]/[s+W] + [p_4Z]/[s+W]$,Evaluate F and V at DFEP

$$\bar{E}(\bar{X},0,\bar{Y},0,0) = ([d_3s]/[qp_1 - d_3]\,,\quad 0,\quad [rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\,,\quad 0,\quad 0)$$

$$F(\bar{E}) = \beta\bar{X} = [\beta d_3s]/[qp_1 - d_3]\,, V(\bar{E}) = t_1 + d_2 + [(p_2\bar{Y})/s] = t_1 + d_2 + \big[\big(rsqp_2(kqp_1 - kd_3 - d_3s)\big)/((qp_1 - d_3)^2ks)\big]$$

which more simplified to $V(\bar{E}) = \{[(qp_1 - d_3)^2(t_1 + d_2)ks + rsqp_2(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2ks]\}$

Then the basic reproduction number of infected prey is

$$R_{01} = FV^{-1} = \{[\beta d_3s]/[qp_1 - d_3]\} * \{[(qp_1 - d_3)^2ks]/[(qp_1 - d_3)^2(t_1 + d_2)ks + rsqp_2(kqp_1 - kd_3 - d_3s)]\}$$

$$R_{01} = \{[(qp_1 - d_3)^2k\beta d_3s^2]/[(qp_1 - d_3)\{(qp_1 - d_3)^2(t_1 + d_2)ks + rsqp_2(kqp_1 - kd_3 - d_3s)\}]\}$$

Theorem 5.2. The basic reproduction number for infected predators at Disease-free equilibrium point (DFEP)

$\bar{E}(\bar{X},0,\bar{Y},0,0) = ([d_3s]/[qp_1 - d_3]\,,\quad 0,\quad [rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\,,\quad 0,\quad 0)$ is given by

$$R_{02} = \{[(qp_1 - d_3)(qp_3d_3)k + \alpha rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2(t_2 + d_4)k]\}$$

Proof: Consider the infected predator model equation in (9):

$\frac{dZ}{dt} = [qp_3XZ]/[s+X] + [qp_4WZ]/[s+W] + \alpha YZ - t_2Z - d_4Z = \{[qp_3X]/[s+X] + [qp_4W]/[s+W] + \alpha Y - (t_2 + d_4)\}Z$. Now let us define functions F &V as follows:

$F = [qp_3X]/[s+X] + [qp_4W]/[s+W] + \alpha Y$, and $V = t_2 + d_4$ Then Evaluate F and V at (DFEP) $\bar{E}(\bar{X},0,\bar{Y},0,0) = ([d_3s]/[qp_1 - d_3]\,,\quad 0,\quad [rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\,,\quad 0,\quad 0)$, Then we have

$$F(\bar{E}) = \{[qp_3d_3s]/[s(qp_1 - d_3)]\} + \{[\alpha rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\}$$
$$= [(qp_1 - d_3)(qp_3d_3)k + \alpha rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2k]\,, \text{and } V(\bar{E}) = t_2 + d_4$$

Therefore the basic reproduction number of infected predator is $R_{02} = FV^{-1}$and hence

$$R_{02} = \{[(qp_1 - d_3)(qp_3d_3)k + \alpha rsq(kqp_1 - kd_3 - d_3s)]/[(qp_1 - d_3)^2(t_2 + d_4)k]\}$$

6. Simulation

In this section, Simulation of model (6)-(10) is carried out using DEDiscover Version: 2.6.4. software. For Simulation, a set of meaningful values are assigned to model parameters and initial value for model variables are given in tables bellow. The model is arranged in such way for Simulation purposes.

```
dX/dt=r*X*(1-(X+W)/k)+r_1*H-beta*X*W-P_1*X*Y/(s+X)-P_3*X*Z/(s+X)//Susceptible prey
dW/dt=beta*X*W-t_1*W-d_2*W-P_2*W*Y/(s+W)-P_4*W*Z/(s+W)//Infected prey
dY/dt=q*P_1*X*Y/(s+X)+q*P_2*W*Y/(s+W)+r_2*H-alpha*Y*Z-d_3*Y//Susceptible predator
```

dZ/dt=q*P_3*X*Z/(s+X)+q*P_4*W*Z/(s+W)-alpha*Y*Z-t_2*Z-d_4*Z//Infected predator
dH/dt=t_1*W+t_2*Z-d_1*H-r_1*H-r_2*H // both infected populations under treatment

Table 4. *Parameter value used for simulation.*

Name	value	Description
r	22.4000	growth rate of susceptible prey
k	1.0000E03	carrying capacity of susceptible prey
r_1	1.0000	Recovery rate of
beta	2.4000	disease transmission rate in prey
P_1	1.0000	predation coefficient of susceptible prey due to susceptible predator
s	1.0000	Half saturated rate
P_3	1.0000	predation coefficient of susceptible prey due to infected predator
t_1	1.0000	Treatment rate of infected prey
d_2	1.0000	Death rate of infected prey
P_2	1.0000	predation coefficient of infected prey due to predators
P_4	1.0000	Predation rate of infected prey due to infected predator
q	1.0000	efficiency of predation
r_2	1.0000	Recovery rate of susceptible predator
alpha	2.6000	Disease transmission rate in predator
d_3	1.0000	Death rate of susceptible predator
t_2	1.0000	Treatment rate of infected predator
d_4	1.0000	Death rate of infected predator
d_1	1.0000	Death rate of both infected and infected predator under treatment

Table 5. *Initial Conditions used for model variables.*

Name	value	Description
X[t0]	1.2000E04	initial # susceptible prey
W[t0]	200.0000	initial # infected prey
H[t0]	1.0000	Initial # under treated prey predator
Y[t0]	160.0000	initial # susceptible predator
Z[t0]	180.0000	initial # of infected predator

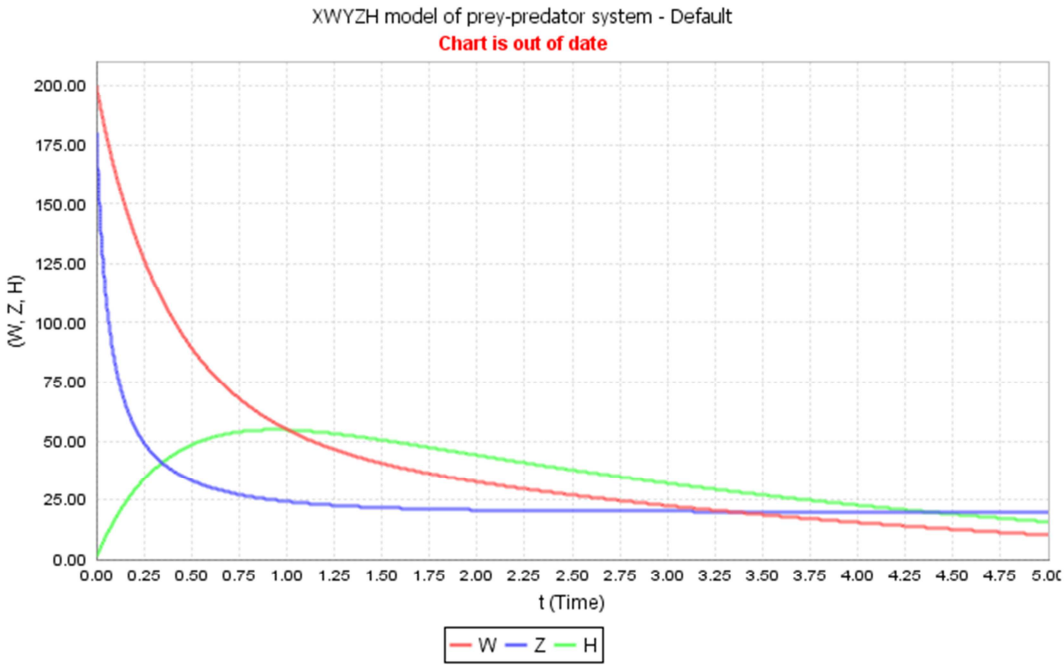


Figure 2. *Infection prey-predator with treatment.*

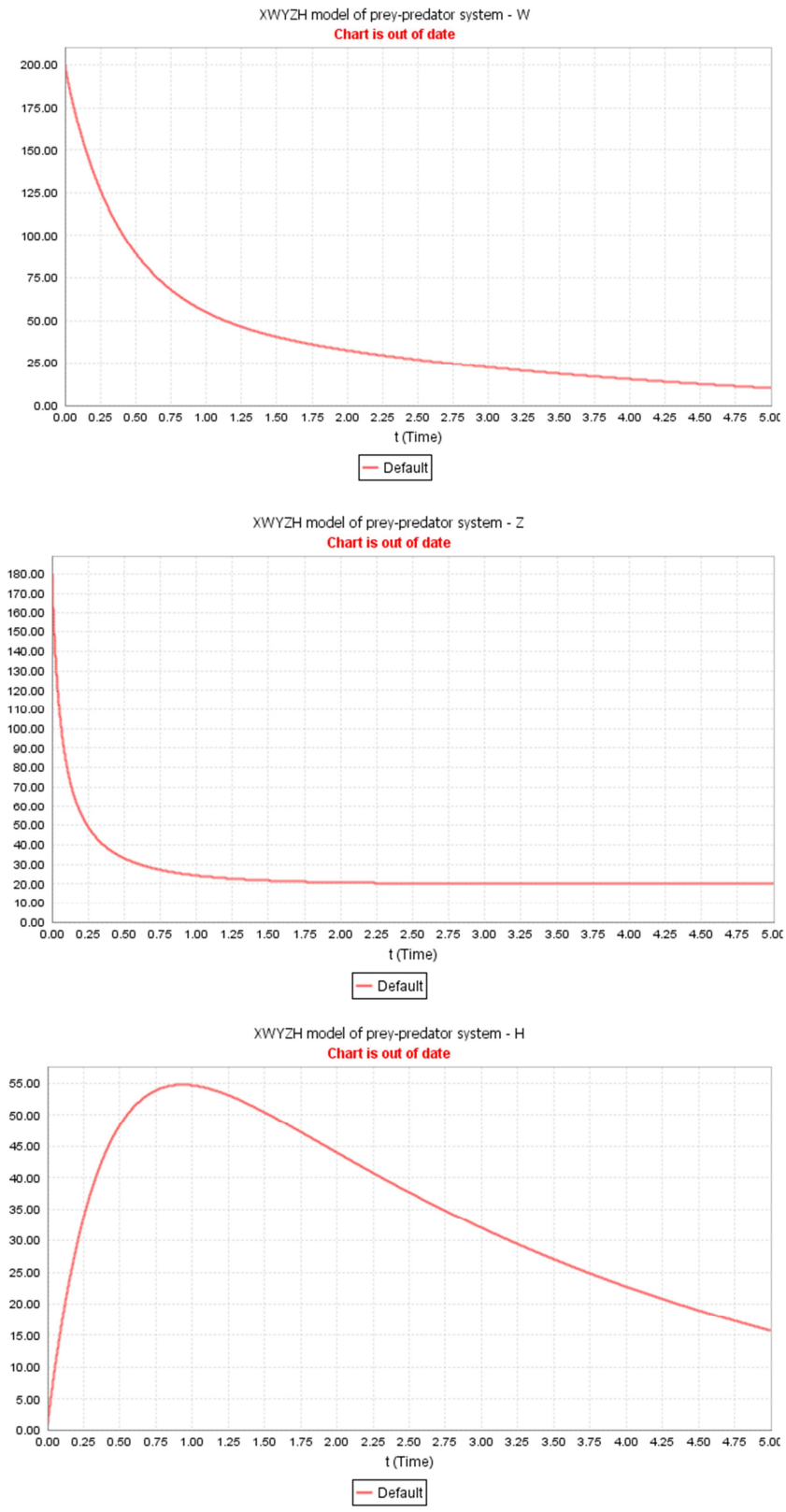


Figure 3. *Individualplots for W, Z, and H.*

From simulation Figure 2 and Figure 3, it can be conclude that treatment is helpful tool to minimize or eradicate infection. it is shown that as treatment rate increase on infected prey-predator, then infected prey-predator population decrease rapidly. This shows due to the fact that infected prey-predator population recovering and move to susceptible classes and that Contributes the susceptible prey-predator population to rise in number.

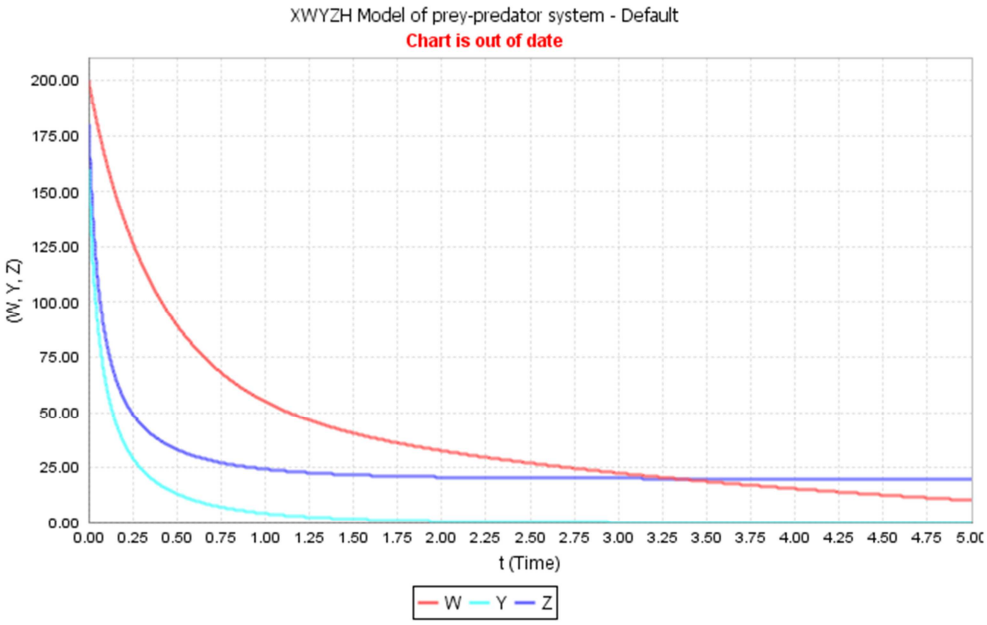
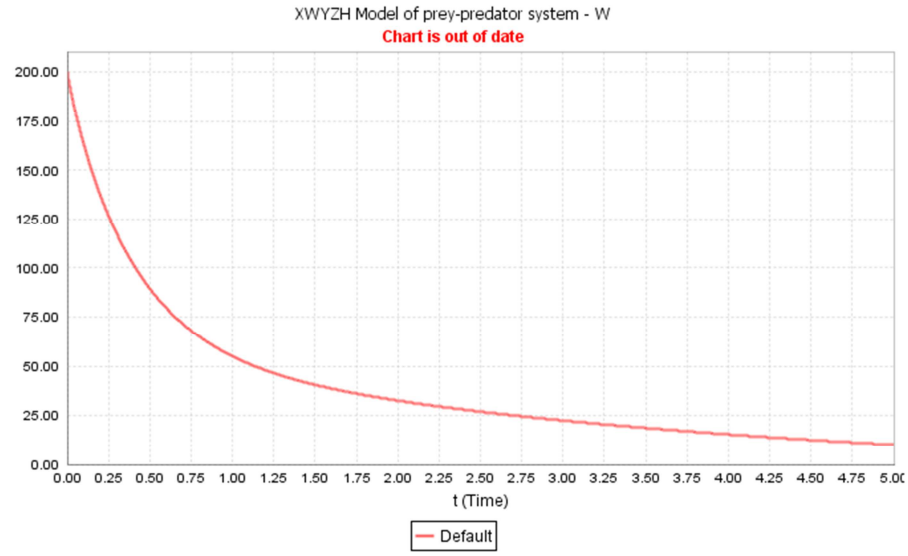
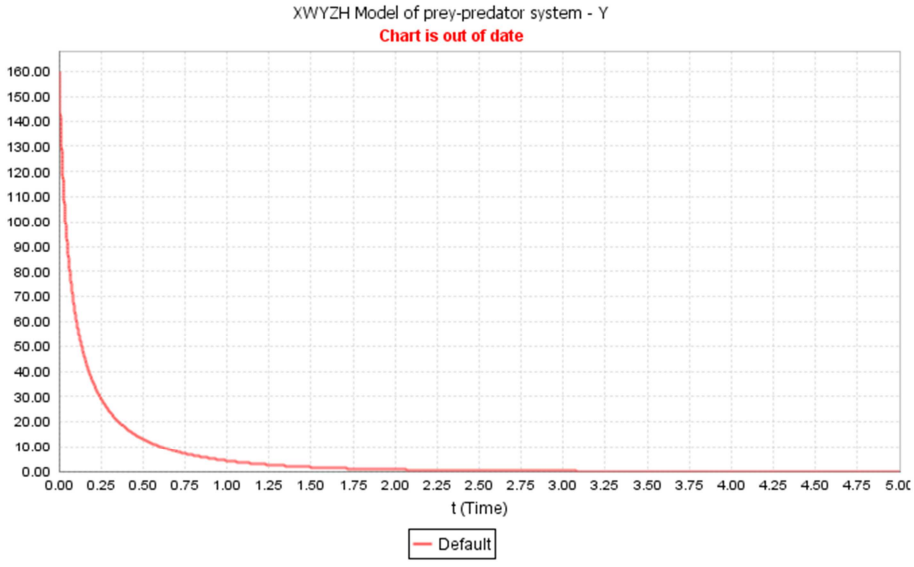


Figure 4. *High infection and predation.*





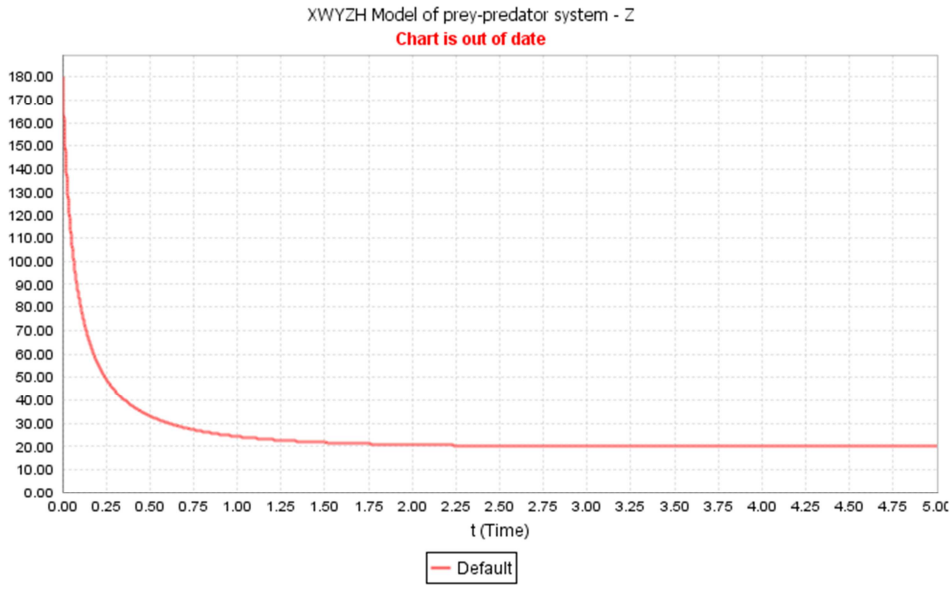


Figure 5. *Individual plot of W, Y and Z.*

In sample Simulation figure 4 and figure 5 shows that high infection and predation results the whole prey-predator population decline to a certain level. Therefore it is better to implement treatment mechanisms to sustain stability of the prey-predator system.

7. Conclusions and Recommendation

In this paper, It can be concluded that the formulated model is Mathematically meaningful, valid, and biologically well posed by proving the boundedness, positivity and existence of the solutions of the model. Trivial, Axial, Disease-free and endemic Equilibrium points are investigated. Moreover, It is observed that in our model trivial equilibrium point is always locally asymptotically unstable. Axial equilibrium point is locally asymptotically stable if and only if the variables satisfy the following three conditions: $(\mathrm{i})\beta k - (t_1 + d_2) < 0$, $(\mathrm{ii})\ qp_1 k - d_3(s + k) < 0$, & $(iii)\ qp_3 k - (t_2 + d_4)(s + k) < 0$. Treatment is helpful tool to minimize or eradicate infection in prey-predator system. Therefore Providing treatment in infected prey-predator system creates opportunity to recover from illness and the prey-predator population can be saved and exist in stable situation. Thus, it is recommended to apply treatment on infected prey-predator to make the whole prey-predator population safe and abundant in nature. One can extend this paper by Assuming the predator grows logistically or by adding parameter like death rate on the prey or by including other variables like vaccination, immigration, migration on prey-predator system, and these things can be considered as limitation of this paper.

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