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An Introduction to Differential Geometry: The Theory of Surfaces

Kande Dickson Kinyua[1, 2, *], Kuria Joseph Gikonyo[2]

[1]Department of Mathematics, Moi University, Eldoret, Kenya
[2]Department of Mathematics, Karatina University, Karatina, Kenya

Email address:
kandesnr@gmail.com (K. D. Kinyua), josephgkuria@gmail.com (K. J. Gikonyo)
[*]Corresponding author

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Abstract: From a mathematical perspective, a *surface* is a generalization of a plane which does not necessarily require being flat, that is, the curvature is not necessarily zero. Often, a surface is defined by equations that are satisfied by some coordinates of its points. A surface may also be defined as the image, in some space of dimensions at least three, of a continuous function of two variables (some further conditions are required to insure that the image is not a curve). In this case, one says that one has a parametric surface, which is parametrized by these two variables, called parameters. Parametric equations of surfaces are often irregular at some points. This is formalized by the concept of manifold: in the context of manifolds, typically in topology and differential geometry, a surface is a manifold of dimension two; this means that a surface is a topological space such that every point has a neighborhood which is homeomorphic to an open subset of the Euclidean plane. A *parametric surface* is the image of an open subset of the Euclidean plane by a continuous function, in a topological space, generally a Euclidean space of dimension at least three. The paper aims at giving an introduction to the theory of surfaces from differential geometry perspective.

Keywords: Curvature, Differential Geometry, Geodesics, Manifolds, Parametrized, *Surface*

1. Introduction

Differential geometry is a discipline of mathematics that uses the techniques of calculus and linear algebra to study problems in geometry. The theory of plane, curves and surfaces in the three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th and the 19th century. Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds. Differential geometry is closely related to differential topology and the geometric aspects of the theory of differential equations. Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces. The theory developed in this study originates from mathematicians of the 18th and 19th centuries. Principal contributors were Euler (1707-1783), Monge (1746-1818) and Gauss (1777-1855), [1, 2, 3, 9].

Mathematical study of curves and surfaces has been developed to answer some of the nagging and unanswered questions that appeared in calculus, such as the reasons for relationships between complex shapes and curves, series and analytic functions. These unanswered questions indicated greater, hidden relationships and symmetries in nature, which the standard methods of analysis could not address. The purpose of this paper is to give an elaborate introduction to the study of curves and surfaces, and those are, in general, curved. Nevertheless, our main tools to understand and analyze these curved objects are (tangent) lines and planes and the way those change along a curve, respective surface. This is why we start with a brief chapter assembling prerequisites from linear geometry and algebra, [4, 6, 7, 8, 10].

The objects that will be studied here are curves and surfaces in two- and three-dimensional space, and they are primarily studied by means of parametrization. The main properties of these objects, which will be studied, are notions related to the shape. The study tangents of curves and tangent spaces of surfaces, and the notion of curvature will be introduced and defined through differentiation of the parametrization, and related to first and second derivatives, respectively. The notion of curvature is quite complicated for surfaces, and the study of this notion will not be ignored. The culmination is a famous theorem of Gauss, which shows that the so-called Gauss curvature of a surface can be calculated directly from quantities which can be measured on the surface itself, without any reference to the surrounding three dimensional space, [4, 5, 7, 8, 10].

2. Manifolds

The core idea of both differential geometry and modern geometrical dynamics lies under the concept of manifold. A manifold is an abstract mathematical space, which locally resembles the spaces described by Euclidean geometry, but which globally may have a more complicated structure, [6, 9]. A manifold can be constructed by 'gluing' separate Euclidean spaces together; for example, a world map can be made by gluing many maps of local regions together, and accounting for the resulting distortions. Therefore, the surface of Earth is a manifold; locally it seems to be flat, but viewed as a whole from the outer space (globally) it is actually round. Another example of a manifold is a circle; small piece of a circle appears to be like a slightly bent part of a straight line segment, but overall the circle and the segment are different one-dimensional manifolds, [2, 9].

Definition 2.1: A manifold is a Hausdorff space M with a countable basis such that for each point $p \in M$ there is a neighborhood U of p that is homeomorphic to R^n for some integer n. If the integer n is the same for every point in M, then M is called a n-dimensional manifold, [1, 2, 9].

Definition 2.2: A topological space X is said to be Hausdorff if for any two distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$, [1, 9].

The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves with the ideas from linear algebra and topology. Certain special classes of manifolds also have additional algebraic structure; they may behave like groups, for instance. From [1, 4, 5], an atlas describes how a manifold is glued together from simpler pieces where each piece is given by a chart (coordinate chart or local coordinate system). The description of most manifolds requires more than one chart. An atlas is a specific collection of charts which covers a manifold. An atlas is not unique as all manifolds can be covered multiple ways using different combinations of charts, [4, 5, 8].

Definition 2.3: An atlas A on a manifold M is said to be maximal if for any compatible atlas A' on M any coordinate chart $(x,U) \in A'$ is also a member of A, [1, 9].

This definition of atlas is exactly analogous to the non–mathematical meaning of atlas. Each individual map in an atlas of the world gives a neighborhood of each point on the globe that is homeomorphic to the plane. While each individual map does not exactly line up with other maps that it overlaps with, the overlap of two maps can still be compared. Different choices for simple spaces and compatibility conditions give different objects. The dimension of the manifold at a certain point is the dimension of the Euclidean space charts at that point map to (number n in the definition), [2, 3, 9]. All points in a connected manifold have the same dimension. In topology and related branches of mathematics, a connected space is a topological space which cannot be written as the disjoint union of two or more nonempty spaces. Connectedness is one of the principal topological properties that is used to distinguish topological spaces. A manifold with empty boundary is said to be closed manifold if it is compact, and open manifold if it is not compact. All one-dimensional manifolds are curves and all two-dimensional manifolds are surfaces, [1, 3, 9].

3. Surfaces

Examples of surfaces abound in everyday life are: balloons, tubes, cans, soap films and the surface of our planet earth are all physical models of surfaces. In order to study the theory of surfaces in these objects, one needs understand the idea of coordinates to make calculations involved. Of course, all these surfaces can be thought of as embedded in Euclidean space E^3. But just as a curve needs only one coordinate, the very definition of a surface is that it is described using just two coordinates: surface of the earth by the longitude and latitude.

Definition 3.1: Let $D \subset R^2$ denote an open subset.

a) A function $f: D \to R$ of two variables is called *smooth* (C^∞), if *all* partial derivatives exist, and, moreover, are continuous functions.

b) A vector function $r: D \to R^3$, $r(u,v) = [x(u,v), y(u,v), z(u,v)]$ of two variables is called *smooth* (C^∞), if its coordinate functions $x, y, z: D \to R$ are all smooth.

The partial derivatives of a smooth function $f: D \to R$ with respect to the variables u and v at a point $(u_0, v_0) \in D$ are denoted $f_u(u_0,v_0), f_v(u_0,v_0) \in R$. Also, the partial derivatives of a vector function $r: D \to R^3$ at $(u_0,v_0) \in D$ are the vectors $r_u(u_0,v_0) = [x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)] \in R^3$, and $r_v(u_0,v_0) = [x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)] \in R^3$, [3, 4, 5, 10].

Definition 3.2: Let $D \subset R^2$ denote an open subset, let $r: D \to R^3$ denote a smooth vector function, and let $u, v: I \to \mathbf{R}$ denote smooth functions such that $(u(t), v(t)) \in D$ for all t in the interval I. Then, the composite function $\mathbf{x}: I \to \mathbf{R}^3$ defined as $\mathbf{x}(t) = \mathbf{r}(u(t), v(t))$ is smooth and has the derivative

$\mathbf{x}'(t) = u'(t)\mathbf{r}u(u(t),v(t)) + v'(t)\mathbf{r}v(u(t),v(t))$, [3, 4, 5, 10].

Definition 3.3: Let $D \subset R^2$ denote an open subset. A smooth vector function $r : D \to R^3$ of two variables is called a parametrization (or coordinate patch) for the surface $S \subset \mathbf{E}^3$ consisting of all points P with $\overrightarrow{OP} = \mathbf{r}(u,v)$ with $(u,v) \in D$ if:

a) r is a one-to-one (injective) map (i.e., every point in S corresponds to a unique point in D);

b) The partial derivatives $r_u(u_0,v_0) = [x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)] \in R^3$ and $r_v(u_0,v_0) = [x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)] \in R^3$ are linearly independent at every point $(u_0,v_0) \in D$.

A subset $S \subset \mathbf{E}^3$ that has a coordinate patch r as above, is called a regular surface, [3, 4, 5, 10].

Definition 3.4: The first parameter curve through $P_0 \in S$ with $\overrightarrow{OP_0} = \mathbf{r}(u_0,v_0)$ arises from parametrization $\mathbf{r}(u_0,v_0)$ looked upon as a vector function of the single variable u (with u in an interval containing u_0). It consists of the points $P(u,v_0)$ with $\overrightarrow{OP}_{(u,v_0)} = \mathbf{r}(u,v_0)$ in the image of the parallel to the u-axis through (u_0,v_0).

Similarly, the second parameter curve through P0 arises from the parametrization $\mathbf{r}(u_0,v)$ as a vector function of the single variable v, [3, 4, 5, 10].

Definition 3.5: A space curve C with parametrization $\mathbf{x} : I \to \mathbf{R}^3$ is called a smooth curve on $\mathbf{r}(D) \subset S$ if and only if there is a smooth parametrization $(u(t),v(t)), t \in I$ of a plane curve in D such that $\mathbf{x}(t) = \mathbf{r}(u(t),v(t))$, [3, 4, 5, 10].

Definition 3.6: Let S denote a regular surface and $P_0 \in S$.

a) The linear tangent plane $T_{P_0}S$ to S at P_0 consists of all velocity vectors to smooth curves on S through P_0. Given a coordinate patch $\mathbf{r} : D \to \mathbf{R}^3$ for S with $\mathbf{r}(u_0,v_0) = \overrightarrow{OP_0}$, it has a parametrization $T_{P_0}S = \{s\mathbf{r}_u(u_0,v_0) + t\mathbf{r}_v(u_0,v_0), s,t \in \mathbf{R}\}$.

The affine tangent plane $\pi_{P_0}S$ to S at P_0 consists of all points $Q \in \mathbf{E}^3$ with $\overrightarrow{P_0Q} \in T_{P_0}S$. It has a parametrization, [3, 4, 5, 10].

$$\pi_{P_0}S = \{Q \in \mathbf{E}^3 \mid \overrightarrow{OQ} = \overrightarrow{OP_0} + s\mathbf{r}_u(u_0,v_0) + t\mathbf{r}_v(u_0,v_0), s,t \in \mathbf{R}\}$$

Definition 3.7: A vector $\mathbf{n} \in \mathbf{R}^3$ is called a normal vector to S at P_0 if n is perpendicular to all tangent vectors $\mathbf{v} \in T_{P_0}S$ [3, 4, 5, 10].

Definition 3.8: Let $\mathbf{r} : D \to \mathbf{R}^3$ denote a coordinate patch for the surface S with $\mathbf{r}(u_0,v_0) = \overrightarrow{OP}, P \in S$. The vector

$$\nu(P) = \nu(u_0,v_0) = \frac{\mathbf{r}_u(u_0,v_0) \times \mathbf{r}_v(u_0,v_0)}{|\mathbf{r}u(u0,v0) \times \mathbf{r}v(u0,v0)|} \in \mathbf{R}3$$

is a unit normal vector to the surface S at the point P, [3, 4, 5, 10].

Definition 3.9: Let $\mathbf{r} : D \to \mathbf{R}^3$ denote a coordinate patch for the surface S. We define three functions $E, F, G : D \to \mathbf{R}$ given by, [3, 4, 5, 10].

$$E(u,v) = \mathbf{r}_u(u,v) \bullet \mathbf{r}_u(u,v)$$
$$F(u,v) = \mathbf{r}_u(u,v) \bullet \mathbf{r}_v(u,v)$$
$$G(u,v) = \mathbf{r}_v(u,v) \bullet \mathbf{r}_v(u,v).$$

Definition 3.10: Let $P \in S$ denote a point on the surface S with $\overrightarrow{OP} = \mathbf{r}(u_0,v_0)$. The length of the tangent vector $\mathbf{w} = a\mathbf{r}_u(u_0,v_0) + b\mathbf{r}_v(u_0,v_0) \in T_pS$ is given by $|\mathbf{w}|^2 = a^2E(u_0,v_0) + 2abF(u_0,v_0) + b^2G(u_0,v_0)$, [3, 4, 5, 10].

Definition 3.11: Let $\mathbf{r} : D \to \mathbf{R}^3$, $D \subset \mathbf{R}^2$ open, denote a coordinate patch for the surface S; let furthermore $\mathbf{x}(t) = \mathbf{r}(u(t),v(t)), a \le t \le b$ denote a parametrization for a curve C on S. The derivative $s'(t) = \frac{ds}{dt}$ of the arc length function $s(t), a \le t \le b$ satisfies:

$$(s'(t))^2 = u'(t)^2 E(u(t),v(t)) + 2u'(t)v'(t)F(u(t),v(t)) + (v'(t))^2 G(u(t),v(t))$$

and thus the length of the segment of the curve C between the points corresponding to the parameters t_0 to t is given (in short form) by, [3, 4, 5, 10].

$$s(t_1) = \int_{t_0}^{t_1} s'(t)dt = \int_{t_0}^{t_1} \sqrt{(u')^2 E + 2u'v'F + (v')^2 G}\ dt.$$

Definition 3.12: The angle α between the curves C_1 and C_2 satisfies, [3, 4, 5, 10]:

$$cos\alpha = \frac{u_1{}'u_2{}'E + (u_1{}'v_2{}' + u_2{}'v_1{}')F + v_1{}'v_2{}'G}{\sqrt{\left((u_1{}')^2E + 2u_1{}'v_1{}'F + (v_1{}')^2G\right)\left((u_2{}')^2E + 2u_2{}'v_2{}'F + (v_2{}')^2G\right)}}$$

Definition 3.13: Let $\mathbf{r}: D \to \mathbf{R}^3$ denote a smooth parametrization for the surface S, and let $R \subset S$ denote a subset whose boundary is a piecewise smooth curve $C \subset S$. The area a(R) of R is given as, [3, 4, 5, 10]:

$$a(R) = \iint_{\mathbf{r}^{-1}(R)} |\mathbf{r}_u \times \mathbf{r}_v| \, dudv = \iint_{\mathbf{r}^{-1}(R)} \sqrt{EG - F^2} \, dudv$$

Definition 3.14: The normal section $C_{\mathbf{v}}(P)$ of the surface S at the point $P \in S$ in direction $\mathbf{v} \in T_P S$ is the curve which arises as the intersection of the normal plane $\pi_{\mathbf{v}}(P)$ and the surface S, i.e., $C_{\mathbf{v}}(P) = S \cap \pi_{\mathbf{v}}(P)$.

The normal curvature $\kappa_n(P;\mathbf{v})$ is then defined as the (plane) curvature of the normal section $C_{\mathbf{v}}(P)$ viewed as a curve in the normal plane $\pi_{\mathbf{v}}(P)$ with orientation given by the basis $\{\mathbf{v}, \nu(P)\}$, [3, 4, 5, 10].

Definition 3.15: Let S be the surface obtained as the graph of a smooth function $f: D \to \mathbf{R}, D \subset \mathbf{R}^2$ open with $\mathbf{0} = [0,\ 0] \in D$ and $f(0,\ 0) = f_x(0,\ 0) = f_y(0,\ 0) = f_{xy}(0,\ 0) = 0$.

Then, $\kappa_1 = f_{xx}(0,\ 0)$ and $\kappa_2 = f_{yy}(0,\ 0)$ are the maximal, resp. minimal normal curvatures of S at O, and the normal curvatures $\kappa_n(\theta)$ in direction $\mathbf{v} = [cos\theta,\ sin\theta,\ 0] \in T_O S$ is given by Euler's formula $\kappa_n(O;\theta) = (cos\theta)^2 \kappa_1 + (sin\theta)^2 \kappa_2$, [3, 4, 5, 10].

Definition 3.16: Let S be the graph of the function f above, let $\kappa_1 = f_{xx}(0,\ 0)$ and $\kappa_2 = f_{yy}(0,\ 0)$. Then, the second order Taylor approximation of f at (0, 0) takes the form $F(x,y) = \kappa_1 x^2 + \kappa_2 y^2$.

The surface T given as the graph of the function F is called the approximating paraboloid of S at O, [3, 4, 5, 10].

Definition 3.17: Normal and geodesic curvature.

Let S denote a surface with a given parametrization $\mathbf{r}: \Omega \to \mathbf{R}^3, \Omega \subset \mathbf{R}^2$. Let C denote a curve on S with parametrization $\mathbf{x}(t) = \mathbf{r}(u(t), v(t))$ with $(u(t), v(t))$ a parametrization for the corresponding curve in Ω. Let Pt denote the point on the curve with $\overrightarrow{OP_t} = \mathbf{r}(u(t), v(t))$. Along the curve, we have the following vector fields ("moving vectors"):

a) the tangent vector field t(t) (or "moving tangent vector") attaching to each point on the curve the unit tangent vector.

b) the normal vectors to the surface S – given by $\nu(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}(u,v)$ – assemble to a normal vector field ("moving normal vector") $\nu(t) = \nu(u(t), v(t))$ along the curve.

c) The vector $\gamma(t) = \nu(t) \times \mathbf{t}(t)$ is contained in the tangent plane $T_{P_t} S$ at P_t and is perpendicular to t(t); it constitutes an "oriented normal vector" to t(t) with respect to $T_{P_t} S$", [3, 4, 5, 10].

Definition 3.18: The coefficients of these two components are called

- $\kappa_g(t) = \kappa(t)\ sin\theta(t)$ – the geodesic curvature at P_t;
- $\kappa_n(t) = \kappa(t)\ cos\theta(t) = \kappa(t)\mathbf{n}(t) \bullet \nu(t)$ – the normal curvature at P_t, [3, 4, 5, 10].

Definition 3.19: The normal curvature of κ satisfies, [3], [4], [5], [10]:

$$\kappa_n(t) = \frac{1}{s'(t)}\left(\mathbf{t}'(t) \bullet \nu(t)\right) = -\frac{1}{s'(t)}\left(\mathbf{t}(t) \bullet \nu'(t)\right).$$

Definition 3.20: Normal curvature depends only on the tangent vector t of the curve at P, and not on the curve itself. For a non-zero tangent vector $\mathbf{t} = a\mathbf{r}_u + b\mathbf{r}_v$, and the definition of the normal curvature of S in direction t, [3, 4, 5, 10]:

$$\kappa_n(\mathbf{t}) = -\frac{1}{|\mathbf{t}|^2}\left((\mathbf{r}_u \bullet \nu_u)a^2 + (\mathbf{r}_u \bullet \nu_v + \mathbf{r}_v \bullet \nu_u)ab + (\mathbf{r}_v \bullet \nu_v)b^2\right)$$

Definition 3.21: The second fundamental form.

Finally, we want to find expressions for the coefficients in the definition 3.12, that are easy to calculate. To this end, we use, [3], [4], [5], [10]:

Definition 3.22: Given a surface S with parametrization $r: D \to \mathbf{R}^3$ as above. We define three real-valued smooth functions $e, f, g: D \to \mathbf{R}$ by

$$e(u,v) = (\mathbf{r}_{uu} \bullet \nu)(u,v) = \frac{\mathbf{r}_{uu} \bullet (\mathbf{r}_u \times \mathbf{r}_v)}{|\mathbf{r}_u \times \mathbf{r}_v|}(u,v)$$

$$f(u,v) = (\mathbf{r}_{uv} \bullet \nu)(u,v) = \frac{\mathbf{r}_{uv} \bullet (\mathbf{r}_u \times \mathbf{r}_v)}{|\mathbf{r}_u \times \mathbf{r}_v|}(u,v)$$

$$g(u,v) = (\mathbf{r}_{vv} \bullet \nu)(u,v) \frac{\mathbf{r}_{vv} \bullet (\mathbf{r}_u \times \mathbf{r}_v)}{|\mathbf{r}_u \times \mathbf{r}_v|}(u,v)$$

and the second fundamental form on a tangent vector $\mathbf{t} = a\mathbf{r}_u + b\mathbf{r}_v \in T_P S$ with $\overrightarrow{OP} = \mathbf{r}(u,v)$ as the quadratic polynomial in the two variables a and b, [3, 4, 5, 10]: $II(\mathbf{t})(u,v) = e(u,v)a^2 + 2f(u,v)ab + g(u,v)b^2$.

Definition 3.23: The normal curvature kn of S at P with $\overrightarrow{OP} = \mathbf{r}(u,v)$ in the tangent direction $\mathbf{t} = a\mathbf{r}_u + b\mathbf{r}_v \in T_P S$ is the quotient of the secondand the first fundamental form (3.15) at t, [3, 4, 5, 10]; i.e.,

$$\kappa_n(\mathbf{t}) = \frac{II(\mathbf{t})}{I(\mathbf{t})} = \frac{e(u,v)a^2+2f(u,v)ab+g(u,v)b^2}{E(u,v)a^2+2F(u,v)ab+G(u,v)b^2}$$

Definition 3.23: Calculation of principal curvatures and principal directions.

Our next aim is to find at every point P on a surface S two principal curvatures such that all other normal curvatures $\kappa_n(\mathbf{t})$ are sandwiched between those two. Moreover, it would be nice to have formulas calculating these entities. Our point of departure is definition 3.23 expressing normal curvatures as the quotient of the two fundamental forms on the tangent direction. What are the maximal, resp. minimal values for this expression (the normal curvature), and in which (tangent) directions do they occur?, [3, 4, 5, 10].

Here is another way to phrase this question: Let $P \in S$ be such that $\overrightarrow{OP} = \mathbf{r}(u_0, v_0)$.

We fix the values of the two fundamental forms at that point, i.e.,

$$E = E(u_0,v_0), F = F(u_0,v_0), G = G(u_0,v_0),$$
$$e = e(u_0,v_0), f = f(u_0,v_0), g = g(u_0,v_0)$$

Now we ask: For which real numbers k does the equation

$$\kappa = \frac{ea^2+2fab+gb^2}{Ea^2+2Fab+Gb^2}$$

have a non-trivial solution [a, b] 6= [0, 0]?

Remark: There exists always a solution for $k = \frac{e}{E}$, i.e., $[a,b] = [1,\ 0]$

Definition 3.24: Let $P \in S$ be a point on a regular surface, and let E, F, G and e, f , g denote the coefficients of the first and second fundamental forms at P in a given parametrization. Then, we define the Gaussian curvature $\kappa(P)$ of S at P as the real number $\kappa(P) = \frac{eg - f^2}{EG - F^2}$, and the mean curvature H(P) as the real number $H(p) = \frac{eG + gE - 2fF}{2(EG - F^2)}$. Remark, that K and H define smooth functions K(u, v) and H(u, v) on their domain, [3, 4, 5, 10].

Definition 3.25: Let $P \in S$ be a point on a surface S, let K(P) and H(P) denote the Gaussian, resp. mean curvature of S at P. Then, the numbers $\kappa_1(P) = H(P) + \sqrt{H(P)^2 - K(P)}$ and $\kappa_2(P) = H(P) - \sqrt{H(P)^2 - K(P)}$ are called the principal curvatures for S at P. The associated principal directions are the tangent directions $\mathbf{t}_1 = a_1\mathbf{r}_u + b_1\mathbf{r}_v$ and $\mathbf{t}_2 = a_2\mathbf{r}_u + b_2\mathbf{r}_v$ with $\kappa_n(\mathbf{t}_i) = \kappa_i(P),\ i = 1,2$. (These are only well-determined for $\kappa_1 \neq \kappa_2$!), [3, 4, 5, 10].

Definition 3.26:

a) The principal curvatures, Gaussian curvature and mean curvature at a point $P \in S$ are connected by the following relations: $K(P) = \kappa_1(P)\kappa_2(P)$ $H(P) = \frac{\kappa_1(P) + \kappa_2(P)}{2}$;

b) the last equation explains the name mean curvature.

c) The principal directions $\mathbf{t}_i = a_i\mathbf{r}_u + b_i\mathbf{r}_v$ can be determined as the solutions of the linear equations $(\kappa_i E - e)a_i + (\kappa_i F - f)b_i = 0$, (or as the solutions of the linear equations $(\kappa_i F - f)a_i + (\kappa_i G - g)b_i = 0$.), [3, 4, 5, 10].

Definition 3.27: The geometric significance of the Gaussian curvature.

The Gaussian curvature appears just as a tool in the calculation of the principal curvatures. In fact, this invariant can tell us much more about the local and global properties of the surface S. First of all, one can see, that Gaussian curvature, mean curvature, and thus the principal curvatures are independent of the chosen parametrization – whereas the coefficients E, F, G and e, f , g clearly depend on parametrizations. The reason is, that the principal curvatures – as the extremal curvatures of the normal sections – are geometric entities that do not depend on parametrization. Gaussian curvature and mean curvature can be calculated from the principal curvatures, [3, 4, 5, 10].

Definition 3.28: Classification of points on a surface.

Already the sign of the Gaussian curvature contains very useful information about the surface S in the neighbourhood of a given point P ∈ S, [3, 4, 5, 10].

Definition 3.29: A point P ∈ S is calledelliptic if K(P) > 0,

a) hyperbolic if K(P) < 0,

b) parabolic if K(P) = 0 and k1(P) 6= 0 or k2(P) 6= 0,

c) planar if k1(P) = k2(P) = 0, [3, 4, 5, 10].

4. Geodesics

In mathematics, the differential geometry of surfaces deals with the differential geometry of smooth surfaces with various additional structures. Surfaces have been extensively studied from various perspectives: extrinsically, relating to their embedding in Euclidean space and intrinsically, reflecting their properties determined solely by the distance within the surface as measured along curves on the surface. One of the fundamental concepts investigated is the Gaussian curvature which was first introduced by Carl Friedrich Gauss (1825-1827)), who showed that curvature was an intrinsic property of a surface, independent of its isometric embedding in Euclidean space, [4, 7, 10].

Surfaces naturally arise as graphs of functions of a pair of variables, and sometimes appear in parametric form or as loci associated to space curves. An important role in their study has been played by Lie groups, namely the symmetry groups of the Euclidean plane, the sphere and the hyperbolic plane. These Lie groups can be used to describe surfaces of constant Gaussian curvature; they also provide an essential ingredient in the modern approach to intrinsic differential geometry through connections. On the other hand, extrinsic properties relying on an embedding of a surface in Euclidean space have also been extensively studied. This is well illustrated by the non-linear Euler-Lagrange equations in the calculus of variations: although Euler developed the one variable equation to understand geodesics, defined independently of an embedding, one of Lagrange's main applications of the two variable equations was to minimal surfaces, a concept that can only be defined in terms of an embedding, [4, 7, 10].

The smooth surfaces equipped with Riemannian metrics are of foundational importance in differential geometry. A Riemannian metric endows a surface with notions of geodesic, distance, angle and area; an important class of such surfaces are the developable surfaces which are surfaces that can be flattened to a plane without stretching (examples the cylinder and the cone). On the other In addition, there are properties of surfaces which depend on an embedding of the surface into Euclidean space. These surfaces are the subject of extrinsic geometry. They include the minimal surfaces which are surfaces that minimize the surface area for given boundary conditions (examples include soap films stretched across a wire frame, catenoids and helicoids) and ruled surfaces which are surfaces with at least one straight line running through every point (examples include cylinder and hyperboloid of one sheet), [4, 7, 10].

5. Conclusion

Starting with the idea of a chart, the paper has developed a step by step introduction of the theory of surfaces upto the curvature of a surface. Using the notions and definitions developed in the this paper, the focus is now classes of interesting surfaces, get new information and develop some advanced concepts on surfaces.

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