**Pure and Applied Mathematics Journal**
2015; 4(5-1): 55-59
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.s.2015040501.21
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

# A Logarithmic Derivative of Theta Function and Implication

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### To cite this article:
Yaling Men, Jiaolian Zhao. A Logarithmic Derivative of Theta Function and Implication. *Pure and Applied Mathematics Journal*. Special Issue: Mathematical Aspects of Engineering Disciplines. Vol. 4, No. 5-1, 2015, pp. 55-59. doi: 10.11648/j.pamj.s.2015040501.21

**Received:** June 26, 2015; **Accepted:** June 28, 2015; **Published:** June 30, 2016

**Abstract:** In this paper we establish an identity involving logarithmic derivative of theta function by the theory of elliptic functions. Using these identities we introduce Ramanujan's modular identities, and also re-derive the product identity, and many other new interesting identities.

**Keywords:** Theta Function, Elliptic Function, Logarithmic Derivative

## 1. Introduction and Definitions

Assume throughout this paper that $q = e^{\pi i\tau}$, when $\Im\tau > 0$. As usual, the classical Jacobi theta functions are defined as follow[1-3],

$$\theta_1(z\mid\tau) = -iq^{\frac{1}{4}}\sum_{n=-\infty}^{+\infty}(-1)^n q^{n(n+1)}e^{(2n+1)iz} \qquad (1.1)$$

$$\theta_2(z\mid\tau) = q^{\frac{1}{4}}\sum_{n=-\infty}^{+\infty}(-1)^n q^{n(n+1)}e^{(2n+1)iz} \qquad (1.2)$$

$$\theta_3(z\mid\tau) = \sum_{n=-\infty}^{+\infty} q^{n^2}e^{2niz} \qquad (1.3)$$

$$\theta_4(z\mid\tau) = \sum_{n=-\infty}^{+\infty}(-1)^n q^{n^2}e^{2niz} \qquad (1.4)$$

The $q-$shifed factorial is defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty}\left(1-aq^n\right)$$

and some times write

$$(a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty$$

With above notation, the celebrated Jacobi triple product identity can be expressed as follow

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}z^n = (q;q)_\infty(z;q)_\infty(q/z;q)_\infty \qquad (1.5)$$

Employing the Jacobi triple product identity, we can derive the infinite product expressions for theta function

Proposition1.1. (Infinite product representations for theta functions)

$$\theta_1(z\mid\tau) = 2q^{\frac{1}{4}}\sin z\left(q^2;q^2\right)_\infty\left(q^2e^{2iz};q^2\right)_\infty\left(q^2e^{-2iz};q^2\right)_\infty$$

$$\theta_2(z\mid\tau) = 2q^{\frac{1}{4}}\left(q^2;q^2\right)_\infty\left(-q^2e^{2iz};q^2\right)_\infty\left(-q^2e^{-2iz};q^2\right)_\infty$$

$$\theta_3(z\mid\tau) = \left(q^2;q^2\right)_\infty\left(-q^2e^{2iz};q^2\right)_\infty\left(-qe^{-2iz};q^2\right)_\infty$$

$$\theta_4(z\mid\tau) = \left(q^2;q^2\right)_\infty\left(qe^{2iz};q^2\right)_\infty\left(qe^{-2iz};q^2\right)_\infty$$

When there is no confusion, We will use $\theta_i(z)$ for $\theta_i(z\mid\tau)$, $\theta_i^{'}(z)$ for $\theta_i^{'}(z\mid\tau)$ to denote the partial derivative with respect to the variable, and $\theta_i$ for $\theta_i(0\mid\tau)$, $i = 1,2,3,4$. From the above equations, the following facts are obvious

$$\theta_1' = 2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)} = 2q^{\frac{1}{4}}\left(q^2;q^2\right)_\infty^3$$

$$\begin{aligned}\theta_2 &= q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^{n(n+1)} = 2q^{\frac{1}{4}}\left(q^2;q^2\right)_\infty\left(-q^2;q^2\right)_\infty^2\\ \theta_3 &= \sum_{n=-\infty}^{\infty}q^{n^2} = \left(q^2;q^2\right)_\infty\left(q;q^2\right)_\infty^2\\ \theta_4 &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \left(q^2;q^2\right)_\infty\left(-q;q^2\right)_\infty^2\end{aligned} \tag{1.6}$$

With respect to the (quasi) period $\pi$ and $\pi\tau$, Jacobi theta functions $\theta_i, i=1,2,3,4$ satisfy the following relations

$$\theta_1(z+\pi\mid\tau) = -\theta_1(z\mid\tau),\quad \theta_2(z+\pi\mid\tau) = -\theta_2(z\mid\tau)$$

$$\theta_3(z+\pi\mid\tau) = \theta_3(z\mid\tau),\quad \theta_4(z+\pi\mid\tau) = \theta_4(z\mid\tau)$$

$$\theta_1(z+\pi\tau\mid\tau) = -q^{-1}e^{-2iz}\theta_1(z\mid\tau)$$

$$\theta_2(z+\pi\tau\mid\tau) = q^{-1}e^{-2iz}\theta_2(z\mid\tau)$$

$$\theta_3(z+\pi\tau\mid\tau) = q^{-1}e^{-2iz}\theta_3(z\mid\tau)$$

$$\theta_4(z+\pi\tau\mid\tau) = -q^{-1}e^{-2iz}\theta_4(z\mid\tau)$$

and

$$\theta_1\left(z+\frac{\pi}{2}\mid\tau\right) = \theta_2(z\mid\tau),\quad \theta_2\left(z+\frac{\pi}{2}\mid\tau\right) = -\theta_1(z\mid\tau)$$

$$\theta_3\left(z+\frac{\pi}{2}\mid\tau\right) = \theta_4(z\mid\tau),\quad \theta_4\left(z+\frac{\pi}{2}\mid\tau\right) = \theta_3(z\mid\tau)$$

also have

$$\begin{aligned}\theta_1\left(z+\frac{\pi\tau}{2}\mid\tau\right) &= iM\theta_4(z\mid\tau)\\ \theta_2\left(z+\frac{\pi\tau}{2}\mid\tau\right) &= M\theta_3(z\mid\tau)\\ \theta_3\left(z+\frac{\pi\tau}{2}\mid\tau\right) &= M\theta_2(z\mid\tau)\\ \theta_4\left(z+\frac{\pi\tau}{2}\mid\tau\right) &= iM\theta_1(z\mid\tau)\end{aligned} \tag{1.7}$$

$$\begin{aligned}\theta_1\left(z+\frac{\pi\tau+\pi}{2}\mid\tau\right) &= M\theta_3(z\mid\tau)\\ \theta_2\left(z+\frac{\pi\tau+\pi}{2}\mid\tau\right) &= -iM\theta_4(z\mid\tau)\\ \theta_3\left(z+\frac{\pi\tau+\pi}{2}\mid\tau\right) &= iM\theta_1(z\mid\tau)\\ \theta_4\left(z+\frac{\pi\tau+\pi}{2}\mid\tau\right) &= M\theta_2(z\mid\tau)\end{aligned} \tag{1.8}$$

Where $M = q^{-\frac{1}{4}}e^{-iz}$.

The following trigonometric series expressions for the logarithmic derivative with respect to $z$ of Jacobi Theta functions will be very useful in this paper,

$$\begin{aligned}\frac{\theta_1'}{\theta_1}(z\mid\tau) &= \cot z + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nz\\ \frac{\theta_2'}{\theta_2}(z\mid\tau) &= -\tan z + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1-q^{2n}}\sin 2nz\\ \frac{\theta_3'}{\theta_3}(z\mid\tau) &= 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{n}}{1-q^{2n}}\sin 2nz\\ \frac{\theta_4'}{\theta_4}(z\mid\tau) &= 4\sum_{n=1}^{\infty}\frac{q^{n}}{1-q^{2n}}\sin 2nz\end{aligned} \tag{1.9}$$

***Theorem 1.1.*** The sum of all the residues of an elliptic function in the period parallelogram is zero.

## 2. Main Theorem and Proofs

***Theorem 2.1.*** For $x$ and $y$, we have

$$\frac{\theta_4'}{\theta_4}(x)+\frac{\theta_4'}{\theta_4}(y)-\frac{\theta_4'}{\theta_4}(x+y) = \theta_2\theta_3\frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_4(x)\theta_4(y)\theta_4(x+y)}$$

Proof. We consider the following function

$$f(z) = \frac{\theta_4(z+x)\theta_4(z+y)\theta_4(z-x-y)}{\theta_1^2(z)\theta_4(z)} \tag{2.1}$$

by the definition of $\theta_i(z\mid\tau)$, we can readily verify that $f(z)$ is an elliptic function with periods $\pi$ and $\pi\tau$,The only poles of $f(z)$ is $0$ and $\frac{\pi\tau}{2}$. Furthermore, $\frac{\pi\tau}{2}$ is its simple pole and 0 is its pole with order two. By virtue of the residue theorem of elliptic functions, we have

$$Res\left(f;\frac{\pi\tau}{2}\right)+Res(f;0) = 0. \tag{2.2}$$

And applying relation of $\theta_1$ and $\theta_4$ in (1.7-1.8) and L'Hospital' rule, we can obtain

$$Res\left(f;\frac{\pi\tau}{2}\right)=\lim_{z\to\frac{\pi\tau}{2}}\left(z-\frac{\pi\tau}{2}\right)f(z)$$

$$=\lim_{z\to\frac{\pi\tau}{2}}\frac{\left(z-\frac{\pi\tau}{2}\right)\theta_4(z+x)\theta_4(z+y)\theta_4(z-x-y)}{\theta_1^2(z)\theta_4(z)}$$

$$=\frac{(iB)^3\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_4'\left(\frac{\pi\tau}{2}\right)\theta_1^2\left(\frac{\pi\tau}{2}\right)} \tag{2.3}$$

$$=-\frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta'_1(0)\theta_4^2(0)}$$

Next we compute $Res(f;0)$,

$$Res(f;0)=\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right)$$

$$=\lim_{z\to 0}z^2 f(z)\left[\frac{2}{z}+\frac{f'}{f}(z)\right] \tag{2.4}$$

$$=\lim_{z\to 0}z^2 f(z)\left[\frac{2}{z}+\frac{\theta'_4}{\theta_4}(z+x)+\frac{\theta'_4}{\theta_4}(z+y)+\frac{\theta'_4}{\theta_4}(z-x-y)-2\frac{\theta'_1}{\theta_1}(z)-\frac{\theta'_4}{\theta_4}(z)\right]$$

$$=\frac{\theta_4(x)\theta_4(y)\theta_4(x+y)}{\theta_1'^2(0)\theta_4(0)}\left[\frac{\theta_4'}{\theta_4}(x)+\frac{\theta_4'}{\theta_4}(y)-\frac{\theta_4'}{\theta_4}(x+y)\right]$$

From Theorem 1.1, substituting (2.3) and (2.4) into (2.2), by performing a little reduction we can complete the proof of Theorem 2.1.

**Corollary 2.1.** For $x$ and $y$, we have

$$\left(\frac{\theta_4'}{\theta_4}\right)'(y)-\left(\frac{\theta_4'}{\theta_4}\right)'(x)=\left(\theta_1'\right)^2\frac{\theta_1(x+y)\theta_1(x-y)}{\theta_4^2(x)\theta_4^2(y)}$$

***Proof.*** We differentiate the formulae of Theorem2.1 with respect to $y$, and then set $y=0$, then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(0)-\left(\frac{\theta_4'}{\theta_4}\right)'(x)=(\theta_2\theta_3)^2\left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2 \tag{2.5}$$

Now we combine with another elementary identity [7, p.467]

$$\theta_1^2(x)\theta_4^2(y)-\theta_1^2(y)\theta_4^2(x)=\theta_4^2\theta_1(x+y)\theta_1(x-y) \tag{2.6}$$

From formula (2.5) and (2.6), we can obtain

$$\left(\frac{\theta_4'}{\theta_4}\right)'(y)-\left(\frac{\theta_4'}{\theta_4}\right)'(x)=(\theta_2\theta_3)^2\left(\left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2-\left(\frac{\theta_1(y)}{\theta_4(y)}\right)^2\right)$$

$$=(\theta_2\theta_3)^2\frac{\theta_1^2(x)\theta_4^2(y)-\theta_1^2(y)\theta_4^2(x)}{\theta_4^2(x)\theta_4^2(y)}$$

$$=(\theta'_1)^2\frac{\theta_1(x+y)\theta_1(x-y)}{\theta_4^2(x)\theta_4^2(y)}$$

This completes the proof of Corollary 2.2.

***Remark 2.1.*** The corollary2.1 is often written in terms of the weierstrass elliptic and sigma functions as [7, p.451]

$$\wp(y)-\wp(x)=\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}$$

***Theorem 2.2.*** For $x$ and $y$ are real, we have

$$\frac{\operatorname{Im}\left[\frac{\theta_1'}{\theta_1}(z-x)-\frac{\theta_4'}{\theta_4}(z-y)\right]}{\left|\left(\frac{\theta_1'}{\theta_1}\right)'(z-x)-\left(\frac{\theta_4'}{\theta_4}\right)'(z-y)\right|}=\frac{i}{2}\frac{\theta_1(z-\bar{z})\theta_4(z+\bar{z}-x-y)}{\theta'_1\left|\theta_4(2z-x-y)\right|} \tag{2.7}$$

where $\operatorname{Im}z$ denotes the imaginary part of the complex number $z$.

***Proof.*** Firstly, we replace $x$ by $z-x+\frac{\pi\tau}{2}$ and $y$ by $x-y+\frac{\pi\tau}{2}$ in Theorem 2.1. it becomes

$$\frac{\theta_1'}{\theta_1}(z-x)+\frac{\theta_1'}{\theta_1}(x-y)-\frac{\theta_4'}{\theta_4}(z-y)=\theta_2\theta_3\frac{\theta_4(z-x)\theta_4(x-y)\theta_1(z-y)}{\theta_1(z-x)\theta_1(x-y)\theta_4(z-y)} \tag{2.8}$$

Sine $x-y$ is real, $\theta_i(x-y), i=1,2,3,4$ is also real valued, then we have

$$\operatorname{Im}\left[\frac{\theta_1'}{\theta_1}(z-x)-\frac{\theta_4'}{\theta_4}(z-y)\right]=\theta_2\theta_3\frac{\theta_4(x-y)}{\theta_1(x-y)}\operatorname{Im}\left(\frac{\theta_4(z-x)\theta_1(z-y)}{\theta_4(z-y)\theta_1(z-x)}\right) \tag{2.9}$$

We note that (2.9) is precisely the numerator of (2.7). We now consider its denominator. In Corollary2.1, replace $y$ by

$z-y$ $z-x+\frac{\pi\tau}{2}$ and $x$ by $z-y$, then obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)'(z-x)-\left(\frac{\theta_4'}{\theta_4}\right)'(z-y)$$
$$=(\theta'_1)^2\frac{\theta_4(2z-x-y)\theta_4(y-x)}{\theta_1^2(z-x)\theta_4^2(z-y)} \quad (2.10)$$

Now from (2.9) and (2.10), the left hand side of (2.7) becomes

$$\frac{1}{\theta_1'\theta_4}\frac{1}{\theta_1(x-y)}\mathrm{Im}\left(\frac{\theta_4(z-x)\theta_1(z-y)}{\theta_4(z-y)\theta_1(z-x)}\right)\times$$
$$\left|\frac{\theta_1^2(z-x)\theta_4^2(z-y)}{\theta_4(2z-x-y)\theta_4(2z-x-y)}\right| \quad (2.11)$$
$$=\frac{\mathrm{Im}\left(\theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y)\right)}{\theta_1'\theta_4\theta(x-y)\left|\theta_4(2z-x-y)\right|}$$

Here we can see that it is crucial that $x$ and $y$ are both real , Since $\overline{\theta_i(z-x)}=\overline{\theta_i(z-y)}$. On the other hand, we can derive a different expression for the imaginary part of the above quantity. Since we note that in Corollary2.1, replacing $y$ by $z-y$, $x$ by $x-z$ and $z$ by $y-\bar{z}$, we can deduce (where $\mathrm{Im}\,z=\frac{z-\bar{z}}{2i}$).

$$2i\,\mathrm{Im}\left(\theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y)\right)$$
$$=\theta_4(z-x)\theta_1(z-y)\theta_1(\bar{z}-x)\theta_4(\bar{z}-y)$$
$$-\theta_1(z-x)\theta_4(z-y)\theta_4(\bar{z}-x)\theta_1(\bar{z}-y)$$
$$=\theta_4\theta_4(z+\bar{z}-x-y)\theta_1(y-x)\theta_1(z-\bar{z})$$

Substituting above equality into (2.11), we can obtain the result (2.7). This complete the proof of Theorem 2.2.

## 3. Implications for Square Sum

In this section, we will re-deduce the Lambert series representations for $\theta_i^2,\theta_i^4,\theta_i^6$ from Theorem 2.1 easily and difference methods from [4-6].

***Theorem 3.1.*** For Jacobi Theta function $\theta_3$, we have

$$\theta_3^2=1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}$$

***Proof.*** We note that $\theta_4\left(x+\frac{\pi\tau+\pi}{2}\right)=q^{\frac{1}{4}}e^{-ix}\theta_2(x)$, then $\frac{\theta'_4}{\theta_4}\left(x+\frac{\pi\tau+\pi}{2}\right)=-i+\frac{\theta'_2}{\theta_2}(x)$

In Theorem 2.1, we replace $x$ by $x+\frac{\pi\tau+\pi}{2}$, then it becomes

$$\frac{\theta'_2}{\theta_2}(x)+\frac{\theta'_4}{\theta_4}(y)-\frac{\theta'_2}{\theta_2}(x+y)=\theta_2\theta_3\frac{\theta_3(x)\theta_1(y)\theta_3(x+y)}{\theta_2(x)\theta_4(y)\theta_2(x+y)}$$

Next,we choose $x=0$ and $y=\frac{\pi}{4}$ with the facts that $\theta_2'(0)=0,\theta_4\left(\frac{\pi}{4}\right)=\theta_3\left(\frac{\pi}{4}\right),\theta_2\left(\frac{\pi}{4}\right)=\theta_1\left(\frac{\pi}{4}\right)$ ,then above equality becomes

$$\theta_3^2=\frac{\theta_4'}{\theta_4}\left(\frac{\pi}{4}\right)-\frac{\theta_2'}{\theta_2}\left(\frac{\pi}{4}\right)=1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}$$

This complete the proof of Theorem 3.1

***Theorem 3.2.*** For Jacobi Theta function $\theta_3$, we have

$$\theta_3^4=1+8\sum_{n=1}^{\infty}\frac{nq^n\left(1-(-1)^nq^n\right)}{1-q^{2n}}$$

***Proof.*** We set $x=0$ in (12 ), then differentiate it with respect to $y$ and set $y=0$, We recall (1.9) for $\frac{\theta_4'}{\theta_4}(z\,|\,\tau)$, then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(z\,|\,\tau)=8\sum_{n=1}^{\infty}\cos 2nz=4\sum_{n=1}^{\infty}\frac{nq^n\left(e^{2inz}+e^{-2inz}\right)}{1-q^{2n}}$$

And from (1.9), we can obtain

$$\left(\frac{\theta_2'}{\theta_2}\right)'(z\,|\,\tau)=-\sec^2 z+\sum_{n=1}^{\infty}\frac{(-1)^n nq^{2n}\cos 2nz}{1-q^{2n}}$$

Hence, we can obtain

$$\theta_3^4=\left(\frac{\theta_4'}{\theta_4}\right)'(0)-\left(\frac{\theta_2'}{\theta_2}\right)'(0)$$
$$=1+8\sum_{n=1}^{\infty}\frac{nq^n\left(1-(-1)^nq^n\right)}{1-q^{2n}}$$

This complete the proof of Theorem 3.2

## Acknowledgements

This paper was partially supported by The Natural sciences funding project of Shaanxi Province (15JK1264) and the Key disciplines funding of Weinan Teacher's University (14TSXK02).

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