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Uncertain relations on a finite set and their properties

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Abstract: There exists some relationships which are difficult to be simply measured by “yes” or “no" in practice, and there must be a certain amount to indicate the extent of this relationship between the two elements. In this paper, the property of the uncertain relations is examined by the uncertainty theory. Firstly it offers the definition of uncertain relation and the concept of property index of the uncertain relation based on uncertain theory; secondly it gives the calculation method of the property index of the uncertain relation; finally, a simple example is presented to illustrate the method.

Keywords: Uncertain Relation, Property Index, Uncertainty Theory

1. Introduction

The relation is a special set, which is an important concept of set theory. According to John N. Warfield, the Theory of relations was developed by Augustus De Morgan [1]. There are a variety of relationships in the world, for example, the "Comrade" relationship, the teacher-student relationship, a superior-subordinate relationship between human; "greater than" relationship, the "equal" relationship, "less than" relationship between the two numbers; "function" relationship between two variables; "call" relationship between the program. Therefore, it is of the great usefulness for mathematics and computer to study the relationship. The relationship of the elements of the set is represented by a structure, which is called the relation. Ordered pair consisting of two related elements can represent the relationship between the two elements, in other words, it is a subset of the Cartesian product between the sets. Relation can be used to solve the problem real-world, for example, to determine the opening of routes between two cities in a network [2].

In fixed environment, for any two elements, there is a relationship between them, or there is not, it must and only be one or the other. This relationship is suitable for the description of the relationship of "clearly identified". However, in practice, there is a lot of relationship which is difficult to be measured simply by "yes" or "no", but it must introduce a certain amount to the extent of this relationship between the two elements. For example, there is a certain relationship between the normal height and weight, this relationship is not clear. For example, for a healthy people of 169 cm high, generally it cannot conclude his weight exactly but estimate his weight based on the relationship between the tables of normal height and weight.

In addition, the relation of “far greater than" and "sufficiently small” belongs to unclear relationship. It causes that some properties of the relationship cannot be verified by the traditional methods. Some researchers believe that it is the randomness factor, and they used the probability theory to study these uncertain phenomena [see 3, 4]. At the same time, some researchers used fuzzy theory to deal with it, and have done a lot of fuzzy relation study (see [5,6,7]). However, the premise of the application of probability theory is that the estimated probability distribution must be sufficiently close to the actual frequency. Unfortunately, the problems we are often faced with are precisely lack of observational data, which not only are unable to calculate the frequency of occurrence but it cannot determine the probability distribution. In this case, we have based on expert experience and knowledge to estimate the occurrence of the event. Because people often overestimate the occurrence of the unlikely event, which makes the reliability of the variance is much larger than the frequency. At this point, if the reliability is regarded as a subjective probability, the result may differ materially from our expectations. In order to study the phenomenon of subjective uncertainty, uncertainty theory [8, 9] is established, and it not only develops as a branch of axiomatic mathematics, but makes a series of successful applications. It provides a motivation for our study of the uncertain relations and provides a new approach to the uncertainty theory for the study of the relationship. Such relation of describing the degree of relationship to supplement the description of the relationship is uncertain relation, while the relationship degree of uncertainty between the two elements is described by the uncertain measure.

An uncertain relationship refers to the relationship between the two elements in the set cannot be completely sure, then how do we determine the property of this relationship? In this paper, the concept of the property index of the uncertain relationship is offered, at the same time, in the uncertain theoretical framework, given the property index of the uncertain relationship calculated.

The remainder of this paper is organized as follows. In the second part it first introduces the basic concepts and the property of the uncertainty theory used in this paper, and then describes the method of determining the property of the relationship. The third part is the main part of this paper, offering the research on the property index and its algorithm method of the uncertain relation. The fourth part is a brief summary of the progress of this research.

2. Preliminaries

2.1. Uncertainty Theory

Uncertainty Theory,founded by Liu [8] in 2007 and refined by Liu [9] in 2010, provides a new approach to deal with indeterminacy factors. Nowadays it has become a branch of mathematics based on the normality, duality, subadditivity, and product axioms. So far, the theory and practice have shown that uncertainty theory is a very effective tool to deal with uncertain information, in particular, the empirical data and subjective estimates.

Here we simply introduce the major developments of uncertainty theory in different areas. Liu [10] introduced the uncertain process and gave the definition of the differential equations of uncertainty. In 2010, Liu [11] established the uncertainty set theory and uncertain reasoning rules which contains a new inference one. As one application of the uncertainty theory, in 2009, Liu [12] proposed uncertainty programming, which is the mathematical programming with uncertain variables. In 2011 Gao and Gao [13] offered the concept of the uncertainty graph and connectivity index of uncertainty graph. In 2012 Gao [14,15] offered the concepts of Cycle index of the uncertainty graph and Tree index of the uncertainty graph. In 2009, Gao [16] proved some properties of continuous uncertainty measure. Gao et al. [17] discussed Liu's inference rule with multiple antecedents and with multiple if-then rules in 2010. You [18] gave some uncertain sequence convergence theorem. In short, it is more extensive for the research and application of uncertainty theory. If the readers want to understand the recent developments of the uncertainty theory, they may consult [19].

In the following part there are some concepts and results of uncertainty theory applied in the text.

Let Γ be a nonempty set, and L a σ-algebra over Γ. Each element $\Lambda \in L$ is called an event. The set function M is called an uncertain measure, if it satisfies the following three axioms [8]:

Axiom 1. (Normality) $M\{\Gamma\}=1$.

Axiom 2. (Duality) $M\{\Lambda\}+M\{\Lambda^c\}=1$ *for any event* Λ.

Axiom 3. (Subadditivity) *For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}M\{\Lambda_i\}.$$

The triplet (Γ, L, M) is called an uncertainty space. An uncertain variable is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of real numbers.

Product uncertain measure was defined by Liu [20] in 2009, thus producing the fourth axiom as follows

Axiom 4. (Product Axiom) *Let* (Γ_k, L_k, M_k) *be uncertainty spaces for* $k=1,2,3,\cdots$. *Then the product uncertain measure* M *is an uncertain measure on the product* σ*-algebra* $L_1\times L_2\times L_3\times\cdots$ *satisfying*

$$M\left\{\prod_{k=1}^{\infty}\Lambda_i\right\}=\bigwedge_{k=1}^{\infty}M\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events for L_k and for $k=1,2,3,\cdots$, respectively.

The uncertain variables $\xi_1,\xi_2,\cdots,\xi_m$ are said to be independent if

$$\mathrm{M}\left\{\bigcap_{i=1}^{m}(\xi_i\in B_i)\right\}=\min_{1\le i\le m}\mathrm{M}\{\xi_i\in B_i\}$$

for any Borel sets $B_1,B_2,\cdots,B_m$ of real numbers.

A function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0, 1\}$. An uncertain variable is said to be Boolean if it takes values either 0 or 1.

Theorem2.1. (Liu [9]) Assume that $\xi_1,\xi_2,\cdots,\xi_n$ are independent Boolean uncertain variables, i.e.,

$$\xi_i=\begin{cases}1 & \text{with uncertain measure } a_i\\ 0 & \text{with uncertain measure } 1-a_i\end{cases}$$

for $i=1,2,\cdots,n$. If f is a Boolean function, then $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ is a Boolean uncertain variable such that

$$M\{\xi=1\}=\begin{cases}\sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\le i\le n}\nu_i(x_i), & if \sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\le i\le n}\nu_i(x_i)<0.5\\ 1-\sup\limits_{f(x_1,\cdots,x_n)=0}\min\limits_{1\le i\le n}\nu_i(x_i), & if \sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\le i\le n}\nu_i(x_i)\ge 0.5\end{cases}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$v_i(x_i) = \begin{cases} a_i, & if \ \ x_i = 1 \\ 1 - a_i, & if \ \ x_i = 0 \end{cases}$$

for $i = 1, 2, \cdots, n$, respectively.

2.2. The Relationship and its Property on the Set

The most direct way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements, which is called a binary relation. A binary relation from set *A* to set *B* is a subset of $A \times B$. In other words, a binary relation of set *A* to set *B* is an ordered set *R*, in *R*, any one of ordered pairs can be recorded as <a,b>, where each ordered pair the first element taken from *A* and the second element from the *B*. We use the notation *aRb* representing <a,b>∈ R. The any ordered couple <a,b> ,which is not in R, can be marked as <*a*,*b*>∉ *R* or $a \overline{R} b$. If *aRb*, we say that *a* and *b* has a relation *R*. The relation is special set that is composed of ordered pairs, in which ∅ called empty relation and $A \times B$ is called the complete relation. The relation on set *A* is a relation from *A* to *A*, that is a set of *A*'s relation is a subset of $A \times A$. If I_A is a binary relation on *A* and satisfies $I_A = \{<x,x>|x \in A\}$, then I_A is called the constant relationship. The matrix as a mathematical tool, can represent a binary relation between the finite set. It is easy for the computer to deal with the relation by the matrix, using 0-1 matrix to indicate a relation between finite set. Assume that *R* is relation from $A = \{x_1, x_2, \ldots, x_n\}$ to $B = \{y_1, y_2, \ldots, y_n\}$, the relation *R* with the matrix $R = (r_{ij})$, called the relations matrix of the *R*, where

$$r_{ij} = \begin{cases} 1 & \text{if } \langle x_i, y_j \rangle \in R \\ 0 & \text{if } \langle x_i, y_j \rangle \notin R \end{cases}$$

For example, $A = \{x_1, x_2, x_3\}$, $B = \{y_1, y_2\}$, *R* is the relation from *A* to *B*, and $R = \{<x_1, y_1>, <x_2, y_2>, <x_3, y_2>\}$, then the relation matrix of *R* as follows

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

If the element of a matrix is either 0 or 1, then the matrix is a Boolean matrix. Obviously, the relation matrix from set *A* to B is a Boolean matrix.

Definition 2.1 Let $A = (a_{ij})$ and $B = (b_{ij})$ be Boolean matrix, Boolean sum $A \vee B$, and the Boolean product $A \circ B$ defined as follows.

$$A \vee B = (c_{ij}), \text{ where } c_{ij} = a_{ij} \vee b_{ij}$$

$$A \circ B = (c_{ij}), \text{ where } c_{ij} = \max_{1 \le k \le n} (a_{ik} \wedge b_{kj})$$

Some of the special property of relations plays a significant role in the relationship research, which is offered in the following introduction.

Definition 2.2 Let R be a binary relation on set A,

(1) If $\langle a,a \rangle \in R$ for all $a \in A$, then R is called reflexive;

(2) If $\langle a,a \rangle \notin R$ for all $a \in A$, then R is called anti-reflexive;

(3) For all $a,b \in A$, if $\langle a,b \rangle \in R$ whenever $\langle b,a \rangle \in R$, then R is called symmetric;

(4) For all $a,b \in A$, if $\langle a,b \rangle \in R$ and $\langle b,a \rangle \in R$, then $a = b$, R is called anti-symmetric;

(5) For all $a,b,c \in A$, if whenever $\langle a,b \rangle \in R$ and $\langle b,c \rangle \in R$, then $\langle a,c \rangle \in R$, R is called transitive;

(6) For all $a,b,c \in A$, if whenever $\langle a,b \rangle \in R$ and $\langle b,c \rangle \in R$, then $\langle a,c \rangle \notin R$, R is called anti-transitive.

By definition, the relation can be determined whether it has the given above property, but it is difficult to determine the property of the relation only by definition, and therefore it can take advantage of the relation matrix. Transitivity and anti-transitivity of the relationship is the most difficult to determine the property of the relation in these six properties. To take advantage of the relation matrix to determine the above-mentioned property of a given relation we first introduce the definition of the *Hadmard* product of the matrix, then the application of the *Hadmard* product of the matrix gives necessary and sufficient conditions of the property of the relation.

Definition 2.3 Let $A = (a_{ij})$ and $B = (b_{ij})$ be *m* × *n* *matrices. The Hadmard product of A and B, denoted by* $C = A \cdot B = (c_{ij})$ *,is the m × n matrix that has*

$$c_{ij} = a_{ij} b_{ij} \quad .$$

for $i = 1,2,\cdots,m$; $j = 1,2,\cdots,n$, *respectively.*

In other words, *Hadmard* product of two matrices of the same size is the matrix with $(i,j)^{th}$ entry equal to the product of the corresponding positions .

The follow theorem is very useful, it can determine the property of the relation according to the *Hadmard* product of relation matrix.

Theorem 2.2 (Li [21]) Let R be a binary relation on set A, then

(1) *R* is reflexive if and only if $R \cdot I = I$;

(2) *R* is anti-reflexive if and only when $R \cdot I = 0$;

(3) *R* is symmetric if and only if $R \cdot R^T = R$;

(4) *R* is anti-symmetric if and only if $R \cdot R^T$, the matrix is diagonal;

(5) *R* is transitive if and only if $R^{(2)} \cdot R = R^{(2)}$;

(6) *R* is the anti- transitive if and only if $R^{(2)} \cdot R = 0$,

Where *I* is an unit matrix with same size as *R*, $R^{(2)}$ is $R \circ R$, the Boolean product of *R* and *R*.

3. Uncertain Relations

3.1. Basic Concepts

In this paper, the indeterminacy factor is that we are not sure whether there exists a relationship between the two elements in a set. If there is no historical data or experimental data, we cannot use random variables to describe the uncertain factors. Usually, we may consult experts to provide a degree of credibility to the existence of such relations. The expert data happens to be the subject of study of the uncertainty theory. Therefore, uncertain variables are applied to describe the indeterminacy factor.

When in the domain of a finite set, the description of the ordinary binary relations is often used to describe the relation matrix. Similarly, the dual uncertainty relation on a finite domain, we use the uncertain relation matrix description.

Definition 3.1 An uncertain matrix is a measurable function from an uncertainty space (Γ, L, M) to the set of real matrix.

Definition 3.2 A relation R on a finite set is said to be an uncertain relation if its relation matrix is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

where r_{ij} represent that the relation R between elements a_i and a_j in the set exist with uncertain measure r_{ij}, for $i, j = 1, 2, \cdots, n$, respectively.

Clearly, the Boolean matrix is a special case of the uncertainty matrix. We are all familiar with 1-1 corresponding relationship in the finite domain between normal relations and the Boolean matrix. It is the same that the 1-1 corresponding relationship between the uncertain relations and the uncertain matrix in the finite domain. The following uncertain relation and the uncertain matrix reciprocal treated as the same are the *R* means.

Obviously, the uncertain relation *R* on the set *A* contains all the information of $A \times A$. It can be known in the definition of the uncertain relation, the elements in the uncertain relation is uncertain Boolean variable, i.e.,

$$R = \left\{\xi_{11}, \xi_{12}, \xi_{13}, \cdots, \xi_{1n,} \xi_{21}, \xi_{22}, \xi_{23}, \cdots, \xi_{2n,} \cdots, \xi_{n1}, \cdots, \xi_{(n-1)n}, \xi_{nn}\right\}$$

where $M(\xi_{ij}{=}1) = r_{ij}$, for any $1 \le i < j \le n$. For simplicity, remove the elements ξ_{ij} satisfying $M(\xi_{ij}{=}1) = 0$, and denote $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$.

In order to explain how much is the chance that an uncertain relation possesses the six properties in definition 2.2, we give the definition of "property index" which is to show whether the uncertain relation has those six properties in definition 2.2.

Definition 3.3 Assume that R is an uncertain relation simply marked as $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$ on set A. The reflexivity function of R is denoted as：

$$f_1(R) = \begin{cases} 1, & \text{if relation } R \text{ is reflexive} \\ 0, & \text{otherwise.} \end{cases}$$

For an uncertain relationship R with $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$, the reflexivity index is defined as

$$\rho_1(R) = M\{f_1(R)) = 1\}$$

That is, reflexivity index of R is the uncertain measure that the relationship R is reflexive.

Definition 3.4 Assume that R is an uncertain relation simply marked as $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$ on set A. The anti-reflexivity function of R is denoted as：

$$f_2(R) = \begin{cases} 1, & \text{if relation } R \text{ is anti-reflexive} \\ 0, & \text{otherwise.} \end{cases}$$

For an uncertain relation R with $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$, the anti-reflexivity index is defined as

$$\rho_2(R) = M\{f_2(R)) = 1\}$$

That is, anti-reflexivity index of R is the uncertain measure that the relation R is anti-reflexive.

Definition 3.5 Assume that R is an uncertain relation simply marked as $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$ on set A. The symmetry function of R is denoted as：

$$f_3(R) = \begin{cases} 1, & \text{if relation } R \text{ is symmetric} \\ 0, & \text{otherwise.} \end{cases}$$

For an uncertain relation R with $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$, the symmetry index is defined as

$$\rho_3(R) = M\{f_3(R)) = 1\}$$

That is, symmetry index of R is the uncertain measure that the relation R is symmetric.

Definition 3.6 Assume that R is an uncertain relation simply marked as $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$ on set A. The anti-symmetry function of R is denoted as：

$$f_4(R) = \begin{cases} 1, & \text{if relation } R \text{ is anti-symmetric} \\ 0, & \text{otherwise.} \end{cases}$$

For an uncertain relation R with $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$,the anti-symmetry index is defined as

$$\rho_4(R) = M\{f_4(R)) = 1\}$$

That is, anti-symmetry index of R is the uncertain measure that the relation R is anti-symmetric.

Definition 3.7 Assume that R is an uncertain relation simply marked as $R = \{\xi_1, \xi_2, \cdots, \xi_m\}$ on set A. The transitivity function of R is denoted as：

$$f_5(R)=\begin{cases}1, & \text{if relation } R \text{ is transitive}\\ 0, & \text{otherwise.}\end{cases}$$

For an uncertain relation R with $R=\{\xi_1,\xi_2,\cdots,\xi_m\}$,the transitivity index is defined as

$$\rho_5(R)=M\{f_5(R))=1\}$$

That is, transitivity index of R is the uncertain measure that the relation R is transitive.

Definition 3.8 Assume that R is an uncertain relation simply marked as $R=\{\xi_1,\xi_2,\cdots,\xi_m\}$ on set A. The anti-transitivity function of R is denoted as:

$$f_6(R)=\begin{cases}1, & \text{if relation } R \text{ is anti-transitive}\\ 0, & \text{otherwise.}\end{cases}$$

For an uncertain relation R with $R=\{\xi_1,\xi_2,\cdots,\xi_m\}$,the anti-transitivity index is defined as

$$\rho_6(R)=M\{f_6(R))=1\}$$

That is, anti-transitivity index of R is the uncertain measure that the relation R is anti-transitive.

3.2. Algorithm and Examples

Theorem3.1. Assume that R is an uncertain relation on set A, and its uncertain relation matrix is

$$R=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n}\\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn}\end{pmatrix},$$

If all elements in R are independent, the reflexivity index, anti-reflexivity index, symmetry index, anti-symmetry index, transitivity index and anti-transitivity index are respectively as

$$\rho_k(R)=\begin{cases}\sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij}), \text{ if } \sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij})<0.5\\ \\ 1-\sup\limits_{f_k(X)=0}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij}), \text{ if } \sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij})\ge 0.5\end{cases}\quad k=1,2,\cdots,6$$

where X is the $n\times n$ uncertain matrix such that

$$X=\begin{pmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix},$$

and x_{ij} *take values either 0 or 1, and* ν_{ij} *are defined by*

$$\nu_{ij}(x_{ij})=\begin{cases}\gamma_{ij}, & \text{if } x_{ij}=1\\ 1-\gamma_{ij}, & \text{if } x_{ij}=0\end{cases}$$

for $i,j=1,2,\cdots,n$,respectively, and

$$f_1(X)=\begin{cases}1, & x_{ii}=1\quad i=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases},$$

$$f_2(X)=\begin{cases}1, & x_{ii}=0\quad i=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_3(X)=\begin{cases}1, & x_{ij}=x_{ji}\quad i,j=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_4(X)=\begin{cases}1, & x_{ij}\ne x_{ji}\quad i\ne j,\ i,j=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_5(X)=\begin{cases}1, & X^{(2)}\cdot X=X^{(2)}\\ 0, & \text{otherwise.}\end{cases}$$

$$f_6(X)=\begin{cases}1, & X^{(2)}\cdot X=0\\ 0, & \text{otherwise.}\end{cases}$$

Proof: Since x_{ij} are independent Boolean uncertain variables, for $i=1,2,\cdots,n;\ j=1,2,\cdots,n$, respectively. Then X is the $n\times n$ uncertain Boolean matrix. Thus we have that the function $f_k(X)$ (*for* k=1,2,⋯,6,*respectively*) is a Boolean function based on Definition 3.3-3.8 . Therefore, based on theorem2.1,we have

$$M(f_k(R)=1)=\begin{cases}\sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij}), \text{ if } \sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij})<0.5\\ \\ 1-\sup\limits_{f_k(X)=0}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij}), \text{ if } \sup\limits_{f_k(X)=1}\min\limits_{1\le i,j\le n}\nu_{ij}(x_{ij})\ge 0.5\end{cases}\quad k=1,2,\cdots,6$$

where X is the $n\times n$ matrix such that

$$X=\begin{pmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix},$$

and x_{ij} take values either 0 or 1, and ν_{ij} are defined by

$$\nu_{ij}(x_{ij})=\begin{cases}\gamma_{ij}, & \text{if } x_{ij}=1\\ 1-\gamma_{ij}, & \text{if } x_{ij}=0\end{cases}$$

for $i=1,2,\cdots,n;\ j=1,2,\cdots,n$,respectively, and

$$f_1(X)=\begin{cases}1, & x_{ii}=1 \quad i=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases},$$

$$f_2(X)=\begin{cases}1, & x_{ii}=0 \quad i=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_3(X)=\begin{cases}1, & x_{ij}=x_{ji} \quad i,j=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_4(X)=\begin{cases}1, & x_{ij}\neq x_{ji} \quad i\neq j,\ i,j=1,2,\cdots,n\\ 0, & \text{otherwise.}\end{cases}$$

$$f_5(X)=\begin{cases}1, & X^{(2)}\cdot X=X^{(2)}\\ 0, & \text{otherwise.}\end{cases}$$

$$f_6(X)=\begin{cases}1, & X^{(2)}\cdot X=0\\ 0, & \text{otherwise.}\end{cases}$$

for $i=1,2,\cdots,n;\ j=1,2,\cdots,n$, respectively. According to definition 3.3-3.8, it is known that the reflexivity index, anti-reflexivity index, symmetry index, anti-symmetry index, transitivity index and anti-transitivity index of R is respectively the uncertain measure that the uncertain relation R is reflexive, anti- reflexive, symmetric, anti-symmetric, transitive, and anti-transitive, respectively. Hence the Theorem 3.1 is proved.

According to Theorem 3.1, we can easily obtain

Corollary 3.1 Assume that R is an uncertain relation on set A, and its uncertain relation matrix is

$$R=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n}\\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn}\end{pmatrix},$$

then the reflexivity index of R is $\min\limits_{1\le i\le n}\{\gamma_{ii}\}$.

Corollary 3.2 Assume that R is an uncertain relation on set A, and its uncertain relation matrix is

$$R=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n}\\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn}\end{pmatrix},$$

then the anti-reflexivity index of R is $\min\limits_{1\le i\le n}\{1-\gamma_{ii}\}$.

In this section, we will give two examples of how to calculate the property index of an uncertain relation.

Example1. Assume that R is an uncertain relation on set A, and its uncertain relation matrix is

$$\begin{pmatrix}0.9 & 0.6\\ 0.2 & 0.3\end{pmatrix}.$$

Then the reflexivity index, anti-reflexivity index, symmetry index, anti-symmetry index, transitivity index and anti-transitivity index of R is 0.3, 0.1,0.4,0.6,0.8, 0.1,respectively.

Following, we calculate reflexivity index, anti-reflexivity index, symmetry index, anti-symmetry index, transitivity index and anti-transitivity index of R by theorem 3.1.

Since the uncertain relation R is composed of 4 uncertain elements, its relation matrix X will be broken down into sixteen cases.

Assume X is one of the following 4 matrices

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

Then we have R is reflexive, i.e., $f_1(X)=1$ and $\sup\limits_{f_1(X)=1}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=0.3$. So its reflexivity index is 0.3 according to Theorem 3.1.

And assume X is one of the following 4 matrices

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Then we have R is anti-reflexive, i.e., $f_2(X)=1$ and $\sup\limits_{f_2(X)=1}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=0.1$. Thus its anti-reflexivity index is 0.1 according to Theorem 3.1.

And assume X is one of the following 8 matrices

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

Then we have R is symmetric, i.e., $f_3(X)=1$ and $\sup\limits_{f_3(X)=1}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=0.4$. Thus its symmetry index is 0.4 according to Theorem 3.1.

And assume X is one of the following 8 matrices

$$\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Then we have R is anti-symmetric, i.e., $f_4(X)=1$ and $\sup\limits_{f_4(X)=1}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=0.6$. So its anti-symmetry index is $1-\sup\limits_{f_4(X)=0}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=1-0.4=0.6$ according to Theorem 3.1.

And assume X is one of the following 13 matrices

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

Then we have R is transitive, i.e., $f_5(X)=1$ and $\sup\limits_{f_5(X)=1}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=0.6$. So its transitivity index is $1-\sup\limits_{f_5(X)=0}\min\limits_{1\le i,j\le 2}v_{ij}(x_{ij})=1-0.2=0.8$ according to Theorem

3.1.

And assume X is one of the following 4 matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have R is anti-transitive, i.e., $f_6(X)=1$ and $\sup\limits_{f_6(X)=1} \min\limits_{1\le i,j\le 2} v_{ij}(x_{ij}) = 0.1$. So its anti-transitivity index is 0.1 according to Theorem 3.1.

Example2. Assume that R is an uncertain relation on set A, and its uncertain relation matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0.4 & 1 \\ 0.6 & 0 & 0 & 1 \\ 0 & 0.7 & 0 & 0 \end{pmatrix}.$$

According to Corollary 1 and Corollary 2,its reflexivity index and anti-reflexivity index both are equal to 0.

Since the uncertain relation R is composed of 3 uncertain elements, its relation matrix X will be broken down into 8 cases.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We can calculate that symmetry index, anti-symmetry index, transitivity index and anti-transitivity index of R is 0, 0.3, 0, 0,separately by theorem 3.1.

4. Conclusions

In real life, we often encounter a variety of indeterminacy factors. Under the framework of uncertainty theory, the six properties of the relation is studied in the paper with the concept of property index of the uncertain relation, its six properties and six algorithms of the property index.

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