Pure and Applied Mathematics Journal
2023; 12(4): 72-78
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20231204.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Some Fixed Point Theorems on b_2 - Metric Spaces

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To cite this article:

Bheem Singh Patel, Zaheer Kareem Ansari, Dharmendra Kumar, Arun Garg. Some Fixed Point Theorems on b_2 - Metric Spaces. *Pure and Applied Mathematics Journal*. Vol. 12, No. 4, 2023, pp. 72-78. doi: 10.11648/j.pamj.20231204.12

Received: August 25, 2023; **Accepted**: September 14, 2023; **Published**: September 27, 2023

Abstract: In this study, we generalize both b-metric spaces and 2-metric spaces into a new class of generalized metric spaces that we call b_2-metric spaces. Then, under various contractive circumstances in partially ordered spaces, we demonstrate a few fixed point theorems in b_2-metric space. Many Mathematician gave the concept of b_2 -metric spaces as a generalization of 2-metric space. The purpose of this research article to established some results of 2-metric space proved by the Arun Garg et al. in b_2 -metric spaces and prove new results.

Keywords: Fixed Point, b - Metric Space, 2-Metric Space, Partial Order Set, Generalized Contractive Mappings

1. Introduction

The idea of metric spaces in functional analysis was initially introduced by Maurice Frechet in 1906 [29]. Mathematicians have since developed the idea of metric spaces in a variety of ways.

Czerwik and many other writers investigated, introduced, and proved various fixed point solutions for single and multi valued mappings in 1993 [1, 2]. Czerwik also studied, introduced, and proved the idea of a b-metric space.

On the other hand, Gähler offered the idea of a 2-metric in [3], using the encouraging example of a triangle's area in R3. For mappings in these spaces, multiple fixed point results were also attained. It is important to keep in mind that 2-metric spaces are not topologically equal to metric spaces, unlike many other recent generalizations of metric spaces, and there is no direct connection between the conclusions produced in 2-metric and in metric spaces.

Different Mathematician studied the various types of mappings in b-Metric Space and 2-Metric Spaces [4-26].

As a generalization of both 2-metric and b-metric spaces, Zead Mustafa et. al. [27] offer the idea of b_2-metric spaces in their study. Then, in partially ordered b_2-metric spaces, he established a few fixed point theorems under various contractive circumstances.

We expand the findings of the b-complete b-metric space in to b_2-metric spaces in this study.

2. Mathematical Preliminaries

The definitions provided by Zead Mustafa et al. [27] are as follows:

Definition 1: Let χ be a nonempty set, $s \geq 1$ be a real number and let $\delta : \chi^3 \to \Re$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in \chi$ there exists a point $z \in \chi$ such that $\delta(x, y, z) \neq 0$
2. If two of three points x, y, z are same then $\delta(x, y, z) = 0$
3. The symmetry:

$$\delta(x,y,z) = \delta(x,z,y) = \delta(y,x,z) = \delta(y,z,x) = \delta(z,x,y) = \delta(z,y,x) \text{ for all } x, y, z \in \chi.$$

4. The rectangle inequality: $\delta(x, y, z) \leq s[\delta(x, y, t) + \delta(y, z, t) + \delta(z, x, t)]$ for all $x, y, z, t \in \chi$.

Definition 2: Let $\{x_n\}$ be a sequence in a b_2-metric space (χ, δ). Then

1. $\{x_n\}$ is said to be b_2-convergent to $x\in\chi$, written as $\lim_{n\to\infty} x_n = x$ if for all $a\in\chi$, $\lim_{n\to\infty}\delta(x_n,x,a)=0$.
2. $\{x_n\}$ is said to be a b_2-Cauchy sequence in χ if for all $a\in\chi$, $\lim_{n\to\infty}\delta(x_n,x_m,a)=0$.
3. (χ,δ) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-Convergent sequence.

Some simple b_2-metric space examples are provided below [27]:

Example 1: Let $\phi=[0,\infty)$ and $\delta(x,y,z)=[xy,yz,zx]^p$ *if* $x\neq y\neq z\neq x$, and otherwise

$\delta(x,y,z)=0$, where $p\geq 1$ is a real number. Evidently, from convexity of function

$f(x)=x^p$ *for* $x\geq o$, then by Jensen inequality, we have

$$(a+b+c)^p\leq 3^{p-1}(a^p+b^p+c^p).$$

So, one can obtain the result that (χ,δ) b_2-metric space with $s\leq 3^{p-1}$.

Example 2: Let a mapping $\delta:\Re^3\to[0,+\infty)$ be defined by

$$\delta(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}$$

Then δ is a 2-metric on $\Re$, *i. e.*, the following inequality holds:

$$\delta(x,y,z)\leq\delta(x,y,t)+\delta(y,z,t)+\delta(z,x,t)$$

for arbitrary real numbers *x*, *y*, *z*, *t*. Using convexity of the function

$f(x)=x^p$ on $[0,+\infty)$ for $p\geq 1$, we obtain that

$\delta_p=\min\{|x-y|,|y-z|,|z-x|\}^p$ is a b_2- *metric on* $\Re$ *with* $s<3^{p-1}$.

Proposition 1: Let (χ,δ) and $(\chi^{\gamma},\delta^{\gamma})$ be two b_2-metric spaces. Then a

Mapping $f:\chi\to\chi^{\gamma}$ is b_2-continuous at a point $x\in\chi$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-

Convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.

Lemma 1 [27]: *Let* (χ,δ) *be a* b_2*-metric space and suppose that* $\{x_n\}$ *and*

$\{y_n\}$ *are* b_2*-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}\delta(x,y,a)\leq \underset{n\to\infty}{Lim}\inf\delta(x_n,y_n,a)\leq \underset{n\to\infty}{Lim}\sup\delta(x_n,y_n,a)\leq s^2\delta(x,y,a)$$

for all "a" in χ. *In particular, if* $y_n=y$ *is constant, then*

$$\frac{1}{s}\delta(x,y,a)\leq \underset{n\to\infty}{Lim}\inf\delta(x_n,y_n,a)\leq \underset{n\to\infty}{Lim}\sup\delta(x_n,y_n,a)\leq s\delta(x,y,a)$$

Proof: It is simple to observe that using the rectangle inequality in the provided b_2-metric space

$$\delta(x,y,a)=\delta(x,a,y)\leq s\delta(x,a,x_n)+s\delta(a,y,x_n)+s\delta(y,x,x_n)$$

$$\leq s\delta(x,a,x_n)+s^2[\delta(a,y,y_n)+\delta(y,x_n,y_n)+\delta(x_n,a,y_n)]+s\delta(y,x_n,y_n)$$

And

$$\delta(x_n,y_n,a)=\delta(x_n,a,y_n)\leq s\delta(x_n,a,x)+s\delta(a,y_n,x)+s\delta(y_n,x,x_n)$$

$$\leq s\delta(x_n,a,x)+s^2[\delta(a,y_n,y)+\delta(y_n,x,y)+\delta(x,a,y)]+s\delta(y_n,x,x_n)$$

We get the desired outcome by using $n\to\infty$ n as the upper limit in the second inequality and $n\to\infty$ as the lower limit in the first inequality.

If $y_n=y$, then

$$\delta(x,y,a)\leq s\delta(x,y,x_n)+s\delta(y,a,x_n)+s\delta(a,x,x_n)$$

And

$$\delta(x_n,y,a)\leq s\delta(x_n,y,x)+s\delta(y,a,x)+s\delta(a,x_n,x)$$

Main Results:

We begin by demonstrating a lemma that states the sequence $\{x_n\}$ is a b_2-Cauchy sequence.

Lemma 2: Let (χ,∂) be a b_2- metric space with coefficient $s\geq 1$ and $\Gamma:\chi\to\chi$ be a mapping.

Suppose that $\{x_n\}$ is a sequence in χ induced by $x_{n+1}=\Gamma x_n$ such that

$$\partial(x_n,x_{n+1},a)\leq\alpha\partial(x_{n-1},x_n,a)\qquad(1)$$

For all $n\in N$, where $\alpha\in[0,1)$ is a constant. Then $\{x_n\}$ is a b_2- Cauchy sequence.

Proof: Suppose $x_0\in\chi$ and $x_{n+1}=\Gamma x_n$ for all $n\in N$. For the lemma's proof, three separate cases are taken into account.

Case I: Let $\alpha \in [0, \frac{1}{s})$. By (1), we have

$$\partial(x_n, x_{n+1}, a) \le \alpha\partial(x_{n-1}, x_n, a)$$

$$\le \alpha^2\partial(x_{n-2}, x_{n-1}, a)$$

$$\le \alpha^3\partial(x_{n-3}, x_{n-2}, a)$$

$$\le \alpha^n\partial(x_0, x_1, a)$$

Thus, for any $n \ge m$ and $n, m \in N$, we have

$$\partial(x_m, x_n, a) \le s[\partial(x_m, x_{m+1}, a) + \partial(x_{m+1}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2[\partial(x_{m+1}, x_{m+2}, a) + \partial(x_{m+2}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2\partial(x_{m+1}, x_{m+2}, a) + s^3[\partial(x_{m+2}, x_{m+3}, a) + \partial(x_{m+3}, x_n, a)]$$

$$\le s\partial(x_m, x_{m+1}, a) + s^2\partial(x_{m+1}, x_{m+2}, a) + s^3\partial(x_{m+2}, x_{m+3}, a) + s^4\partial(x_{m+3}, x_{m+4}, a) + ------ + s^{n-m-1}\partial(x_{n-2}, x_{n-1}, a) + s^{n-m-1}\partial(x_{n-1}, x_n, a)$$

$$\le s\alpha^m\partial(x_0, x_1, a) + s^2\alpha^{m+1}\partial(x_0, x_1, a) + s^3\alpha^{m+2}\partial(x_0, x_1, a) + s^4\alpha^{m+3}\partial(x_0, x_1, a) + ... + s^{n-m-1}\alpha^{n-2}\partial(x_0, x_1, a) + s^{n-m-1}\alpha^{n-1}\partial(x_0, x_1, a)$$

$$\le s\alpha^m[1 + s\alpha + s^2\alpha^2 + s^3\alpha^3 + s^4\alpha^4 + ... + s^{n-m-1}\alpha^{n-m-2} + s^{n-m-1}\alpha^{n-m-1}]\partial(x_0, x_1, a)$$

$$\le s\alpha^m[\sum_{i=0}^{\infty}(s\alpha)^i]\partial(x_0, x_1, a)$$

$= \frac{s\alpha^m}{1-s\alpha}\partial(x_0, x_1, a)$, as $m \to \infty$, which implies that $\{x_n\}$ is a b_2- Cauchy sequence.

In other words $\{\Gamma^n x_0\}$ is a b_2- Cauchy sequence.

Case II: Now, let $\alpha \in [\frac{1}{s}, 1), (s > 1)$. In this case, we have $\alpha^n \to 0$ as $n \to \infty$, so there is $n_0 \in N$, such that $\alpha^{n_0} < s$. Thus, by case I, we claim that $\{(\Gamma^{n_0})^n x_0\}_{n=0}^{\infty} := \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+n} ....\}$ is a b_2- Cauchy sequence. Then

$\{x_n\}_{n=0}^{\infty} := \{x_0, x_1, x_2, ..., x_{n_0-1}, ...\} \cup \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+n} ....\}$

is a b_2- Cauchy sequence in X.

Case III: Let s=1, then the proof of lemma is similar to case I.

Now we prove the theorems of Arun Garg et. al [28] in b_2-metric spaces:

Theorem 3: Let $(\chi, \le)$ be a partially ordered set and suppose that there exist a b_2- metric ∂ on χ such that (χ, ∂) is a b_2- complete metric space with coefficient s≥1 and $\Gamma : \chi \to \chi$ be a mapping such that

$$s\partial(\Gamma x, \Gamma y, a) \le \alpha_1\partial(x, y, a) + \alpha_2\frac{\partial(x, \Gamma x, a)\partial(y, \Gamma y, a)}{1 + \partial(x, y, a)} + \alpha_3\frac{\partial(x, \Gamma y, a)\partial(y, \Gamma x, a)}{1 + \partial(x, y, a)} + \alpha_4\frac{\partial(x, \Gamma x, a)\partial(x, \Gamma y, a)}{1 + \partial(x, y, a)} + \alpha_5\frac{\partial(y, \Gamma x, a)\partial(y, \Gamma y, a)}{1 + \partial(x, y, a)} \tag{2}$$

Where α_1, α_2, α_3, α_4 and α_5 are positive constant with $(\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5) < 1$. Then Γ has a unique fixed point in χ . Moreover, for any $x \in \chi$, the iterative sequence $\{\Gamma^n x\}$ ($n \in N$) b_2- converges to fixed point.

Proof: Assuming $x_0 \in \chi$ such, we create an iterative sequence $\{x_n\}$ by $x_{n+1} = \Gamma x_n$ ($n \in N$).

If there exist $n_0 \in N$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = \Gamma x_{n_0}$, I e. x_{n_0} is a fixed point of Γ .

Without losing generality, let's move on, suppose $x_n \ne x_{n+1}$ for all ($n \in N$), then by (2)

$$
\begin{aligned}
&s\partial(x_n, x_{n+1}, a) = s\partial(\Gamma x_{n-1}, \Gamma x_n, a) \\
&\le \alpha_1 \partial(x_{n-1}, x_n, a) + \alpha_2 \frac{\partial(x_{n-1}, \Gamma x_{n-1}, a)\partial(x_n, \Gamma x_n, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_3 \frac{\partial(x_{n-1}, \Gamma x_n, a)\partial(x_n, \Gamma x_{n-1}, a)}{1+\partial(x_{n-1}, x_n, a)} + \\
&\alpha_4 \frac{\partial(x_{n-1}, \Gamma x_{n-1}, a)\partial(x_{n-1}, \Gamma x_{n-1}, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_5 \frac{\partial(x_n, \Gamma x_{n-1}, a)\partial(x_n, \Gamma x_n, a)}{1+\partial(x_{n-1}, x_n, a)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_1 \partial(x_{n-1}, x_n, a) + \alpha_2 \frac{\partial(x_{n-1}, x_n, a)\partial(x_n, x_{n+1}, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_3 \frac{\partial(x_{n-1}, x_{n+1}, a)\partial(x_n, x_n, a)}{1+\partial(x_{n-1}, x_n, a)} + \\
&\alpha_4 \frac{\partial(x_{n-1}, x_n, a)\partial(x_{n-1}, x_n, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_5 \frac{\partial(x_n, x_n, a)\partial(x_n, x_{n+1}, a)}{1+\partial(x_{n-1}, x_n, a)}
\end{aligned}
$$

$$\le \alpha_1 \partial(x_{n-1}, x_n, a) + \alpha_2 \partial(x_n, x_{n+1}, a) + \alpha_4 s[\partial(x_{n-1}, x_n, a) + \partial(x_n, x_{n+1}, a)]$$

$$s\partial(x_n, x_{n+1}, a) \le (\alpha_1 + s\alpha_4)\partial(x_{n-1}, x_n, a) + (\alpha_2 + s\alpha_4)\partial(x_n, x_{n+1}, a)$$

$$(s - \alpha_2 - s\alpha_4)\partial(x_n, x_{n+1}, a) \le (\alpha_1 + s\alpha_4)(x_{n-1}, x_n, a) \tag{3}$$

Again,

$$
\begin{aligned}
&s\partial(x_n, x_{n+1}, a) = s\partial(\Gamma x_{n-1}, \Gamma x_n, a) \\
&\le \alpha_1 \partial(x_{n-1}, x_n, a) + \alpha_2 \frac{\partial(x_{n-1}, \Gamma x_{n-1}, a)\partial(x_n, \Gamma x_n, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_3 \frac{\partial(x_n, \Gamma x_{n-1}, a)\partial(x_{n-1}, \Gamma x_n, a)}{1+\partial(x_{n-1}, x_n, a)} + \\
&\alpha_4 \frac{\partial(x_n, \Gamma x_n, a)\partial(x_n, \Gamma x_{n-1}, a)}{1+\partial(x_{n-1}, x_n, a)} + \alpha_5 \frac{\partial(x_{n-1}, \Gamma x_{n-1}, a)\partial(x_{n-1}, \Gamma x_n, a)}{1+\partial(x_{n-1}, x_n, a)}
\end{aligned}
$$

$$s\partial(x_n, x_{n+1}, a) \le \alpha_1 \partial(x_{n-1}, x_n, a) + \alpha_2 \partial(x_n, x_{n+1}, a) + \alpha_5 s[\partial(x_{n-1}, x_n, a) + \partial(x_n, x_{n+1}, a)]$$

$$(s - \alpha_2 - s\alpha_5)\partial(x_n, x_{n+1}, a) \le (\alpha_1 + s\alpha_5)\partial(x_{n-1}, x_n, a) \tag{4}$$

Adding (3) and (4), we get

$$(2s - 2\alpha_2 - s\alpha_4 - s\alpha_5)\partial(x_n, x_{n+1}, a) \le (2\alpha_1 + s\alpha_4 + s\alpha_5)\partial(x_{n-1}, x_n, a)$$

$$\partial(x_n, x_{n+1}, a) \le \frac{(2\alpha_1 + s\alpha_4 + s\alpha_5)}{(2s - 2\alpha_2 - s\alpha_4 - s\alpha_5)}\partial(x_{n-1}, x_n, a)$$

$$\alpha = \frac{(2\alpha_1 + s\alpha_4 + s\alpha_5)}{(2s - 2\alpha_2 - s\alpha_4 - s\alpha_5)}$$

In view of $(\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5) < 1$, then $(0 \le \alpha < 1$. Thus by the Lemma (2), $\{x_n\}$ is a b$_2$-Cauchy sequence in χ. Since (χ, ∂) is a b$_2$-complete, then there exists some point x* such that $x_n \to x^*$ as $n \to \infty$, then by (2), we can easily see that

$$
\begin{aligned}
&s\partial(\Gamma x, \Gamma y, a) \le \alpha_1 \partial(x_n, x^*, a) + \alpha_2 \frac{\partial(x_n, \Gamma x_n, a)\partial(x^*, \Gamma x^*, a)}{1+\partial(x, y, a)} + \alpha_3 \frac{\partial(x_n, \Gamma x^*, a)\partial(x^*, \Gamma x^*, a)}{1+\partial(x, y, a)} + \\
&\alpha_4 \frac{\partial(x_n, \Gamma x_n, a)\partial(x_n, \Gamma x^*, a)}{1+\partial(x, y, a)} + \alpha_5 \frac{\partial(x^*, \Gamma x_n, a)\partial(x^*, \Gamma x^*, a)}{1+\partial(x, y, a)}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_1 \partial(x_n, x^*, a) + \alpha_2 \frac{\partial(x_n, x_{n+1}, a)\partial(x^*, Tx^*, a)}{1+\partial(x_n, x^*, a)} + \alpha_3 \frac{\partial(x_n, Tx^*, a)\partial(x^*, Tx^*, a)}{1+\partial(x_n, x^*, a)} + \\
&\alpha_4 \frac{\partial(x_n, x_{n+1}, a)\partial(x^*, Tx^*, a)}{1+\partial(x_n, x^*, a)} + \alpha_5 \frac{\partial(x^*, x_{n+1}, a)\partial(x^*, Tx^*, a)}{1+\partial(x_n, x^*, a)}
\end{aligned} \tag{5}
$$

Taking the limit as $n \to \infty$ both the sides of (5), we get $\underset{n\to\infty}{Limit}(x_{n+1}, \Gamma x^*) = 0$ i e. $x_n \to \Gamma x^* as, n \to \infty$.

It demonstrates that $\Gamma x^* = x^*$ by virtue of the limit of the b_2-convergent sequence's uniqueness.

x* is a fixed point of Γ as a result.

To demonstrate the fixed point's uniqueness, we assume that if there is a second fixed point y*, then by (2), we get

$$s\partial(\Gamma x^*, \Gamma y^*, a) \le \alpha_1 \partial(x^*, y^*, a) + \alpha_2 \frac{\partial(x^*, \Gamma x^*, a)\partial(y^*, \Gamma y^*, a)}{1+\partial(x^*, y^*, a)} + \alpha_3 \frac{\partial(x^*, \Gamma y^*, a)\partial(y^*, \Gamma x^*, a)}{1+\partial(x^*, y^*, a)} +$$
$$\alpha_4 \frac{\partial(x^*, \Gamma x^*, a)\partial(x^*, \Gamma y^*, a)}{1+\partial(x^*, y^*, a)} + \alpha_5 \frac{\partial(y, \Gamma x, a)\partial(y, \Gamma y, a)}{1+\partial(x^*, y^*, a)}$$

$$s\partial(\Gamma x^*, \Gamma y^*, a) \le \alpha_1 \partial(x^*, y^*, a) + \alpha_3 \partial(x^*, \Gamma y^*, a)$$

$$\partial(x^*, y^*, a) \le \frac{(\alpha_1 + \alpha_3)}{s}(x^*, y^*, a) \tag{6}$$

As $(\alpha_1 + \alpha_2 + \alpha_3 + s\alpha_4 + s\alpha_5) < 1$, this implies that $(\alpha_1 + \alpha_3) < 1, as, s \ge 1$.

We conclude from (6) that $\partial(x^*, y^*, a) = 0 \Rightarrow x^* = y^*$.

We now generalize Naidu's [6] finding.

Theorem 4: Let $(\chi, \le)$ be a partially ordered set and suppose that there exist a b_2- metric ∂ on χ such that (χ, ∂) is a b_2- complete b_2- metric space with coefficient s≥1 and $\Gamma_1 \& \Gamma_2$ be a pair of self mapping from χ to χ satisfying the following conditions:

$$\begin{aligned}&(a)\\ &s[\partial^2(\Gamma_1 x, \Gamma_2 y, a)] \le \alpha\partial(x, \Gamma_1 x, a)\partial(y, \Gamma_2 y, a) + \beta\partial(y, \Gamma_1 x, a)\partial(x, \Gamma_2 y, a) - \min\{\partial(x, y, a)\partial(y, \Gamma_2 y, a),\\ &\partial(x, \Gamma_1 x, a)\partial(y, \Gamma_2 y, a), \partial(x, \Gamma_1 x, a)\partial(y, \Gamma_1 x, a), \partial(x, \Gamma_2 y, a)\partial(y, \Gamma_1 x, a), \partial(y, \Gamma_1 x, a)\partial(y, \Gamma_2 y, a)\}\end{aligned} \tag{7}$$

(b) $\Gamma_1 \& \Gamma_2$ are compatible pair for every x, y, a $\in \chi$ and for some non-negative $\begin{matrix}\alpha, \beta, with\\ 0 \le \alpha, \beta \prec 1\end{matrix}$ and $\frac{\alpha + 2\beta s}{s} = \lambda < 1$. Then $\Gamma_1 \& \Gamma_2$ have a common fixed point in χ. Further, if $\frac{\beta}{s} \prec 1$, then $\Gamma_1 \& \Gamma_2$ have a unique fixed point.

Proof: Let $\frac{\alpha + 2\beta s}{s} = \lambda$, we define a sequence $\{x_n\}$ subset of χ for an arbitrary point $x_0 \in \chi$ such that $\Gamma_1 x_n = x_{n+1}$, $\Gamma_2 x_{n+1} = x_{n+2}$, n=0, 1, 2, 3,……

$$s[\delta^2(x_n, x_{n+1}, a)] = s[\delta^2(\Gamma_1 x_{n-1}, \Gamma_2 x_n, a)]$$

$$\le \alpha[\delta(x_{n-1}, \Gamma_1 x_{n-1}, a)\delta(x_n, \Gamma_2 x_n, a)] + \beta[\delta(x_n, \Gamma_1 x_n, a)\delta(x_{n-1}, \Gamma_2 x_n, a)]$$

$$\text{-min}\left\{\begin{aligned}&\delta(x_{n-1}, x_n, a)\delta(x_n, \Gamma_2 x_n, a), \delta(x_{n-1}, \Gamma_1 x_{n-1}, a)\delta(x_n, \Gamma_2 x_n, a),\\ &\delta(x_{n-1}, \Gamma_1 x_{n-1}, a)\delta(x_n, \Gamma_1 x_{n-1}, a), \delta(x_{n-1}, \Gamma_2 x_n, a)\delta(x_n, \Gamma_1 x_{n-1}, a),\\ &\delta(x_n, \Gamma_2 x_n, a)\delta(x_n, \Gamma_1 x_{n-1}, a)\}\end{aligned}\right.$$

$$s[\delta^2(x_n, x_{n+1}, a)] \le \alpha[\delta(x_{n-1}, x_n, a))\delta(x_n, x_{n+1}, a)] + \beta[\delta(x_n, x_{n+1}, a)s\{\delta(x_{n-1}, x_n, a) + \delta(x_n, x_{n+1}, a)\}]$$

$$\text{-min}\left\{\begin{aligned}&\delta(x_{n-1}, x_n, a)\delta(x_n, x_{n+1}, a), \delta(x_{n-1}, x_n, a)\delta(x_n, x_{n+1}, a),\\ &\delta(x_{n-1}, x_n, a)\delta(x_n, x_n, a), \delta(x_{n-1}, x_{n+1}, a)\delta(x_n, x_n, a),\\ &\delta(x_n, x_{n+1}, a)\delta(x_n, x_n, a)\}\end{aligned}\right.$$

$$s[\delta^2(x_n, x_{n+1}, a)] \le \alpha[\delta(x_{n-1}, x_n, a))\delta(x_n, x_{n+1}, a)] + \beta s[\delta(x_n, x_{n+1}, a)\delta(x_{n-1}, x_n, a)] + \beta s[\delta^2(x_n, x_{n+1}, a)]$$

$$\text{-min}\left\{\begin{aligned}&\delta(x_{n-1}, x_n, a)\delta(x_n, x_{n+1}, a), \delta(x_{n-1}, x_n, a)\delta(x_n, x_{n+1}, a),\\ &0, 0, 0\}\end{aligned}\right.$$

$$s(1-\beta)\delta(x_n, x_{n+1}, a) \le (\alpha + s\beta)\delta(x_{n-1}, x_n, a)$$

$$\delta(x_n, x_{n+1}, a) \le \frac{(\alpha + s\beta)}{s(1-\beta)}\delta(x_{n-1}, x_n, a)$$

$$(\alpha + s\beta) \prec s(1-\beta) \Rightarrow \lambda = \frac{\alpha + s\beta}{s} \prec 1$$

$$\delta(x_n, x_{n+1}, a) \le \lambda\delta(x_{n-1}, x_n, a) \tag{8}$$

Replacing x, y, a by x_{n-1}, x_n, a respectively, we have

$$\delta(x_{n-1}, x_n, a) \le \lambda\delta(x_{n-2}, x_{n-1}, a) \tag{9}$$

From (8) and (9), we have

$$\delta(x_n, x_{n+1}, a) \le \lambda^2\delta(x_{n-1}, x_n, a) \tag{10}$$

Continue in same manner, n times, we have

$$\delta(x_n, x_{n+1}, a) \le \lambda^n\delta(x_0, x_1, a)$$

Thus, for some $m, n \succ 0, m \succ n,$ we have

$$\delta(x_n, x_m, a) \le \delta(x_n, x_{n+1}, a) + \delta(x_{n+1}, x_{n+2}, a) + \delta(x_{n+2}, x_{n+3}, a) + ... + \delta(x_{m-1}, x_m, a)$$

$$\le (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1})\delta(x_0, x_1, a)$$

$$\le \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda}\delta(x_0, x_1, a).$$

Letting $m, n \to \infty,$ we have $\delta(x_n, x_m, a) \to 0.$

$\Rightarrow \{x_n\}$ is a Cauchy b$_2$- sequence in χ.

Again Γ_1, Γ_2 are compatible pair and $\{x_n\} \subseteq \chi$ is a sequence then $\{\delta(\Gamma_1\Gamma_2 x_n, \Gamma_2\Gamma_1 x_n)\} \to 0$ as $\{\Gamma_1 x_n\}$ and $\{\Gamma_2 x_n\}$ converges to same limit. So,

$\lim_{n\to\infty}\Gamma_1\Gamma_2 x_n = \lim_{n\to\infty}\Gamma_2\Gamma_1 x_n \Rightarrow \Gamma_1(\lim_{n\to\infty}\Gamma_2 x_n) = \Gamma_2(\lim_{n\to\infty}\Gamma_1 x_n) \Rightarrow \Gamma_1 u = \Gamma_2 u = u$, for some u i e. u is a common fixed point for $\Gamma_1 and \Gamma_2$

To demonstrate the fixed point's exclusivity, we assume that if there is a second fixed point v, then by (7), we obtain

$$s[\partial^2(\Gamma_1 u, \Gamma_2 v, a)] \le \alpha\partial(u, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a) + \beta\partial(v, \Gamma_1 u, a)\partial(u, \Gamma_2 v, a) - \min\{\partial(u, v, a)\partial(v, \Gamma_2 v, a),$$
$$\partial(u, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a), \partial(u, \Gamma_1 u, a)\partial(v, \Gamma_1 u, a), \partial(u, \Gamma_2 v, a)\partial(v, \Gamma_1 u, a), \partial(v, \Gamma_1 u, a)\partial(v, \Gamma_2 v, a)\}$$

$$\partial(u, v, a) \le \frac{\beta}{s}\partial(u, v, a)$$

$\Rightarrow \partial(u, v, a) = 0$ as $\dfrac{\beta}{s} \prec 1.$

$\Rightarrow u = v$. This completes the proof.

3. Conclusion

Zead Mustafa et al. [27] introduce the notion of b$_2$-metric spaces as a generalization of both 2-metric and b-metric spaces. He then proved a few fixed point theorems in a variety of contractive settings in partially ordered b$_2$-metric spaces.

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