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Characterizations and Representations of the Core-EP Inverse and Its Applications

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Abstract: Generalized inverse matrices are an important branch of matrix theory, have a wide range of applications in many fields, such as mathematical statistics, system theory, optimization computing and cybernetics etc. This paper mainly studies the correlation properties and applications of the Core-ep inverse. Firstly, we present the characterizations of the Core-EP inverse by the matrix equations, and an example is given for analysis. Secondly, we present a representation for computing the Core-EP inverse, get a representation of $A_{ij}{}^{\oplus}$ by Cramer rule , and an example is given for analysis. Finally, we study the constrained matrix approximation problem in the Frobenius norm by using the Core-EP inverse: $\| Ax-b \|_F$=min subject to $x\in R(A^k)$, where $A\in C_{m,m}$, we obtain the unique solution to the problem.

Keywords: Core-EP Inverse, Characterizations, Representations, Frobenius Norm

1. Introduction

Let $C_{m,m}$ be the set of $m\times m$ complex matrices. The symbols A^*, $R(A)$, and $\mathrm{rk}(A)$ denote the conjugate transpose, range (column space), and rank, respectively, of $A\in C_{m,m}$. Moreover, I_m is the identity matrix of order m.

For an $m\times m$ matrix A, the index of A is the smallest nonnegative integer k such that $\mathrm{rk}\left(A^{k+1}\right)=\mathrm{rk}\left(A^k\right)$, denoted as Ind (A).

The Moore-penrose inverse denoted by $A^\dagger$ of A is the unique matrix satisfying

$$(1) AA^\dagger A=A,\ (2)\ A^\dagger AA^\dagger=A^\dagger,\ (3)\ (AA^\dagger)^*=AA^\dagger,\ (4)\ (A^\dagger A)^*=A^\dagger A. [1]$$

The Drazin inverse denoted by A^D of A is the unique matrix satisfying

$$(1) A^D AA^D=A^D, (2) AA^D=A^D A, (3) A^{k+1}A^D=A^k,$$

where k is the index of A ,when A 's index is one, A^D is called the group inverse of A and is denoted by $A^\#$ [2, 4] .

The Core-EP inverse denoted by $A^{\oplus}$ of A is the unique matrix satisfying

$$(1)\ A^{\oplus}A^{k+1}=A^k,\ (2)\ A^{\oplus}AA^{\oplus}=A^{\oplus},\ (3)\ (AA^{\oplus})^*=AA^{\oplus}$$

and $R(A)\subseteq R(A^k)$, where k is the index of A [3]. When A 's index is one, $A^{\oplus}$ is called the core inverse of A and is denoted by $A^{\otimes}$.

For any complex $m\times m$ matrix A of index k, there exists an $m\times m$ unitary matrix U such that

$$A=U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{1}$$

and

$$A^{\oplus}=U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^{*} \tag{2}$$

where $T\in C_{k,k}$ is invertible, $S\in C_{k,m-k}$, $N\in C_{m-k,m-k}$ is nilpotent, and $N^k=0$ [3].

Consider the following equation

$$Ax=b. \tag{3}$$

Let $A\in C_{m,m}$ with $\mathrm{Ind}(A)=k$, and $b\in R(A^k)$. Campell and Meyer[2]has shown that $x=A^D b$ is the unique solution of (3) concerning $x\in R(A^k)$. It is noteworthy that Morikuni and Rozloznik [5] study the equation (3) by the generalized minimal residual method in the case of $A\in C_{m,m}$, Ind (A) = 1 and $b\in R(A)$.

When $b\notin R(A)$, (3) is unsolvable, it has least-squares solutions. Motivated by the above-mentioned work, it is natural to consider the least-squares solutions of (3) under the particular condition $x\in R(A)$, i.e.,

$$\|Ax-b\|_F=\min \text{ subject to } x\in R(A), \tag{4}$$

where $A\in C_{m,m}$, Ind (A) = 1, $\mathrm{rk}(A)=r<m,$ and $b\in C_m$. In Wang and Zhang [6]obtained $x=A^{\otimes}b$ is the unique solution of (4). In this paper, we study the constrained matrix approximation problem in the Frobenius norm by employing the Core-EP inverse:

$$\|Ax-b\|_F=\min \text{ subject to } x\in R(A^k), \tag{5}$$

where $A\in C_{m,m}$, Ind (A) = k, $\mathrm{rk}(A)=r<m,$ and $b\in C_m$.

2. Characterizations of Core-EP Inverses

The characterizations for the Moore-Penrose inverse, the Drazin inverse and the Core inverse have been studied[8-10]. And now we present characterizations for the Core-EP inverse. It is well-known that if A is a nonsingular matrix of order n, then $CA^{-1}B$ is the unique matrix X for which

$$\mathrm{rk}\begin{pmatrix} A & B \\ C & X \end{pmatrix}=rk(A),$$

Groβ [14] generalized this rank relation when A is a rectangular matrix. This section presents a generalization of this fact to a singular matrix A to obtain a similar result for the Core-EP inverse $A^{\oplus}$.

LEMMA 2.1. [7] For $P=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and A is nonsingular. Then $\mathrm{rk}(P)=\mathrm{rk}(A)$ if only if $D=CA^{-1}B.$

LEMMA 2.2. [11] For $A\in C_{m,m}$, $\mathrm{Ind}(A)=k$, and $\mathrm{rk}(A^k)=r$. Then there exist a unique matrix X such that

$$A^{k+1}AX=0, XA^k=0, X^2=X, rank(X)=n-r,$$

a unique matrix Y such that

$$YA^k=0, Y^2=Y, Y^*=Y, rank(Y)=n-r,$$

and a unique Z such that

$$\mathrm{rk}\begin{pmatrix} A & I-Y \\ I-X & Z \end{pmatrix}=\mathrm{rk}(A).$$

The matrix Z is the Core-EP inverse $A^{\oplus}$of A , and $X=I-A^{\oplus}A$, $Y=I-AA^{\oplus}$.

THEOREM 2.3. Let $A\in C_{m,m}$ be of rank r, $\mathrm{Ind}(A)=k$ and have representation (1). Then $X=A^{\oplus}$ is the solution to

$$\mathrm{rk}\begin{pmatrix} A & B \\ C & X \end{pmatrix}=\mathrm{rk}(A),$$

when

$$B=\begin{pmatrix} TG^{-1}T^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \text{and } C=\begin{pmatrix} G & GT^{-1}S \\ 0 & 0 \end{pmatrix},$$

where G is $r\times r$ real positive definite diagonal matrix.

Proof By applying the equations (1) and (2), we get

$$\mathrm{rk}\begin{pmatrix} A & B \\ C & X \end{pmatrix}=rk\begin{pmatrix} T & S & TG^{-1}T^{-1} \\ 0 & N & 0 \\ G & GT^{-1}S & T^{-1} \end{pmatrix}=rk\begin{pmatrix} T & S & TG^{-1}T^{-1} \\ 0 & N & 0 \\ G & 0 & T^{-1} \end{pmatrix}=rk\begin{pmatrix} T & S & TG^{-1}T^{-1} \\ 0 & N & 0 \\ 0 & 0 & 0 \end{pmatrix}=rk\begin{pmatrix} T & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 0 \end{pmatrix}=\mathrm{rk}(A).$$

THEOREM 2.4. Let $A\in C_{m,m}$, $\mathrm{rk}(A)=r,$ $A[\alpha|\beta]$ is $r\times r$ nonsingular submatrix of A . and B,C is mentioned above. Then we have

$$A^{\oplus}=C[N\,|\,\beta]\left(A[\alpha\,|\,\beta]\right)^{-1}B[\alpha\,|\,N],$$

where $\alpha=\{i_1,i_2,i_3,\cdots i_r\}, \beta=\{j_1,j_2,j_3,\cdots j_r\}$.

Proof Let

$$P=\begin{pmatrix} A[\alpha|\beta] & B[\alpha|N] \\ C[N|\beta] & A^{\oplus} \end{pmatrix}$$

then we have $\mathrm{rk}(P)\ge \mathrm{rk}(A[\alpha|\beta])=r=rk(A)$. from theorem 2.3 we have

$$\mathrm{rk}(P)\le \mathrm{rk}\begin{pmatrix} A & B \\ C & A^{\oplus} \end{pmatrix}=\mathrm{rk}(A),$$

so we obtain $\mathrm{rk}(P)=rk(A)=\mathrm{rk}(A[\alpha|\beta])$. In addition, from lemma 2.1 we obtain

$$A^{\oplus}=C[N\,|\,\beta](A[\alpha\,|\,\beta])^{-1}B[\alpha\,|\,N].$$

THEOREM 2.5. Let $A\in C_{m,m}$ be of rank r, $\mathrm{Ind}(A)=k$ and have representation (1). Then $X=A^{\oplus}$ is the solution to

$$\mathrm{rk}\begin{pmatrix} A^k & B \\ C & X \end{pmatrix}=\mathrm{rk}(A^k),$$

when

$$B=\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},\ C=\begin{pmatrix} T^{-1}M^{-1}T^k & T^{-1}M^{-1}H \\ 0 & 0 \end{pmatrix},$$

where M is $r\times r$ real positive definite diagonal matrix.

Proof Applying (1) and (2), we get

$$\mathrm{rk}\begin{pmatrix} A^k & B \\ C & X \end{pmatrix}=rk\begin{pmatrix} T^k & H & M \\ T^{-1}M^{-1}T^k & T^{-1}M^{-1}H & T^{-1} \\ 0 & 0 & 0 \end{pmatrix}=rk\begin{pmatrix} T^k & 0 & M \\ T^{-1}M^{-1}T^k & 0 & T^{-1} \\ G & 0 & 0 \end{pmatrix}=rk\begin{pmatrix} T^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\mathrm{rk}(A^k).$$

where $H=T^{k-1}S+T^{k-2}SN+T^{k-3}SN^2+\cdots+SN^{k-1}$.

Example 2.1. Let

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have

$$T=\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix},\ S=\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \text{ and } N=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $G=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, by caculating we get

$$TG^{-1}T^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & 1 \end{pmatrix},\ GT^{-1}S=\begin{pmatrix} -\frac{1}{5} & 0 \\ \frac{2}{5} & 4 \\ \frac{3}{5} & 0 \end{pmatrix}, \text{ and } T^{-1}=\begin{pmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{1}{5} & 1 & -\frac{2}{5} \\ \frac{3}{5} & 0 & -\frac{1}{5} \end{pmatrix}.$$

We have

$$rk\begin{pmatrix} T & S & TG^{-1}T^{-1} \\ 0 & N & 0 \\ G & GT^{-1}S & T^{-1} \end{pmatrix} = rk\begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & -\frac{1}{10} & \frac{1}{2} & \frac{1}{5} \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 2 & 0 & \frac{2}{5} & 4 & \frac{1}{5} & 1 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{3}{5} & 0 & \frac{3}{5} & 0 & -\frac{1}{5} \end{pmatrix} = 4 = \mathrm{rk}(A).$$

Example 2.2. Let

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

by caculating we get $\mathrm{Ind}(A) = k = 2.$ From A we have

$$T = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \text{ and } N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, by caculating we get

$$C = \begin{pmatrix} T^{-1}M^{-1}T^k & T^{-1}M^{-1}H \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 & -\frac{1}{5} \\ 0 & \frac{1}{2} & -1 & -\frac{1}{2} & \frac{6}{5} \\ 3 & 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{\oplus} = B = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and } X = A^{\oplus} = \begin{pmatrix} -\frac{1}{5} & 0 & \frac{2}{5} & 0 & 0 \\ \frac{1}{5} & 1 & -\frac{2}{5} & 0 & 0 \\ \frac{3}{5} & 0 & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, we have

$$rk\begin{pmatrix} A^2 & B \\ C & X \end{pmatrix} = rk\begin{pmatrix} 7 & 0 & 4 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 & 2 & 0 \\ 6 & 0 & 7 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & -\frac{1}{5} & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & \frac{1}{2} & -1 & -\frac{1}{2} & \frac{5}{6} & \frac{1}{5} & 1 & -\frac{2}{5} \\ 3 & 0 & 1 & 0 & \frac{3}{5} & \frac{3}{5} & 0 & -\frac{1}{5} \end{pmatrix} = 3 = \mathrm{rk}(A^2).$$

3. Representations of Core-EP Inverses

In this section, we present some representations for Core-EP inverse.

THEOREM 3.1. For $A, B \in C_{m,m}$ and $\mathrm{Ind}(A) = k$, we have

$$A^{\oplus} = \left(A^{k+1}\Big|_{R(A^k)}\right)^{-1} A^k, \tag{6}$$

where $A^{k+1}\Big|_{R(A^k)}$ is restriction on $R(A^k)$, and it holds that

$$\left(A \otimes B\right)^{\oplus} = A^{\oplus} \otimes B^{\oplus}.$$

Proof From[12], we obtain

$$A^{\oplus} = A^k \left(A^{k+1}\right)^{\otimes},$$

notice that $A^{k+1}\Big|_{R(A^k)}$ is one-to-one mapping of $R\left(A^k\right)$ onto $R\left(A^k\right)$. Suppose that $A^{k+1}x = 0$, where $x \in R\left(A^k\right)$. It is obvious that

$$A^k x \in N\left(A\right), A^k x \in R\left(A\right),$$

and

$$A^k x \in N\left(A\right) \cap R\left(A\right) = 0,$$

i.e. $A^k x = 0$.

On the other hand, if $A^k x = 0$ and $x \in R\left(A^k\right)$, then $x \in R\left(A^k\right) \cap N\left(A^k\right) = 0$. Thus $A^{k+1}\Big|_{R(A^k)}$ is nonsingular on $R\left(A^k\right)$ and

$$A^{\oplus} = A^k\left(A^{k+1}\right)^{\otimes} = A^k\left(A^{k+1}\right)^{\otimes} A A^{\oplus} = \left(A^{k+1}\Big|_{R(A^k)}\right)^{-1} A^k A^{k+1}(A^{k+1})^{\otimes} A A^{\oplus} = \left(A^{k+1}\Big|_{R(A^k)}\right)^{-1} A^{k+1} A^{\oplus} A A^{\oplus}$$

$$= \left(A^{k+1}\Big|_{R(A^k)}\right)^{-1} A^{k+1} A^{\oplus} = \left(A^{k+1}\Big|_{R(A^k)}\right)^{-1} A^k.$$

Form [10], we know $\left(A \otimes B\right)^{\otimes} = A^{\otimes} \otimes B^{\otimes}$, so we have

$$\left(A \otimes B\right)^{\oplus} = \left(A \otimes B\right)^k \left[\left(A \otimes B\right)^{k+1}\right]^{\otimes} = A^k\left(A^{k+1}\right)^{\otimes} \otimes B^k\left(B^{k+1}\right)^{\otimes} = A^{\oplus} \otimes B^{\oplus}.$$

LEMMA 3.2. [11] Let $A \in C_{m,m}$ and let B and C^* be of full column ranks such that

$$N\left(\left(A^k\right)^*\right) = R(B), R\left(A^k\right) = N(C).$$

Then the bordered matrix

$$Z = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is nonsingular and

$$Z^{-1} = \begin{pmatrix} A^{\oplus} & \left(I - A^{\oplus}A\right)C^{\dagger} \\ B^{\dagger}\left(I - A^{\oplus}A\right) & -B^{\dagger}\left(I - A^{\oplus}A\right)A\left(I - A^{\oplus}A\right)C^{\dagger} \end{pmatrix}$$

THEOREM 3.3. Let $A \in C_{m,m}$ and let B and C^* be of full column ranks such that

$$N\left(\left(A^k\right)^*\right) = R(B), R\left(A^k\right) = N(C)$$

we have

$$A_{ij}{}^{\oplus} = \det\begin{pmatrix} A(j \leftarrow e_i) & B \\ C(j \leftarrow 0) & 0 \end{pmatrix} / \det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, i, j = 1, 2, \cdots m.$$

Where $B = (I - A^{\oplus} A)C^{\dagger}$, $C = B^{\dagger}(I - A^{\oplus} A)$.

Proof From Lemma 3.2 we have

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A^{\oplus} & B \\ C & -CAB \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

by Cramer rule, we have

$$\begin{pmatrix} A^{\oplus} & B \\ C & -CAB \end{pmatrix}_{ij} = \det\left(\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}(j \leftarrow e_i)\right) / \det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

so

$$A_{ij}{}^{\oplus} = \det\begin{pmatrix} A(j \leftarrow e_i) & B \\ C(j \leftarrow 0) & 0 \end{pmatrix} / \det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Since $N\left(\left(A^k\right)^*\right) = R(B), R\left(A^k\right) = N(C)$, and applying the decomposition (1) of A, then we have

$$B = \begin{pmatrix} 0 & b_2 \end{pmatrix}U, C = U^*\begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

Let $C = B^*$, we have

$$A_{ij}{}^{\oplus} = \det\begin{pmatrix} A(j \leftarrow e_i) & B \\ B^*(j \leftarrow 0) & 0 \end{pmatrix} / \det\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}.$$

EXAMPLE 3.1 Let $A=\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, and we know $\text{Ind}(A)=k=3$. By calculating we get

$$A_{ij}{}^{\oplus}=\begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $k = 3$, we get

$$\left(A^4\big|_{R(A^3)}\right)^{-1}A^3=\begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hencn,

$$A_{ij}{}^{\oplus}=\left(A^{k+1}\big|_{R(A^k)}\right)^{-1}A^k.$$

4. Applications

THEOREM 4.1. Let $A\in C_{m,m}$, $\text{Ind}(A)=k$ and $b\in C_m$. Then

$$x=A_{ij}{}^{\oplus}b \tag{7}$$

is the unique solution of (5).

Proof From $x\in R\left(A^k\right)$, it follows that there exists $y\in C_m$ for which $x=A^k y$. Let the decomposition of A is as shown in (1). Now we denote

$$U^*y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, U^*b=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, and\ A^{\oplus}b=U\begin{pmatrix} T^{-1}b_1 \\ 0 \end{pmatrix}, \tag{8}$$

where y_1, b_1 and $T^{-1}b_1\in C_{rk(A)}$. It follows that

$$\begin{aligned}\|Ax-b\|_F^2=\|AA^k y-b\|_F^2=\left\|U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^*U\begin{pmatrix} T^k & H \\ 0 & 0 \end{pmatrix}U^*y-UU^*b\right\|_F^2 &=\left\|U\begin{pmatrix} T^{k+1} & TH \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}-U\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right\|_F^2 \\ &=\left\|\begin{pmatrix} T^{k+1}y_1+THy_2-b_1 \\ -b_2 \end{pmatrix}\right\|_F^2 \\ &=\left\|T^{k+1}y_1+THy_2-b_1\right\|_F^2+\|b_2\|_F^2.\end{aligned}$$

where $H = T^{k-1}S + T^{k-2}SN + T^{k-3}SN^2 + \cdots + SN^{k-1}$. Since T is invertible, we have min $\left\|T^{k+1}y_1 + THy_2 - b_1\right\|_F^2 = 0$ if

$$y_1 = T^{-(k+1)}b_1 - T^{-k}Hy_2.$$

Therefore,

$$x = A^k y = U\begin{pmatrix} T^k & H \\ 0 & 0 \end{pmatrix}U^* y = U\begin{pmatrix} T^k y_1 + Hy_2 \\ 0 \end{pmatrix} = U\begin{pmatrix} T^{-1}b_1 \\ 0 \end{pmatrix} = A_{ij}{}^{\oplus} b,$$

that is, (7) is the unique solution of (5).

THEOREM 4.2. Let $A \in C_{m,m}$ and let B and C^* be of full column ranks such that

$$N\left(\left(A^k\right)^*\right) = R(B), R\left(A^k\right) = N(C).$$

Let $b \in R\left(A^k\right)$, then the unique solution $x = A_{ij}{}^{\oplus} b$ of (5) can be expressed componentwise, by

$$x_j = \det\begin{pmatrix} A(j \leftarrow e_i) & B \\ C(j \leftarrow 0) & 0 \end{pmatrix} / \det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, j = 1, 2, \cdots m.$$

Proof Since $x = A_{ij}{}^{\oplus} b \in R\left(A^k\right)$ and $R\left(A^k\right) = N(C)$, we have $Cx = 0$. The solution of (5) satisfies

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Form Theorem 4.1 we have $x = A_{ij}{}^{\oplus} b$ and its components follow from the Cramer rule.

5. Conclusion

This paper mainly studies the correlation properties and applications of the Core-ep inverse, firstly, we present the characterizations of the Core-EP inverse by the matrix equations. and then We present a representation for computing the Core-EP inverse, and an example is given for analysis, and finally the Core-EP inverse is used to study the solution of the equation, i.e $\|Ax-b\|_F$=min subject to $x \in R(A^k)$ where $A \in C_{m,m}$, we obtain the unique solution to the problem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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