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On Geodesic Flow and Energy Functional on Riemannian Manifolds

Yang Liu

Shenzhen Technology University, Shenzhen, China

Email address:
yliu1mat@gmail.com

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Abstract: In this paper, we study the geodesic flow and the energy functional on a Riemannian manifold and show that the geodesics have minimal energy, in other words, are the minimizers of the energy functional, from the new perspective of involution, and that the geodesic flow is a Hamiltonian flow which has a close connection with the canonical symplectic structure on the tangent bundle of a Riemannian manifold.

Keywords: Geodesic Flow, Energy Functional, Hamiltonian Flow, Vector Bundle

1. Introduction

It is well known that geodesics are the shortest lines connecting points in manifolds equipped with metrics, on which one can see, for instance, the researches [3, 13, 2, 16, 7], and on the other hand, geodesics can also be viewed as minimal hypersurfaces of two-dimensional Riemannian manifolds. Further, in the next section, we will show that geodesic flows on a Riemannian manifold are Hamiltonian flows of the energy functional, and it has a close connection with the canonical symplectic structure on the tangent bundle of Riemannian manifolds. On the other hand, the related complex structure in the differential aspect was studied in many researches [17, 11, 10], [12, 14, 4]. Recently, it has been found that the symplectic measurements induced by a symplectic form can potentially be used to address problems in convex geometry, on which one can see, for example, the researches [1, 15, 6, 8, 5]. In this paper, we study the geodesic flow and the energy functional on a Riemannian manifold. The main contribution of this paper is to show that the geodesics have minimal energy, in other words, are minimizers of the energy functional, from the novel perspective of involution, and that the geodesic flow is a Hamiltonian flow, which has a close connection with the canonical symplectic structure on the tangent bundle of a Riemannian manifold.

2. Geodesic Flow and Energy Functional

Let M be an n-dimensional Riemannian manifold with metric g, ξ_χ be any vector in the tangent bundle *TM*, and $c(t)$ be a curve in *TM* starting at $c(0) = \xi_\chi$ with initial velocity

$$c'(0) = (X, \Xi) \in T_{\xi_x}TM, \tag{1}$$

then

$$(X, \Xi) \in H_{\xi_x}TM, \tag{2}$$

Where $H_{\xi_x}TM$ is the horizontal space, for which one can see Shoshichi Kobayashi and Katsumi Nomizu's study [9], if and only if $\nabla_{\xi_x}X = 0$, where $H_{\xi_x}TM$ is the horizontal space of the tangent bundle *TM*.

In terms of coordinates $(x^1, \cdots, x^n, \xi^1, \cdots, \xi^n)$, let

$$X := \sum_{i=1}^{n} X_i \frac{\partial}{\partial x^i} \tag{3}$$

and

$$\Xi := \sum_{i=1}^{n} \Xi_i \frac{\partial}{\partial \xi^i}. \tag{4}$$

Then we have

$$\Xi_k + \sum_{i,j=1}^{n} \Gamma_{ij}^k X_i \xi_j = 0 \tag{5}$$

by $\nabla_{\xi_x}X = 0$.

On the other hand, for vertical space, we have that $(X, \Xi) \in V_{\xi_x}TM$ holds if and only if X = 0, i.e. $X_i = 0$ for all i.

Here is a direct way to show that a geodesic flow is a Hamiltonian flow of the energy functional. In terms of coordinates, we know that for the geodesic vector field

$$\chi(\xi_x) := \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x^k} - \sum_{k=1}^{n}\sum_{i,j=1}^{n} \Gamma_{ij}^k \xi_i \xi_j \frac{\partial}{\partial \xi^k}, \tag{6}$$

on *TM* by [5], we have

$$\begin{aligned} i_{\chi(\xi_x)} d\alpha_{\xi_x}(\cdot) &= d\alpha_{(\xi_x}(\chi(\xi_x),\cdot) \\ &= dg_{(\xi_x}\left(\xi_x, \pi_*(\sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x^k} - \sum_{k=1}^{n}\sum_{i,j=1}^{n} \Gamma_{ij}^k \xi_i \xi_j \frac{\partial}{\partial \xi^k}))(\cdot)\right) \\ &= dg(\xi_x, \xi_x)(\cdot) \\ &= dE_{\xi_x}(\cdot) \end{aligned} \tag{7}$$

So if $\omega := d\alpha$ is the natural symplectic form on *TM*, we then have the following lemma.

Lemma 1. Geodesic flows on a Riemannian manifold are Hamiltonian flows of the energy functional. More precisely, $i\chi\omega = dE$.

There are some relations between the energy functional and the canonical differential forms on Riemannian manifolds. We first consider the case $\Gamma_{ij}^k = 0$, and in this case it means $\Xi_k = 0$ for all k by [5] if $(X, \Xi) \in H_{\xi_x}TM$. There is a natural involution between the horizontal space and the vertical space, which is

$$\iota : H_{\xi_x}TM \to V_{\xi_x}TM$$

$$\iota\left(\sum k_i \frac{\partial}{\partial x^i}\right) = \sum k_i \frac{\partial}{\partial \xi^i}. \tag{8}$$

Thus

$$\begin{aligned} \alpha_{\xi_x} \circ \iota\left(\sum_{i=1}^{n}\left(k_i \frac{\partial}{\partial \xi^i}\right)\right) &= \alpha_{\xi_x}\left(\sum_{i=1}^{n} k_i \frac{\partial}{\partial x^i}\right) \\ &= g\left(\xi_x, \pi_*(\sum_{i=1}^{n}(k_i \frac{\partial}{\partial x^i}))\right) \\ &= \sum_{i,j=1}^{n} g_{ij}\, \xi_i \xi_j \end{aligned} \tag{9}$$

On the other hand, for *dE*, we have

$$\begin{aligned} dE_{\xi_x}\left(\sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i}\right) &= \sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i} E(\xi_x) \\ &= \sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i} g(\xi_x, \xi_x) \\ &= \sum_{i,j=1}^{n} g_{ij}\, \xi_i k_j \end{aligned} \tag{10}$$

Therefore

$$dE_{\xi_x}((0,\Xi)) = \alpha_{\xi_x} \circ \iota((0,\Xi)) \tag{11}$$

for any $(0,\Xi) \in V_{\xi x}TM$.

Let us now consider the general case. For any $(X, \Xi) \in H_{\xi_x}TM$, we have $\Xi_k = -\sum_{i,j=1}^{n} \Gamma_{ij}^k \chi_i \xi_j$, analogous to the case $\Gamma_{ij}^k = 0$, an involution we can have is

$$\iota : V_{\xi_x}TM \to H_{\xi_x}TM$$

$$\iota\left(\Sigma + k_i \frac{\partial}{\partial \xi^i}\right) = \sum k_i \frac{\partial}{\partial x^i} - \sum \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k}. \tag{12}$$

Therefore,

$$\begin{aligned} \alpha_{\xi_x} \circ \iota\left(\sum_{i=1}^{n}\left(k_i \frac{\partial}{\partial \xi^i}\right)\right) &= \alpha_{\xi_x}\left(\sum_{i=1}^{n} k_i \frac{\partial}{\partial x^i} - \sum_{k=1}^{n}\sum_{i,j=1}^{n} \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k}\right) \\ &= g\left(\xi_x, \pi_*(\sum_{k=1}^{n} k_i \frac{\partial}{\partial x^i} - \sum_{k=1}^{n}\sum_{i,j=1}^{n} \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k})\right) \\ &= \sum_{i,j=1}^{n} g_{ij} \xi_j\, k_j \end{aligned} \tag{13}$$

and

$$\begin{aligned} dE_{\xi_x}\left(\sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i}\right) &= \sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i} E(\xi) \\ &= \sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i} g(\xi, \xi) \\ &= \sum_{i,j=1}^{n} g_{ij} \xi_j\, k_j. \end{aligned} \tag{14}$$

Hence, it follows that

$$\alpha_{\xi_x} \circ \iota\left(\sum_{i=1}^{n}\left(k_i \frac{\partial}{\partial \xi^i}\right)\right) = dE_{\xi_x}\left(\sum_{i=1}^{n} k_i \frac{\partial}{\partial \xi^i}\right). \tag{15}$$

In other words, we have proved the following theorem:

Theorem 2. Let E be the energy functional on the Riemannian manifold M and let α be the natural 1-form on *TM* whose differential is the natural symplectic form on *TM*. Then we have

$$dE = \alpha \circ \iota \tag{16}$$

on the vertical bundle of *TTM*.

Remark 3. Since both sides onthe horizontal bundle of *TTM* are zeros, then $dE = \alpha \circ \iota$ on *TTM*.

3. Conclusion

The geodesics have minimal energy, in other words, are the minimizers of the energy functional, from the new perspective of involution, and that the geodesic flow is a Hamiltonian flow which has a close connection with the canonical symplectic structure on the tangent bundle of a Riemannian manifold.

Declaration of Interest

The author declares that there is no conflict of interest.

Data Availability Statement

The author confirms that the data supporting the findings of this study are available within the article or its supplementary materials.

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