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Methodology Article

Partial Differential Equation Formulations from Variational Problems

Uchechukwu Opara

Department of Mathematics & Statistics, Veritas University, Abuja, Nigeria

Email address:
ucmiop@yahoo.com

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Abstract: The calculus of variations applied in multivariate problems can give rise to several classical Partial Differential Equations (PDE's) of interest. To this end, it is acknowledged that a vast range of classical PDE's were formulated initially from variational problems. In this paper, we aim to formulate such equations arising from the viewpoint of optimization of energy functionals on smooth Riemannian manifolds. These energy functionals are given as sufficiently regular integrals of other functionals defined on the manifolds. Relevant Banach domains which contain the optimal functional solutions are identified by preliminary analysis, and then necessary optimality conditions are discovered by differentiation in these Banach spaces. To determine specific optimal functionals in simple settings, smaller target domains are taken as appropriate subsets of the Banach (Sobolev) spaces. Briefings on analytical implications and approaches proffered are included for the aforementioned simple settings as well as more general case scenarios.

Keywords: Riemannian Manifolds, Calculus of Variations, Energy Functionals, Ricci Flow, Potential Theory

1. Introduction

A wide class of Partial Differential Equations are formulated initially from problems in the calculus of variations. In the bid to estimate critical points of functionals on subsets of infinite dimensional linear spaces, PDE's are often generated by formulation laws of the calculus of variations. Local and global minima of such functionals have profound implications in models of the physical universe. For instance, critical points of a functional usually correspond to the equilibrium stages of associated physical systems, and a non-exaggerated perturbation of a system from a state of equilibrium tends to return it to the same equilibrium state. Two energy functionals (Dirichlet and Perelman) are probed for their critical points in simple settings. The local minimum of Perelman's Energy as included could have new contributions towards the collection of static Ricci solitons currently known. Hence, this paper serves as an appendage to the very modern study of Ricci Flow. The section on Dirichlet's Energy probes into the very foundations of Sobolev and Lebesgue spaces.

In due course, we will invoke two fundamental tools during formulation which are outrightly stated in this section. The first is a general theorem in functional optimization theory and the second is a lemma of variational calculus.

Optimization Theorem [2] – Let E be a real reflexive Banach space, and the functional $f\colon E \to \mathbb{R}\quad \{+\infty\}$ be convex, lower semi-continuous and proper. Then,

i.) For any non-empty subset K of E that is weakly compact (closed, convex and norm-bounded), there exists $\alpha \in K$ such that $f(\alpha) = \min_{x \in K} f(x)$;

ii.) If in addition f is coercive, there exists $\alpha \in E$ such that $f(\alpha) = \min_{x \in E} f(x)$.

Lemma of The Calculus of Variations [6] – Let $\Omega \subset \mathbb{R}^n$ be a regular, orientable and bounded submanifold-with-boundary. Let $h \in C(\Omega)$ and assume that

$$\int_{\Omega} h.\phi \, d\mu = 0$$

for all $\phi \in C_0^1(\Omega)$. Then $h \equiv 0$ on Ω, where Ω is of dimension m and $d\mu$ is the geometric m-volume on Ω.

It is rather straightforward to prove the above stated lemma by contradiction, as hence shown. Assume $|h(x_0)| > 0$ for some $x_0 \in \Omega\backslash\partial\Omega$, where $\partial\Omega$ is the boundary of the closure of Ω defined in the geometric sense. Then for some $\delta > 0$, we have $d(x_0, \partial\Omega) < \delta$ and $|h(x)| \geq \frac{|h(x_0)|}{2}$ $\forall x \in B(x_0, \delta) \cap \Omega$. Setting:

$$\phi(x) = \begin{cases} (\delta^2 - \left||x - x_0|\right|^2)^2; \ if\ x \in B(x_0, \delta) \cap \Omega \\ 0; \ if\ x \in \Omega\backslash B(x_0, \delta) \end{cases}$$

then $\phi \in C_0^1(\Omega)$. To confirm this claim, we justify differentiability of ϕ by first taking Ω to be $\mathbb{R}^n$. Differentiability of this function is obvious everywhere but on $\partial B(x_0, \delta)$. The gradient of this function for $x \in B(x_0, \delta)$ is given by $\phi'(x) = 4\left(\left||x - x_0|\right|^2 - \delta^2\right).(x - x_0)$. Thus, for any sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n \backslash\ \partial B(x_0, \delta)$ approaching some point $x \in \partial B(x_0, \delta)$, we get $\phi'(x_j)$ to be approaching zero. This gives us that the gradient function ϕ' is continuous on $\mathbb{R}^n$. Moreover,

$$|\int_\Omega h(x).\phi(x)d\mu\,| \geq \frac{|h(x_0)|}{2}\int_{B(x_0,\delta)\cap\Omega} \phi(x)d\mu > 0$$

which contradicts our initial assumption.

If Ω is not open, then it becomes expedient for us to define the gradient function intrinsically, using a covariant derivative for ϕ on Ω. In this event, we take

$$\phi'|_{T\Omega} = \nabla\phi = (\langle\phi', E_1\rangle, \langle\phi', E_2\rangle, \ldots, \langle\phi', E_m\rangle) \coloneqq (\nabla_{E_i}\phi)_{i=1}^m$$

where $\{E_1, E_2, \ldots, E_m\}$ is an orthonormal basis of functions for the tangent spaces to Ω at each point and $[E_1, E_2, \ldots, E_m]_p$ is the usual orientation for $T_p\Omega$ at each $p \in \Omega$. We can draw the same conclusions by adjusting certain conditions of the lemma, such as taking Ω to be a submanifold-without-boundary and using $\phi \in C^1(\overline{\Omega})$ as the test functions, when appropriate.

The function spaces $C^1(\overline{\Omega})$ and $C_0^1(\Omega)$ are not reflexive, which will prompt us to consider instead weak formulations of the optimization problems during computations. That is to say, we will target solutions in larger reflexive Sobolev spaces, reckoning with the fact that $C^1(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for any natural k and for $1 < p < \infty$. Occasionally, weak solutions also turn out to be solutions of the classical problems. Following our presentation of the fundamental tools, we now give two methods of formulation of classical PDE's associated to optimization of differentiable functionals.

2. Methods of Formulation

2.1. Method 1 [6]

Let $\Omega \subseteq \mathbb{R}^n$ be a connected, orientable and bounded submanifold of class C^2, and we are to minimize the functional

$$\begin{aligned} E:\ & V \longrightarrow \mathbb{R} \\ & v \mapsto \int_\Omega F(x, v, \nabla v)d\mu \end{aligned}$$

on a subset $V \subseteq C^1(\overline{\Omega})$ for a sufficiently regular function F. Let $\bar{v}$ be a minimizer of E on V. Then for some real positive number *r*, we have

$$E(\bar{v}) \leq E(v)\ \forall v \in V \cap B(\bar{v}, r)$$

We will choose V such that it accommodates appropriate tangent cones, which are variations of the form $v + \tau\phi$ for $v \in V$ and $\phi \in C_0^1(\Omega)$ (resp $C^1(\overline{\Omega})$). There exists a positive real number δ such that for any $\tau \in (-\delta, \delta)$, we have

$$\bar{v} + \tau\phi \in B(\bar{v}, r)$$

Defining $\gamma(\tau) := E(\bar{v} + \tau\phi)$, then 0 is a minimizer of γ in $(-\delta, \delta)$ which means

$$\gamma'(0) = 0 \implies E'(\bar{v}).\phi = 0$$

We find the derivative of E at v in the direction of ϕ;

$$\begin{aligned} E'(\bar{v}).\phi &= \lim_{\alpha\to 0}\frac{E(\bar{v} + \alpha\phi) - E(\bar{v})}{\alpha} = \int_\Omega \lim_{\alpha\to 0}\frac{F\big(x, \bar{v}(x) + \alpha\phi(x), \nabla\bar{v}(x) + \alpha\nabla\phi(x)\big) - F(x, \bar{v}, \nabla\bar{v})}{\alpha}d\mu \\ &= \int_\Omega F'(x, v, \nabla v); (0, \phi, \nabla\phi)d\mu = \int_\Omega [F_v(x, v, \nabla v).\phi + F_{\nabla v}(x, v, \nabla v).\nabla\phi]d\mu. \end{aligned}$$

The regularity of F at $\bar{v}$ is necessary for passing the limit into the above integral as we have done in the second line. This gives us uniform convergence of the integrand by way of the mean value inequality, since the integrand equals $F'(\zeta); (0, \phi, \nabla\phi)$ for some $\zeta \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m$, with $||F'||$ having a finite upper bound independent of α in a neighborhood of ζ.

We hereby explain certain used notations by again having $\{E_1, E_2, \ldots, E_m\}$ as an orthonormal basis of functions for the tangent bundle $T\Omega$ at each point of the manifold. We represent the gradient functions intrinsically to Ω:

$$\nabla v = (\nabla_{E_i}v)_{i=1}^m \text{ and } \nabla\phi = (\nabla_{E_i}\phi)_{i=1}^m$$

This way, $F_{\nabla v}$ has m scalar components, each of which we will denote $F_{v_{E_i}}$. We then implement Green's theorem of multivariate integration to re-evaluate a term in the formulation –

$$\int_\Omega [F_{\nabla v}.\nabla\phi]d\mu = \int_\Omega (F_{v_{E_i}})_{i=1}^m.(\nabla_{E_i}\phi)_{i=1}^m d\mu = -\int_\Omega (div\,F_{\nabla v})\phi d\mu$$

The reader ought to reckon with the divergence operator (*div*) invoked above, which will be used explicitly in the included illustrations. As such, we assume in addition that F is of class $C^2$and we get that $\forall\ \phi \in C_0^1(\Omega)$ (or $C^1(\overline{\Omega})$ if $\overline{\Omega}$ is without boundary),

$$\int_\Omega [F_v(x,\bar{v},\nabla\bar{v}) - divF_{\nabla v}].\phi\, d\mu = 0 \implies F_v(x,\bar{v},\nabla\bar{v}) = divF_{\nabla v}(x,\bar{v},\nabla\bar{v}) \quad (1)$$

at any local or global minimizer $\bar{v}$ of E on $V,$ which gives us a necessary optimality condition. If Ω is an open subset of $\mathbb{R}^n$, then the above is simply written as

$$F_v(x,\bar{v},\nabla\bar{v}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{v_{x_i}}(x,\bar{v},\nabla\bar{v})$$

2.2. Method 2

Now, assume that $\Omega\ \subseteq \mathbb{R}^n$ is open, bounded with a regular topological boundary, and we are to minimize the functional

$$\begin{aligned} E\colon\ \ V \subseteq C^1(\overline{\Omega}) \quad &\longrightarrow \quad \mathbb{R} \\ v \quad &\mapsto \quad \int_\Omega F(x,v,\nabla \mathrm{v})dV - \int_{\partial\Omega} g(x,v)dS. \end{aligned}$$

For this case, we work with test functions $\in C^1(\overline{\Omega})$ because of the contributor to the functional from the boundary, and our formulation yields the necessary optimality condition

$$\int_\Omega [F_v(x,\bar{v},\nabla\bar{v}).\phi + F_{\nabla v}(x,\bar{v},\nabla\bar{v}).\nabla\phi]dV - \int_{\partial\Omega} g_v(x,\bar{v})\phi\ dS = 0$$

at any minimizer $\bar{v}$ of E in V. Of course, we need the functions F and g to be sufficiently regular. Applying Green's theorem to a term in the above formulation;

$$\int_\Omega [F_{\nabla v}.\nabla\phi]dV = \int_{\partial\Omega} \left(\frac{\partial F_{\nabla v}}{\partial N}.\phi\right)dS - \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{v_{x_i}}\right).\phi dV$$

where N is the outward unit normal or Gauss map evaluated on $\partial\Omega$ and

$$\frac{\partial F_{\nabla v}}{\partial N} := \langle F_{\nabla v}, N\rangle$$

Substituting this in our formulation, we get

$$\int_\Omega \left(F_v - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{v_{x_i}}\right).\phi dV + \int_{\partial\Omega} (\langle F_{\nabla v}, N\rangle - g_v).\phi dS\ = 0$$

at any local or global minimizer $\bar{v}$ of $E.$ By the fundamental lemma of the calculus of variations, we conclude –

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{\bar{v}\ x_i} = F_{\bar{v}}\ in\ \Omega$$

$$\textstyle\sum_{i=1}^{n} F_{\bar{v}_{x_i}} N_i =\ g_{\bar{v}}\ on\ \partial\Omega \quad (2)$$

which is a boundary value PDE problem of Neumann type. By slightly adjusting prior hypotheses, it is straightforward to generalize that the formulations from (1) and (2) give not only necessary optimality conditions for minimizers, but for local critical points at large.

The given lemma of the calculus of variations applies to larger reflexive Sobolev spaces, and this we can infer from a generalization of the lemma called the du Bois – Reymond lemma. It gives us that for any $h \in W^{k,p}(\Omega)$ satisfying $\int_\Omega h(x).\phi(x)d\mu = 0$ for all $\phi \in C_0^1(\Omega)$ then $h \equiv 0$ almost everywhere on Ω. However, it is most convenient to work with subsets of the reflexive Hilbert space $W^{k,2}(\Omega)$ because of continuity of the partial inner product –

$$\langle h, \phi\ \rangle = \int_\Omega h(x).\phi(x)d\mu\ \ for\ h,\phi \in W^{k,2}(\Omega)$$

and the availability of other analytical solution tools such as the Lax-Milgram theorem [3]. Here, given $h \in W^{k,2}(\Omega)$ satisfying $\int_\Omega h(x).\phi(x)d\mu = 0$ for all $\in W^{k,2}(\Omega)$, then $h \equiv 0$ almost everywhere on Ω. We will consider the weakly formulated methods in these larger reflexive Sobolev spaces in order to investigate existence and/or uniqueness of solutions to the optimization problems. Given the possibility of solution existence from the optimization theorem, we proceed to solve the weakly formulated PDE in a Sobolev space, and finally check whether the weak solutions are also solutions in $C^1(\overline{\Omega})$. It may turn out that the weakly formulated problem has a solution, while the classical problem does not. In the succeeding examples, we will illustrate the theoretical framework laid out above.

3. Results

3.1. Result 1: Perelman Energy Functional

Let $S \subset \mathbb{R}^n$ be a regular, compact and connected hypersurface. Perelman's energy functional on S is formally analogous to heat flow along the manifold and we give it by

$$\begin{array}{llll} E: & C^1(S) & \longrightarrow & \mathbb{R} \\ & v & \mapsto & \int_S (R + ||\nabla v||^2)\exp(-v)dS \end{array}$$

where R is the scalar curvature of S. In any setting, the functional E lacks coercivity, and usually it also lacks a global minimizer.

As an illustration of this statement, let S be a regular surface in $\mathbb{R}^3$ consisting only of elliptic points, in which case the scalar curvature equals twice the Gaussian curvature and the Perelman energy is strictly positive for any v. We consider the problem in weak settings in order to investigate arguments using our optimization theorem. W can be any of the classical Sobolev spaces containing $C^1(S)$, and we see that $\inf_{v \in W} E(v) = 0$ by taking $||v||_W$ to infinity along the positive direction of constant functionals $v \equiv k$, where $k \in \mathbb{R}^+$ is a constant.

Nevertheless, the formulation of method 1 in subsection 2.1 above provides weak local critical points of E which we hereby discuss. This formulation gives us

$$-(R + ||\nabla \bar{v}||^2)\exp(-\bar{v}) = 2div(\exp(-\bar{v})\nabla \bar{v}) \tag{3}$$

at any critical point $\bar{v}$ of E. Observe that (3) is set as a non-linear second order P. D. E, as the divergence operation on the right hand side produces a second order differential of $\bar{v}$. As a simple computational illustration, we will have S to be the two-dimensional unit sphere $S^2 \subset \mathbb{R}^3$. This is a compact surface consisting only of elliptic points embedded in the real Euclidean 3-space with unit Gaussian curvature $K \equiv 1$ so that its scalar curvature is also constant: $R \equiv 2$. Solving for $\bar{v}$ in this case, we have

$$\left(2 + (\nabla_{E_1}\bar{v})^2 + (\nabla_{E_2}\bar{v})^2\right)\exp(-\bar{v}) + 2div(\exp(-\bar{v})\nabla \bar{v}) \Rightarrow \left(2 + \sum_{j=1,2}\left(\frac{\nabla_{\Phi_{u_j}}\bar{v}}{\left|\left|\Phi_{u_j}\right|\right|}\right)^2\right)\exp(-\bar{v})$$

$$+2div\left(\exp(-\bar{v})\left(\frac{\nabla_{\Phi_{u_j}}\bar{v}}{\left|\left|\Phi_{u_j}\right|\right|}\right)^2\right) = 0 \Rightarrow \left(2 + \frac{1}{g_{jj}}\left(\frac{\partial f}{\partial u_j}\right)^2\right)\exp(-f) + 2\sqrt{|g^{-1}|}\frac{\partial}{\partial u_j}\left(\sqrt{|g|}\exp(-f)\sqrt{\frac{1}{g_{jj}}}\frac{\partial f}{\partial u_j}\right) = 0$$

where $f = \bar{v} \circ \Phi$ and the parametrization

$$\Phi: U \subseteq \mathbb{R}^2 \Rightarrow S^2;\ (u_1, u_2) \mapsto \Phi(u_1, u_2) := p \in S^2$$

is used to pull back the differential equation to an open subset $U \subseteq \mathbb{R}^2$and solve. Using the spherical co-ordinate system, we have

$$\begin{array}{lll} \Phi: U = (0,2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \longrightarrow & S^2 \\ (u_1, u_2) & \mapsto & (cosu_1 cosu_2, sinu_1 cosu_2, sinu_2) \end{array}$$

so that $g_{11} = ||\Phi_{u_1}||^2 = cos^2 u_2;\ g_{22} = ||\Phi_{u_2}||^2 = 1.$

We have E_1, E_2 to be the unit vectors in the directions of the partial derivatives $\Phi_{u_1}$and Φ_{u_2}respectively. Observe that we have used the elementary property of directional derivatives; $\nabla_{\alpha X} Y = \alpha \nabla_X Y$ for a scalar field α. Hence, the formulation in (3) becomes

$$2 + sec^2 u_2\left(\frac{\partial f}{\partial u_1}\right)^2 - \left(\frac{\partial f}{\partial u_2}\right)^2 - 2secu_2\left(\frac{\partial f}{\partial u_1}\right)^2 + 2secu_2\frac{\partial^2 f}{\partial u_1^2} - 2tanu_2\frac{\partial f}{\partial u_2} + 2\frac{\partial^2 f}{\partial u_2^2} = 0$$

In all consequent computations for this example, we will fix $\frac{\partial f}{\partial u_1} := f_{u_1}$ to be zero for simplicity. In so doing, we obtain a second order non-linear O. D. E:

$$2 - [f'(u_2)]^2 - 2\tan(u_2).f'(u_2) + 2f''(u_2) = 0$$

Substitution of the variable $f'(u_2) = w$ gives an O. D. E of the first order. Although solutions exist on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, specified initial value solutions may not be bounded or unique on the interval because the tangent function is only continuous, and not uniformly continuous therein. For instance, we must use the initial condition $w\left(-\frac{\pi}{2}\right) = 0$ in order to avoid a singularity at the point $u_2 = -\frac{\pi}{2}$, but the solution for w obtained under this specification is unbounded because graphically generated schemes reveal $\lim_{u_2 \to \frac{\pi}{2}^-} w(u_2) = -\infty$. For this case, we clearly have $\bar{v} \in L^1_{loc}(S^2)$, which is the largest of function spaces covered by the du Bois-Reymond lemma. As such $\bar{v}$ is almost everywhere locally integrable since it is continuously differentiable except at $\Phi\left(u_1, \frac{\pi}{2}\right)$, which is a point. However, continuous differentiability on the entire compact unit sphere is not obtainable.

The nature of critical functions can be investigated using the second variation test [7] of the functional E. We can judge the nature of $\bar{v}$ by considering variations of the form $\bar{v} + \varepsilon\phi$ for $\varepsilon > 0$ small enough to judge how E acts locally around $\bar{v}$. We may examine the most significant terms of an associated Taylor Series expansion for this purpose. The smoothness of the metric of the hypersurface S guarantees that E is more than twice differentiable, so that we can deduce the required terms as follows.

$$E(v+\varepsilon\phi)=\int_{S}F(x,v+\varepsilon\phi,\nabla v+\varepsilon\nabla\phi)dS=\int_{S}F[(x,v,\nabla v)+\varepsilon(0,\phi,\nabla\phi)]dS.$$

We now expand the integrand partially via a Taylor Series varying with respect to ϕ to get

$$F(x,v,\nabla v)+\ \varepsilon F'(x,v,\nabla v);(0,\phi,\nabla\phi)+\frac{\varepsilon^2}{2}[F''(x,v,\nabla v);(0,\phi,\nabla\phi)];(0,\phi,\nabla\phi)+O(\varepsilon^3)$$

$$=F(x,v,\nabla v)+\ \varepsilon F'(x,v,\nabla v);(0,\phi,\nabla\phi)+\frac{\varepsilon^2}{2}(F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi)+O(\varepsilon^3)$$

Therefore, $E(v+\varepsilon\phi)-E(v)$ equals

$$\int_{S}\left(\varepsilon F'^{(x,v,\nabla v);(0,\phi,\nabla\phi)}+\frac{\varepsilon^2}{2}(F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi)+O(\varepsilon^3)\right)dS$$

$$\Longrightarrow\ E(\bar{v}\ +\varepsilon\phi)-E(\bar{v}\ )=\int_{S}\left(\frac{\varepsilon^2}{2}(F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi)+O(\varepsilon^3)\right)dS\ |_{v=\bar{v}}$$

The sign of the result obtained above is determined by the leading term of the integrand; which is $\frac{\varepsilon^2}{2}(F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi)$. This is to say, if the critical function $(\bar{v}\ )$is a strict local minimizer of E, then $E(\bar{v}+\varepsilon\phi)-E(\bar{v}\ )>0$ for ε small enough, meaning $F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi$ is positive for any non-zero test function $\phi\in C^1(S)$. By a similar line of reasoning, we can make the opposite conclusion in the event whereby $\bar{v}$ is a strict local maximizer of E. Critical points for which $E(\bar{v}\ +\varepsilon\phi)-E(\bar{v}\ )$ is neither positively nor negatively defined on any ε-neighborhood of $\bar{v}$ in the ambient Sobolev space are saddle points.

The procedure just outlined above is that of examining the second variation of E. Indeed, we can readily view this as the infinite-dimensional analogue of the behavior of the second derivative of functionals, by way of the quadratic form

$$E''(v)(\phi)(\phi)=\int_{S}(F_{vv}\phi^2+2\phi F_{v\nabla v}.\nabla\phi+[F_{\nabla v\nabla v};\nabla\phi].\nabla\phi)dS$$

With the given analytical observation, we now make remarks based on the computation initiated above on the Perelman energy functional.

$$E''(v)(\phi)(\phi)=\int_{S^2}[(\ 2+||\nabla v||^2)e^{-v}\phi^2-4e^{-v}\phi\langle\nabla\phi,\nabla v\rangle+2||\nabla\phi||^2e^{-v}]dS$$

$$=\int_{S^2}e^{-v}[(\ 2+||\nabla v||^2)\phi^2-4\phi\langle\nabla\phi,\nabla v\rangle+2||\nabla\phi||^2]dS \tag{4}$$

Observe also that

$$2||\phi\nabla v-\nabla\phi||^2=[\ 2\phi^2||\nabla v||^2-4\phi\langle\nabla\phi,\nabla v\rangle+2||\nabla\phi||^2\ ]\geq 0 \tag{5}$$

If everywhere on S^2, we have $||\nabla\bar{v}||^2$ strictly less than 2, then whenever $\phi\neq 0$, we also get

$$(\ 2+||\nabla v||^2)\phi^2-4\phi\langle\nabla\phi,\nabla v\rangle+2||\nabla\phi||^2>\ 2\phi^2||\nabla v||^2-4\phi\langle\nabla\phi,\nabla v\rangle+2||\nabla\phi||^2\Longrightarrow E''(v)(\phi)(\phi)>0$$

from (4) and (5).

In such an event, the critical function $\bar{v}$ would be a strict local minimizer for the Perelman energy. Using the weak solution discussed above, (wherein the spherical coordinate system was employed and f_{u_1} set to zero) this situation can be created simply by truncating parallels u_2^* of S^2 for which $|w(u_2^*)|>\sqrt{2}$, for $f'(u_2)=w$ as previously defined. The resulting manifold would be a connected spherical section with boundary, and the critical function would then be a classical solution to (3) above.

If $||\nabla\bar{v}\ ||^2>2$ on a dS-non-negligible subset $S^*$of S^2, then $\bar{v}$ would be a saddle point for E. This is confirmed by using an appropriate Urysohn test function ϕ, satisfying

$\phi=\exp(\bar{v})$ on a non-negligible subset of S^* and

$\phi=0$ whenever $||\nabla\bar{v}\ ||^2<2$,

for which $E''(\bar{v}\ )(\phi)(\phi)<0$. It is easy to verify that the integrand of $E''(\bar{v}\ )(\phi)(\phi)$ is negative whenever $||\nabla\bar{v}\ ||^2>2$ and $\phi=\exp(\bar{v})$.

E has no strict local maximizers, as $E''(v)(\kappa)(\kappa)>0$ for any non-zero constant function κ.

In the classical applications of Perelman's energy, the functional v is time dependent and critical points of E with respect to time characterize steady *Ricci Solitons.* Therefore, Perelman's entropy can be regarded as a variational tool for the study of *Ricci flow,* which is an advanced current area of interest in pseudo-Riemannian geometry. It is also worthy of note that the same functional also occurs quite importantly in a branch of theoretical physics known as string theory [8].

3.2. Result 2: Dirichlet Energy Functional

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with C^1 topological boundary. The Dirichlet energy functional on Ω is given by

$$\begin{aligned} E:\quad V \subset C^1(\overline{\Omega}) &\longrightarrow \mathbb{R} \\ v \quad &\mapsto \int_\Omega ||\nabla v||^2 dV \end{aligned}$$

where

$$V = \{v \in C^1(\overline{\Omega}) : v|_{\partial\Omega} = h\}$$

and h is a particular differentiable function defined on the compact set $\partial\Omega$. The classical problem is to minimize E over V, but we first consider the weak setting in the reflexive Sobolev space $W^{1,2}(\Omega) \coloneqq H^1(\Omega)$ to effectively perform analysis using the given optimization theorem in the Introduction section. In particular, this setting is made appropriate by continuous differentiability of the functional E on $H^1(\Omega)$. Hence, the domain V' of E in this setting is the pre-image of the singleton $h \in L^2(\partial\Omega)$ under the continuous trace operator;

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$$

so V' is (norm-) closed in $H^1(\Omega)$. Moreover, the set V' is convex because

$$\lambda u + (1-\lambda)v \in V' \ \forall \lambda \in [0,1], \forall u, v \in V'$$

For any function v_0 in V', it is easy to check that the set $B = \{v \in V' : E(v) \le E(v_0)\}$ is bounded due to the coercivity of E, giving us existence of a minimizer for E on B and thus also on V'. The critical function $\bar{v}$ will exist uniquely due to strict convexity of E.

Given the minimizer $\bar{v} \in H^1(\Omega)$, the weak formulation for this problem is the following boundary value PDE:

$$\sum_{i=1}^{n} \frac{\partial^2 \bar{v}}{\partial {x_i}^2} = 0 \ in\ \Omega;$$

$$\bar{v} = h \ on\ \partial\Omega \tag{6}$$

This is obtained by implementing the formulation method 2 (subsection 2.2) with test functions in $H_0^1(\Omega)$ instead of $H^1(\Omega)$, because there is no contribution to the functional E from the boundary of Ω. Problem (6) is known as Laplace's equation, as

$$\sum_{i=1}^{n} \frac{\partial^2 \bar{v}}{\partial {x_i}^2} \coloneqq \Delta \bar{v}$$

Is called the Laplacian of $\bar{v}$. (6) is a linear second-order elliptic P. D. E of Dirichlet type, of which the solutions constitute an interesting category of functions. Its solutions are called harmonic functions and they are the chief ingredients in the study of potential theory. We will hereby give just a brief analysis of this formulation and possible solutions.

Symmetries of the Laplace Equation

One of the most efficient approaches to tackling Laplace's equation is exploiting its symmetries. This equation is known to accommodate the Lie groups of conformal transformations on $\mathbb{R}^n$, which are precisely the non-degenerate symmetry Lie groups of invariance transformations which preserve angles between vectors in their domains. Any such group can be decomposed into one-parameter subgroups. Each member P_λ of a conformal one-parameter Lie group $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ can be seen as a re-parametrization of $\mathbb{R}^n$;

$$\begin{aligned} P_\lambda: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x = (x_1, \dots, x_n) &\longmapsto (y_i(x,\lambda))_{i=1}^n \end{aligned}$$

characterized by

$$\left\|\frac{\partial P_\lambda}{\partial x_i}\right\| = \left\|\frac{\partial P_\lambda}{\partial x_j}\right\|$$

and

$$\langle \frac{\partial P_\lambda}{\partial x_i}, \frac{\partial P_\lambda}{\partial x_j} \rangle = 0 \ for\ i \ne j, 1 \le i, j \le n$$

In this event,

$$\sum_{i=1}^{n} \frac{\partial^2 \bar{v}}{\partial {x_i}^2} = 0 \ \Rightarrow \ \sum_{i=1}^{n} \frac{\partial^2 \bar{v}}{\partial {y_i}^2} = 0$$

When P_λ is linear, its action can be faithfully represented by an appropriate non-degenerate linear map $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, meaning $P_\lambda(x) = Ax \ \forall x \in \mathbb{R}^n$. Specifically, any transformation represented by a subgroup of the orthogonal group

$$O(n) \coloneqq \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^T = A^{-1}\}$$

exhibits the required properties and these suffice for simplifying equation (6), provided that they leave the boundary condition invariant. Except for the case on $n = 2, O(n)$ has infinitely many one-parameter subgroups.

In addition to being conformal transformations, parametrizations by the orthogonal group are *isometries* meaning that

$$\left\|\frac{\partial P_\lambda}{\partial x_i}\right\| = 1$$

and

$$||Ax|| = ||x||$$

for every one-parameter subgroup $\{P_\lambda\}_{\lambda \in \mathbb{R}} \subseteq O(n)$ and $A \in O(n)$. Of course, we can deduce invariance of Laplace's equation (without boundary constraints) under the smaller rotational group $SO(n) \subseteq O(n)$ [5].

For an illustration of this example, we will consider a simple solution of Laplace's equation, using the n −ball of radius a; $\Omega = B^n(0, a)$ for convenience, with the boundary constraint $h(x) = \varphi(||x||)$ on $\partial\Omega$. Consider φ to be a continuously differentiable real-valued function defined on an open subinterval of the reals containing $\{a\}$. In this setting, any element of $O(n)$ leaves equation (6) invariant, and we can effectively deduce the Lie group invariant $r(x) = ||x||$ which eliminates the group parameter λ since

$$r(P_\lambda(x)) = r(x)\ \forall\{P_\lambda\}_{\lambda\in\mathbb{R}} \subseteq O(n)$$

We hereby seek a solution of functional form $\bar{v}(x) = \psi(r)$.

$$\frac{\partial\bar{v}}{\partial x_i} = \frac{d\psi}{dr}\frac{\partial r}{\partial x_i} = \frac{d\psi}{dr}\frac{\partial\left[(\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}\right]}{\partial x_i} = \psi'(r)\frac{x_i}{r}$$

$$\begin{aligned}\frac{\partial^2\bar{v}}{\partial {x_i}^2} &= \psi''(r)\left(\frac{\partial r}{\partial x_i}\right)^2 + \psi'(r)\frac{\partial^2 r}{\partial {x_i}^2}\\ &= \psi''(r)\left(\frac{x_i}{r}\right)^2 + \psi'(r)\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)\\ &= \psi''(r)\left(\frac{x_i}{r}\right)^2 + \psi'(r)\left(\frac{r - \frac{{x_i}^2}{r}}{r^2}\right)\end{aligned}$$

$$\sum_{i=1}^{n}\frac{\partial^2\bar{v}}{\partial {x_i}^2} = \sum_{i=1}^{n}\left(\psi''(r)\left(\frac{x_i}{r}\right)^2 + \psi'(r)\left(\frac{r - \frac{{x_i}^2}{r}}{r^2}\right)\right) = \psi''(r) + \psi'(r)\left(\frac{n-1}{r}\right)$$

$$\sum_{i=1}^{n}\frac{\partial^2\bar{v}}{\partial {x_i}^2} = 0$$

$$\Rightarrow \psi''(r) = \psi'(r)\left(\frac{1-n}{r}\right)$$

$$\Rightarrow \int\frac{\psi''(r)}{\psi'(r)}dr = \int\left(\frac{1-n}{r}\right)dr$$

$$\Rightarrow ln|\psi'(r)| = (1-n)\ln(r) + c$$

$$\Rightarrow \psi'(r) = \alpha r^{(1-n)}$$

where $\alpha \in \mathbb{R}$ is a constant of integration. For a further constant k of integration, we have the solutions:

$\psi(r) = \alpha\ln(r) + k$ for $n = 2$, and $\psi(r) = \alpha\frac{r^{(2-n)}}{2-n} + k$ for $n \geq 3$. Hence, the corresponding weak solutions in $H^1(B^n(0,a))$ are

$$\bar{v}(x) = \begin{cases}\alpha\ln(||x||) + k \text{ for } n = 2\\ \alpha\dfrac{||x||^{(2-n)}}{2-n} + k \quad \text{for } n \geq 3\end{cases}$$

for $x \neq 0$. Due to uniqueness of the weak solution, it becomes clear that $(\mathbf{6})$ often lacks a classical solution for this case, taking k to be zero and the function φ to be the identity for instance, recalling $h(x) = \varphi(||x||)$ on $\partial\Omega$. This is because $\bar{v}(x)$ as computed is not continuously extendable at $x = 0 \in B^n(0,a)$. Nevertheless, the above solutions $\bar{v}(x)$ are harmonic functions on $B^n(0,a)\backslash\{0\}$ and details about such functions are seen in classical potential theory.

The solutions derived above to both (3) and (6) have been obtained after placing certain restrictions in either case. It is useful to determine characteristics of these two differential equations in other settings. In particular, Laplace's equation (6) can be restructured in several ways after implementing its *potential transforms* from conservation laws [5] to give optional vantage points for determination of subgroups of the overall admitted Lie symmetry group. This is instrumental in solving for group invariant solutions to (6), other than the fundamental solutions given here.

4. Conclusion

In many everyday applications, analogous quantified functionals are not differentiable, unlike the continuously differentiable examples considered above. In order to embrace a broader scope of functionals, we test instead for their lower semi-continuity and convexity. Given a functional

$$E: V \to \mathbb{R} \cup \{+\infty\}$$

the domain D of E is the active region for our analysis;

$$D = \{v \in V: E(v) \in \mathbb{R}\}$$

E is lower semi-continuous at $v_0 \in D$ if for every sequence $\{v_n\}_{n \in \mathbb{N}} \subset D$ which converges in norm to v_0, we have $\liminf_{n\to\infty} E(v_n) \geq E(v_0)$. E is lower semi-continuous on D if it is lower semi-continuous at every $v \in D$. E is said to be convex if

$$E(\lambda u + (1-\lambda)v) \leq \lambda E(u) + (1-\lambda)E(v)\ \forall u, v \in D\ and\ \forall \lambda \in [0,1]$$

If E is both convex and lower semi-continuous but not differentiable, then E is still termed subdifferentiable and we invoke the Fenchel subdifferential of E at $u \in D$:

$$\partial E(u) = \{v^* \in V^* : \langle v^*, v-u\rangle \leq E(v) - E(u)\ \forall v \in V\}$$

We have the subdifferential $\partial E(u)$ to be nonempty upon subdifferentiablity of E on D and elements of $\partial E(u)$ are called subgradients of E at u. $\partial E(u)$ is always a convex set. If E is in addition differentiable at u, then $\partial E(u)$ is a singleton which coincides with the classical differential of E at u; $\partial E(u) \in V^*$, where V^* is the dual of V. A necessary and sufficient condition for $\bar{v}$ to be a local minimizer of E is that $0 \in \partial E(\bar{v})$, and this is how we initiate weak formulations in this setting. As we have done in our included illustrations, we may first consider the problem set in a larger reflexive space to make arguments about existence and uniqueness of solutions using the same given optimization theorem in the introduction.

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