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Boundedness for Sublinear Operators with Rough Kernels on Weighted Grand Morrey Spaces

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Abstract: In this paper, we study the boundedness of some sublinear operators with rough kernels, satisfied by most of the operators in classical harmonic analysis, on the generalized weighted grand Morrey spaces. More specifically, we show that the sublinear operators with rough kernels are bounded on these spaces under the conditions that the operators and the kernel functions satisfy some size conditions, and the operators are bounded on Lebesgue spaces. This is done by exploiting the well-known boundedness of sublinear operators with rough kernels on Lebesgue spaces, a more explicit decomposition of the generalized weighted grand Morrey spaces and the good properties of the weight functions and the kernel functions. Through combining some properties of A_p weight with the relevant lemmas of operators with rough kernel, we obtain the boundedness for sublinear operators with rough kernels on weighted grand morrey spaces. Furthermore, using the equivalent norm and the properties of BMO functions, an application of the boundedness of the sublinear operators with rough kernels to the corresponding commutators formed by certain operators and BMO functions are also considered. And the boundedness of commutator is obtained by the lemma of function BMO.

Keywords: Weighted Grand Morrey Space, Sublinear Operator, Rough Kernel, Commutator

1. Introduction

Morrey [1] first introduced the classical Morrey spaces to investigate the local behavior of solutions to second order elliptic partial differential equations (PDE). Morrey found that many properties of the solutions to PDE can be attributed to the boundedness of some operators on Morrey spaces. Since then, many researchers became interested in studying the norm inequalities of operators on Morrey type spaces, the following form of which was the one they usually adopted (see for example [2]):

$$M_{p,q}(\mathbb{R}^n)=\left\{f:\|f\|_{M_{p,q}(\mathbb{R}^n)}=\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|^{1-\frac{p}{q}}}\int_B|f(x)|^p\,dx\right)^{\frac{1}{p}}<\infty\right\}$$

where $f\in L^p_{loc}(\mathbb{R}^n)$ and $1\le p\le q<\infty$. Here and in what follows, we denote by B any balls in $\mathbb{R}^n$, $B=B(x_0,r)$ centered at $x_0\in\mathbb{R}^n$ with radius $r>0$ and $\lambda B=B(x_0,\lambda r)$ with $\lambda>0$. $M_{p,q}(\mathbb{R}^n)$ is an expansion of $L^p(\mathbb{R}^n)$ in the sense that $M_{p,p}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$.

Some inequalities (for example, the $L^p(\mathbb{R}^n)(1\le p<\infty)$ inequalities) for several kinds of operators remain true when Lebesgue measure dx is replaced by certain measure $w(x)dx$. It is worth pointing out that many authors are interested in the inequalities when $w(x)$ belongs to the Muchenhoupt classes. For some related works, we refer to two classical books [3-4]. The Muckenhoupt classes A_p and $A_{(p,q)}$ [5] contain the functions w which satisfy

$$A_p : \sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \le C, 1 < p < \infty$$

and

$$A_{(p,q)} : \sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{1/p'} \le C, 1 < p, q < \infty,$$

respectively, where $1/p + 1/p' = 1$. The A_p theory has found applications in several branches of Analysis, from Complex function theory to PDE [3].

Komori and Shirai [6] introduced weighted Morrey spaces, which are natural generalization of weighted Lebesgue spaces, and investigated the boundedness of classical operators in harmonic analysis including the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

the Calderón-Zygmund singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) dy,$$

where K is the Calderón-Zygmund kernel and the fractional integral which is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Let $1 \le p < \infty$, $0 < \lambda < 1$ and w be a function. Then the weighted Morrey space $M_{p,\lambda}(w)$ is defined by

$$M_{p,\lambda}(w) = \left\{ f : \|f\|_{M_{p,\lambda}(w)} = \sup_B \left(\frac{1}{w(B)^\lambda} \int_B |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\},$$

where $w(B) = \int_B w(x)dx$. It is obvious that if $w = 1$, $\lambda = 1 - p/q$, then $M_{p,\lambda}(w) = M_{p,q}(\mathbb{R}^n)$. For $w \in A_p (1 \le p < \infty)$, if $\lambda = 0$, then $M_{p,0}(w) = L^p(w)$ and if $\lambda = 1$, then $M_{p,1}(w) = L^\infty(w)$.

The corresponding Morrey spaces related to the boundedness for I_α are the weighted Morrey spaces $M_{p,\lambda}(w_1, w_2)$ with two weights which are also introduced by Komori and Shirai [6]. Let $1 \le p < \infty$, $0 < \lambda < 1$. For two weights w_1 and w_2,

$$M_{p,\lambda}(w_1, w_2) = \left\{ f : \|f\|_{M_{p,\lambda}(w_1,w_2)} = \sup_B \left(\frac{1}{w_2(B)^\lambda} \int_B |f(x)|^p w_1(x) dx \right)^{1/p} < \infty \right\}.$$

If $w_1 = w_2 = w$, we denote by $M_{p,\lambda}(w_1, w_1) = M_{p,\lambda}(w_2, w_2) = M_{p,\lambda}(w)$.

Meskhi [7] first introduced the grand Morrey spaces in a bounded domain and derived the boundedness of a class of integral operators in the frame of quasi-metric measure spaces with doubling measures. Inspired by the definition of $M_{p,\lambda}(w)$, we adopt the definition of the grand Morrey space [7] and consider the following generalized weighted grand Morrey spaces $M_{p),\theta,\lambda}(w)$ defined on the whole spaces $\mathbb{R}^n$ instead of bounded domains as

$$\|f\|_{M_{p),\theta,\lambda}(w)} = \sup_{0<\varepsilon<p-1} \Phi_{\theta,\varepsilon}^{p,\lambda}(f; w) < \infty,$$

where

$$\Phi_{\theta,\varepsilon}^{p,\lambda}(f; w) = \varepsilon^{\theta/(p-\varepsilon)} \|f\|_{M_{p-\varepsilon,\lambda}(w)}.$$

In the fractional case, we need to consider the corresponding generalized weighted grand Morrey spaces with two weights

$$\|f\|_{M_{p),\theta,\lambda}(w_1,w_2)} = \sup_{0<\varepsilon<p-1} \Phi_{\theta,\varepsilon}^{p,\lambda}(f; w_1, w_2) < \infty,$$

where

$$\Phi_{\theta,\varepsilon}^{p,\lambda}(f; w_1, w_2) = \varepsilon^{\theta/(p-\varepsilon)} \|f\|_{M_{p-\varepsilon,\lambda}(w_1,w_2)}.$$

If $\theta = 0$, then $M_{p),0,\lambda}(w) = M_{p,\lambda}(w)$ and $M_{p),0,\lambda}(w_1, w_2) = M_{p,\lambda}(w_1, w_2)$ are the classical weighted Morrey spaces of Komori and Shirai. Using Hölder's inequality, it is easy to have the following embedding

$$M_{p,\lambda}(w) \subset M_{p),\theta_1,\lambda}(w) \subset M_{p),\theta_2,\lambda}(w) \subset M_{p-\varepsilon,\lambda}(w),$$

where $\theta_1 < \theta_2$ and $0 < \varepsilon < p-1$.

If $\lambda = 0$, then $M_{p),\theta,0}(w) = L_{p),\theta}(w)$ is the generalized weighted grand Lebesgue space defined on $\mathbb{R}^n$ while if furthermore $w(x)=1$ and $\theta = 1$, $L_{p),1}(\mathbb{R}^n)$ is the grand Lebesgue space (also called small Lebesgue space) appeared in the paper by Greco, Iwaniec and Sbordone [8], where the existence and uniqueness of the nonhomogeneous n-harmonic equations were established. However, this topic exceeds the scope of this paper. For the structural properties of these spaces, we refer the reader to [9-14].

Given a function Ω over the unit sphere S^{n-1} of $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$ and $x' = x/|x|$, a Calderó n-Zygmund singular integral operator with rough kernel was given by

$$T_\Omega f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

and a related maximal operator

$$M_\Omega f(x) = \sup_{B \ni x} \int_B \Omega(x-y) f(y)dy,$$

where Ω is homogeneous of degree zero and satisfies

$$\Omega \in L^r(S^{n-1}), \quad 1 < r \leq \infty \tag{1}$$

and

$$\int_{S^{n-1}} \Omega(x')dx' = 0. \tag{2}$$

When $r = \infty$, Ω can be seen as a smooth kernel and T_Ω a standard Calderó n-Zygmund singular integral operator, which has been fully studied by many papers, a classical survey work, see for example [4].

For simplicity of notation, Ω is always homogeneous of degree zero and satisfies (1) and (2) throughout this paper if there are no special instructions. When Ω satisfies some size conditions, the kernel of the operator T_Ω has no regularity, and so the operator T_Ω is called rough singular integral operator. In recent years, a variety of operators related to the singular integrals for Calderó n-Zygmund, but lacking the smoothness required in the classical theory, have been studied. Duoandikoetxea [15] studied the norm inequalities for T_Ω in homogeneous case on weighted $L^p(1<p<\infty)$ spaces. For more corresponding works, we refer the reader to [16-21] and the references therein.

Hu, Lu and Yang [22] considered some more general sublinear operators with rough kernels which satisfy

$$|\mathcal{T}_\Omega f(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)f(y)|}{|x-y|^n} dy, \quad x \notin \text{supp } f \tag{3}$$

for any integral function $f \in L^1(\mathbb{R}^n)$ with compact support. Condition (3) was first introduced by Soria and Weiss [23] and was satisfied by many operators with rough kernels in classical harmonic analysis, such as T_Ω [24] and the oscillatory singular integral operator

$$\bar{T}_\Omega f(x) = p.v.\int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy, \quad x \notin \text{supp } f$$

where the phase is a polynomial. The boundedness of $\bar{T}_\Omega$ on weighted $L^p(\mathbb{R}^n)(1 \leq p < \infty)$ spaces were fully studied by Ojanen in his doctoral dissertation [25].

Let $D_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = D_k \setminus D_{k-1}$ for $k \in Z$. Throughout this paper, we will denote by χ_E the characteristic function of the set E. Inspired by the works of [26] and [19], we consider some size conditions (the following (4) and (5)) which are more general than (3) on the generalized weighted grand Morrey spaces.

$$|\mathcal{T}_\Omega f(x)| \leq C|x|^{-n} \int_{\mathbb{R}^n} |\Omega(x-y)f(y)| dy \tag{4}$$

when $\text{supp } f \subseteq A_k$ and $|x| \geq 2^{k+1}$ with $k \in Z$ and

$$|\mathcal{T}_\Omega f(x)| \leq C 2^{-kn} \int_{\mathbb{R}^n} |\Omega(x-y)f(y)| dy \tag{5}$$

when $\text{supp } f \subseteq A_k$ and $|x| \leq 2^{k-1}$ with $k \in Z$. It is worth pointing out that M_Ω satisfies conditions (4) and (5) (see [27]). Also, condition (3) implies the size conditions (4) and (5) since $|x-y| > |x|/2$ when $|x| \geq 2^{k+1}$ and $\text{supp } f \subseteq A_k$ while $\text{supp } f \subseteq A_k$ and $|x| \leq 2^{k-1}$ imply $|x-y| > |y|/2$.

The boundedness of parabolic sublinear operators with rough kernel generated by parabolic Calderó n-Zygmund operators and their commutators on the parabolic generalized local Morrey spaces have been investigated [28]. Recently, Zheng, Zhang and Shi [29] introduced the boundedness for sublinear operators on generalized weighted grand Morrey spaces. They studied the boundedness of some sublinear operators, satisfied by most of the operators in classical harmonic analysis, on the generalized weighted grand Morrey spaces. And they also considered the applications to the corresponding commutators formed by certain operators and bounded mean oscillations (BMO) functions. Inspired by the above, we study some sublinear operators with rough kernels on generalized weighted grand Morrey spaces.

We end this section with the outline of this paper. Section 2 contains Theorem 2.1-Theorem 2.4 and the proofs of them. In Section 3, we extend the corresponding results to the commutators of certain sublinear operators. In Section 4, we give a conclusion.

2. Method and Result

2.1. Boundedness of Sublinear Operators

The topic of this paper is intended as an attempt to study the boundedness of some sublinear operators with rough kernels which satisfy (4) and (5) on generalized weighted grand Morrey spaces and give some criterions to deduce the boundedness of the sublinear operators on certain spaces.

Now, we formulate our major results of this paper as

Theorem 2.1. *Let* $0<\lambda<1$, $0<\theta<\infty$, $1<r<\infty$, $r't\le s<p<\infty$ *with* t *be in* (15) *of Lemma* 2.4 *and a sublinear operator* $\mathcal{T}_\Omega$ *satisfies* (4) *and* (5). *If* $\mathcal{T}_\Omega$ *is bounded on* $L^s(w)$ *with* $w\in A_{p/r'}$, *then* $\mathcal{T}_\Omega$ *is bounded on* $M_{p),\theta,\lambda}(w)$.

Theorem 2.2. *Let* λ,θ,r,w *be in Theorem 2.1,* $r'\le p<\infty$ *and a sublinear operator* $\mathcal{T}_\Omega$ *satisfies* (4) *and* (5). *If* $\mathcal{T}_\Omega$ *is bounded on* $L_{p),\theta}(w)$, *then* $\mathcal{T}_\Omega$ *is bounded on* $M_{p),\theta,\lambda}(w)$.

We can get similar results for fractional integrals following the line of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. Let $0<\alpha<n$, r , t, p be in Theorem 2.1, $0<\lambda_i<1$ and $0<\theta_i<1$ $i=1,2$. Suppose that a sublinear operator $\mathcal{T}_{\alpha,\Omega}$ satisfies size conditions

$$|\mathcal{T}_{\alpha,\Omega}f(x)|\le C|x|^{-(n-\alpha)}\int_{\mathbb{R}^n}|\Omega(x-y)f(y)|\,dy \qquad (6)$$

when supp $f\subseteq A_k$ and $|x|\ge 2^{k+1}$ with $k\in Z$ and

$$|\mathcal{T}_{\alpha,\Omega}f(x)|\le C2^{-k(n-\alpha)}\int_{\mathbb{R}^n}|\Omega(x-y)f(y)|\,dy \qquad (7)$$

when supp $f\subseteq A_k$ and $|x|\le 2^{k-1}$ with $k\in Z$. Then we have

(a) If $\mathcal{T}_\alpha$ maps $L^{s_2}(w^p)$ into $L^{s_1}(w^q)$ with $w\in A_{(p/r',q)}$, then $\mathcal{T}_\alpha$ is bounded from $M_{p),\theta_2,\lambda_2}(w^p,w^q)$ to $M_{q),\theta_1,\lambda_1}(w^q)$, where $1<s_2<n/\alpha$, $r't\le s_1<q<\infty$, $s_1-s_2=p-q$, $\lambda_1=\lambda_2 t/(p-q+t)$, $\theta_1=\theta_2 t/(p-q+t)$, $p<q<p+r$ and $1/s_1=1/s_2-\alpha/n$.

Theorem 2.4. Let $p,q,\alpha,w,t,r,s_i,\lambda_i (i=1,2)$ be in Theorem 2.3, $0<\theta<\infty$ and the sublinear operator $\mathcal{T}_\alpha$ satisfies size conditions (6) and (7). If $\mathcal{T}_\alpha$ maps $L_{p),\theta}(w^p)$ into $L_{q),\theta}(w^q)$, then $\mathcal{T}_\alpha$ is bounded from $M_{p),\theta,\lambda_2}(w^p,w^q)$ to $M_{q),\theta,\lambda_1}(w^q)$.

Theorem 2.1-Theorem 2.4 can be seen as an extension of the related results in [6-7]. We emphasize that (6) and (7) are weaker conditions than the following condition

$$|\mathcal{T}_{\alpha,\Omega}f(x)|\le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}dy,\quad 0<\alpha<n \qquad (8)$$

for any integral function f with compact support. Condition (8) is satisfied by most fractional integral operators with rough kernels, such as the fractional integral operators of Muchenhoupt and Wheeden [30]

$$\mathcal{T}_{\alpha,\Omega}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)f(y)}{|x-y|^{n-\alpha}}dy, 0<\alpha<n.$$

Some mapping properties of $\mathcal{T}_{\alpha,\Omega}$ on various kinds of function spaces, see [31-33] and the references therein.

Proofs of Theorem Theorem 2.1-Theorem 2.4 depend heavily on some properties of A_p weights, which can be found in any papers or any books dealing with weighted boundedness for operators in harmonic analysis, such as [4]. For the convenience of the reader we collect some relevant properties of A_p weights without proofs, thus making our exposition self-contained.

Lemma 2.5. Let $1\le p<\infty$ and $w\in A_p$. Then the following statements are true

(a) There exists a constant C such that

$$w(2B)\le Cw(B) \qquad (9)$$

When w satisfies this condition, we say w satisfies the doubling condition.

(b) There exists a constant $C>1$ such that

$$w(2B)\ge Cw(B) \qquad (10)$$

When w satisfies this condition, we say w satisfies the reverse doubling condition.

(c) There exist two constants C and $r>1$ such that the following reverse Hölder inequality holds for every ball $B\subset\mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r}\le C\left(\frac{1}{|B|}\int_B w(x)dx\right) \qquad (11)$$

(d) For all $\lambda>1$, we have

$$w(\lambda B)\le C\lambda^{np}w(B) \qquad (12)$$

(e) There exist two constants C and $\delta>0$ such that for any measurable set $Q\subset B$

$$\frac{w(Q)}{w(B)}\le C\left(\frac{|Q|}{|B|}\right)^{\delta} \qquad (13)$$

If w satisfies (13), we say $w\in A_\infty$.

(f) For all $p<q<\infty$, we have

$$A_\infty=\cup_p A_p,\quad A_p\subset A_q \qquad (14)$$

Lemma 2.6. (a) If $p\ge r'$, $w\in A_{p/r'}$ There exists a constant $1\le t\le p/r'$ such that

$$w\in A_t \qquad (15)$$

(b) If $p>1$, $w\in A_{(p,q)}$, then we have

$$w^{p}\in A_{p},\quad w^{q}\in A_{q} \tag{16}$$

Lemma 2.7. Let $1<p<\infty$, $0<\lambda<1$, $0<\theta<\infty$ and $w(x)$ be a function. Then there exists a constant C such that for any $0<\delta<p-1$,

$$\|f\|_{M_{p),\theta,\lambda}(w)}\le C\sup_{0<\varepsilon<\delta}\Phi_{\theta,\varepsilon}^{p,\lambda}(f,w),$$

where C is independent of ε and f.

Proof. For fixed $0<\delta<p-1$, it is obvious that

$$\|f\|_{M_{p),\theta,\lambda}(w)}=\max\left\{\sup_{0<\varepsilon\le\delta}\Phi_{\theta,\varepsilon}^{p,\lambda}(f,w),\ \sup_{\delta<\varepsilon<p-1}\Phi_{\theta,\varepsilon}^{p,\lambda}(f,w)\right\}$$

$$:=\max\{B_1,B_2\}.$$

For the term B_2, by the facts that $\sup_{\delta<\varepsilon<p-1}\varepsilon^{1/(p-\varepsilon)}=p-1$, $1/(p-\varepsilon)>1/(p-\delta)$ and Hö lder's inequality we can obtain

$$B_2=\sup_{\delta<\varepsilon<p-1}\varepsilon^{\theta/(p-\varepsilon)}\sup_{B}\left(\frac{1}{w(B)^{\lambda}}\int_B|f(x)|^{p-\varepsilon}\ w(x)dx\right)^{1/(p-\varepsilon)}$$

$$\le(p-1)^{\theta}\sup_{\delta<\varepsilon<p-1}\sup_{B}w(B)^{(1-\lambda)/(p-\varepsilon)}\left(\frac{1}{w(B)}\int_B|f(x)|^{p-\delta}\ w(x)dx\right)^{1/(p-\delta)}$$

$$\le(p-1)^{\theta}\sup_{B}w(B)^{(1-\lambda)/(p-\delta)}\left(\frac{1}{w(B)}\int_B|f(x)|^{p-\delta}\ w(x)dx\right)^{1/(p-\delta)}$$

$$\le(p-1)^{\theta}\delta^{-\theta/(p-\delta)}\sup_{B}\left(\frac{1}{w(B)^{\lambda}}\int_B|f(x)|^{p-\delta}\ w(x)dx\right)^{1/(p-\delta)}$$

$$\le C\sup_{0<\varepsilon\le\delta}\Phi_{\theta,\varepsilon}^{p,\lambda}(f,w).\quad\square$$

The following lemma about the operators with rough kernel is essential to our proofs.

Lemma 2.8. [34] Let $\Omega\in L^{r}(S^{n-1})$ with $1\le r<\infty$. Then the following statements are true

(a) If $x\in A_k$ and $j\ge k+1$, then $\int_{A_j}|\Omega(x-y)|^{r}\ dy\le C2^{jn}$;

(b) If $y\in A_k$ and $k\ge j+1$, then $\int_{A_j}|\Omega(x-y)|^{r}\ dx\le C2^{k(n-1)+j}$.

Proof of Theorem 2.1. Let $r't\le s<p<\infty$, $w\in A_{p/r'}$ and $0<\lambda<1$. By Lemma 2.7, it suffices to show that

$$\sup_{0<\varepsilon<\min\{p-r't,p/r\}}\sup_{B}\left(\frac{\varepsilon^{\theta}}{w(B)^{\lambda}}\int_B|\mathcal{T}_{\Omega}f(x)|^{p-\varepsilon}\ w(x)dx\right)\le C\|f\|_{M_{p),\theta,\lambda}(w)}^{p-\varepsilon}.$$

For a fixed ball $B=B(x_0,r)$ and $0<\varepsilon<\min\{p-r't,p/r\}$, there is no loss of generality in assuming $r=1$. We decompose $f=f\chi_{2B}+f\chi_{(2B)^{c}}:=f_1+f_2$. Since $\mathcal{T}_{\Omega}$ is a sublinear operator, so we get

$$\frac{\varepsilon^{\theta}}{w(B)^{\lambda}}\int_B|\mathcal{T}_{\Omega}f(x)|^{p-\varepsilon}\ w(x)dx$$

$$\le\frac{\varepsilon^{\theta}}{w(B)^{\lambda}}\int_B|\mathcal{T}_{\Omega}f_1(x)|^{p-\varepsilon}\ w(x)dx+\frac{\varepsilon^{\theta}}{w(B)^{\lambda}}\int_B|\mathcal{T}_{\Omega}f_2(x)|^{p-\varepsilon}\ w(x)dx$$

$$:=I+II.$$

By the assumption on $\mathcal{T}_{\Omega}$ and (14), we can obtain

$$I \le \frac{\varepsilon^{\theta}}{w(B)^{\lambda}} \int_{\mathbb{R}^n} |\mathcal{T}_{\Omega} f_1(x)|^{p-\varepsilon} w(x)dx \le \frac{C\varepsilon^{\theta}}{w(B)^{\lambda}} \int_{2B} |f(x)|^{p-\varepsilon} w(x)dx \le C\| f\|_{M_{p),\theta,\lambda}(w)}^{p-\varepsilon} \tag{17}$$

For the term II, by (5) we have

$$II \le \frac{C\varepsilon^{\theta}}{w(B)^{\lambda}} \int_{B} \left| \sum_{k=1}^{\infty} 2^{-kn} \mathcal{T}_{\Omega,k} f(x) \right|^{p-\varepsilon} w(x)dx, \tag{18}$$

where

$$\mathcal{T}_{\Omega,k} f(x) = \int_{A^{k+1}} |\Omega(x-y) f(y)|\, dy.$$

We distinguish two cases according to the size of $p-\varepsilon$ and r to get the estimates for $\mathcal{T}_{\Omega,k}$.

Case 1. $p-\varepsilon > r'$. In this case, (15), $w \in A_{p/r'}$ and $\varepsilon < p - tr'$ imply that $w \in A_{(p-\varepsilon)/r'}$, hence

$$\int_{B} w^{-r'/(p-\varepsilon-r')} dy \le \frac{|B|^{(p-\varepsilon)/(p-\varepsilon-r')}}{w(B)^{r'/(p-\varepsilon-r')}}. \tag{19}$$

By (19), Hölder's inequality and Lemma 2.8, we have

$$\begin{aligned}
\mathcal{T}_{\Omega,k} f(x) &\le C\left(\int_{A^{k+1}} |\Omega(x-y)|^{r} dy\right)^{1/r} \left(\int_{2^{k+1}B} |f(y)|^{r'} dy\right)^{1/r'} \\
&\le C2^{(k+1)n/r} \left(\int_{2^{k+1}B} |f(y)|^{r'} w(y)^{r'/(p-\varepsilon)} w(y)^{-r'/(p-\varepsilon)} dy\right)^{1/r'} \\
&\le C2^{(k+1)n/r} \left(\int_{2^{k+1}B} |f(y)|^{p-\varepsilon} w(y)dy\right)^{1/(p-\varepsilon)} \\
&\quad \times \left(\int_{2^{k+1}B} w(y)^{-r'/(p-\varepsilon-r')} dy\right)^{(p-\varepsilon-r')/(p-\varepsilon)r'} \\
&\le C\left(\frac{1}{w(2^{k+1}B)^{\lambda}} \int_{2^{k+1}B} |f(y)|^{p-\varepsilon} w(y)dy\right)^{1/(p-\varepsilon)} \frac{2^{(k+1)n/r} |2^{k+1}B|^{1/r'}}{w(2^{k+1}B)^{(1-\lambda)/(p-\varepsilon)}} \\
&\le C\left(\frac{1}{w(2^{k+1}B)^{\lambda}} \int_{2^{k+1}B} |f(y)|^{p-\varepsilon} w(y)dy\right)^{1/(p-\varepsilon)} \frac{2^{(k+1)n}}{w(2^{k+1}B)^{(1-\lambda)/(p-\varepsilon)}}.
\end{aligned} \tag{20}$$

Case 2. $p-\varepsilon = r'$. In this case, $w \in A_1$ implies that

$$(ess \inf_{x\in 2^{k+1}B} w(x))^{-1} \le \frac{|2^{k+1}B|}{w(2^{k+1}B)} \tag{21}$$

which combining with the Hölder inequality and Lemma 2.8 yield that

$$\begin{aligned}
\mathcal{T}_{\Omega,k} f(x) &\le C2^{(k+1)n/r} \left(\int_{2^{k+1}B} |f(y)|^{p-\varepsilon} w(y)w(y)^{-1} dy\right)^{1/(p-\varepsilon)} \\
&\le C\left(\frac{1}{w(2^{k+1}B)^{\lambda}} \int_{2^{k+1}B} |f(y)|^{p-\varepsilon} w(y)dy\right)^{1/(p-\varepsilon)} \frac{2^{(k+1)n}}{w(2^{k+1}B)^{(1-\lambda)/(p-\varepsilon)}}
\end{aligned} \tag{22}$$

Substituting (20) and (22) into (18), we can assert that

$$II \le C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)} \left(\sum_{k=1}^{\infty} \frac{w(B)^{(1-\lambda)/(p-\varepsilon)}}{w(2^{k+1}B)^{(1-\lambda)/(p-\varepsilon)}} \right)^{(p-\varepsilon)} \le C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)},$$

where we have used (10) in the last inequality. Combining (17) and (18), we obtain the proof of Theorem 2.1. □

Proof of Theorem 2.2. The proof of Theorem 2.2 is straightforward by the method in the proof of Theorem 2.1. The only difference is that we use the $L_{p),\theta}(w)$ boundedness of $\mathcal{T}$ instead of the $L^s(w)$ boundedness. We omit it's proof here.

Proof of Theorem 2.3. We can use the similar arguments as in the proof of Theorem 2.1. For fixed $0<\varepsilon<\min\{p-r't, p/r\}$, it suffices to show that

$$\frac{\varepsilon^{(q-\varepsilon)\theta_2/(p-\varepsilon)}}{w^q(B)^{(q-\varepsilon)\lambda_2/(p-\varepsilon)}} \int_B |\mathcal{T}_{\alpha,\Omega} f(x)|^{q-\varepsilon}\ w(x)^q\,dx \le C\|f\|^{q-\varepsilon}_{M_{p),\theta_2,\lambda_2}(w^p,w^q)}.$$

For a fixed ball $B=B(x_0,1)$, we decompose $f=f\chi_{2B}+f\chi_{(2B)^c}:=f_1+f_2$. Since $\mathcal{T}_{\alpha,\Omega}$ is a sublinear operator, we get

$$\frac{\varepsilon^{(q-\varepsilon)\theta_2/(p-\varepsilon)}}{w^q(B)^{(q-\varepsilon)\lambda_2/(p-\varepsilon)}} \int_B |\mathcal{T}_{\alpha,\Omega} f(x)|^{q-\varepsilon}\ w(x)^q\,dx$$

$$\le \frac{\varepsilon^{(q-\varepsilon)\theta_2/(p-\varepsilon)}}{w^q(B)^{(q-\varepsilon)\lambda_2/(p-\varepsilon)}} \int_B (|\mathcal{T}_{\alpha,\Omega} f_1(x)|^{q-\varepsilon} + |\mathcal{T}_{\alpha,\Omega} f_2(x)|^{q-\varepsilon}) w^q(x)dx$$

$$:= J + JJ.$$

Using the assumption on $\mathcal{T}_{\alpha,\Omega}$, we can get

$$J \le C\|f\|^{p-\varepsilon}_{M_{p),\theta_2,\lambda_2}(w^p,w^q)}.$$

For the term JJ, by similar argument as that of Theorem 2.1, we obtain

$$JJ \le \frac{C\varepsilon^{(q-\varepsilon)\theta_2/(p-\varepsilon)}}{w^q(B)^{(q-\varepsilon)\lambda/(p-\varepsilon)}-1} \sum_k \left(2^{-k(n-\alpha)} \int_{A_k} |\Omega(x-y)f(y)|\,dy \right)^{q-\varepsilon} w^q(B)^{1-(q-\varepsilon)\lambda/(p-\varepsilon)}$$

$$\le C\|f\|^{q-\varepsilon}_{M_{p),\theta_2,\lambda_2}(w^p,w^q)} \left(\sum_{k=1}^{\infty} \frac{w^q(B)^{(1/(q-\varepsilon)-\lambda/(p-\varepsilon))}}{w^q(2^{k+1}B)^{(1/(q-\varepsilon)-\lambda/(p-\varepsilon))}} \right)^{q-\varepsilon}$$

$$\le C\|f\|^{q-\varepsilon}_{M_{p),\theta_2,\lambda_2}(w^p,w^q)}$$

The estimates for J and JJ imply the proof of Theorem 2.3. □

Proof of Theorem 2.4. We omit the proof of Theorem 2.4 for the similarity as that of Theorem 1.3.

2.2. Boundedness of Commutators

We say that b is a $BMO(\mathbb{R}^n)$ function if the following sharp maximal function is finite

$$b^{\#}(x) = \sup_B \frac{1}{|B|} \int_B |b(y)-b_B|\,dy,$$

where the supreme is taken over all balls $B\subset\mathbb{R}^n$ and $f_B = \frac{1}{|B|}\int_B f(y)dy.$ This means $\|b\|_{BMO(\mathbb{R}^n)} = \|b^{\#}\|_{L^\infty} < +\infty$. An early work about $BMO(\mathbb{R}^n)$ space can attribute to John and Nirenberg [35]. For $1<p<\infty$, there is a close relation between $BMO(\mathbb{R}^n)$ and A_p weights

$$BMO(\mathbb{R}^n) = \{\alpha \log w : w\in A_p, \alpha \ge 0\}.$$

Given an operator T acting on a generic function f and a function b, the commutator T_b is formally defined as

$$T_b f = [b,T]f = bT(f) - T(bf).$$

Since $L^{\infty}(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$, the boundedness of T_b is worse than T (for example, the singularity, see also [36]). Therefore, many authors want to know whether T_b enjoys the similar boundedness with T. There are a lot of articles that deal with the topic of commutators of different operators with BMO functions on Lesbugue spaces. The first results on this commutator were obtained by Coifman, Rochberg and Weiss [37] in their study of certain factorization theorems for generalized Hardy spaces. In the present section, we will extend the boundedness of $\mathcal{T}_{\Omega}$ and $\mathcal{T}_{\alpha,\Omega}$ to $\mathcal{T}_{\Omega,b}$ and $\mathcal{T}_{\alpha,\Omega,b}$ on $M_{p),\theta,\lambda}$, respectively.

Theorem 2.9. *Let* r,p,λ,θ,t,s *and* w *be in Theorem* 2.1. *Suppose that the sublinear operator* $\mathcal{T}_{\Omega}$ *satisfies condition* (3) *for any integral function* f *with compact support. If* $\mathcal{T}_{\Omega,b}$ *is bounded on* $L^s(w)$ *with* $b\in BMO(\mathbb{R}^n)$, *then* $\mathcal{T}_{\Omega,b}$ *is bounded on* $M_{p),\theta,\lambda}(w)$.

Theorem 2.10. *Let* r,p,λ,θ *and* w *be in Theorem* 2.1. *Suppose that the sublinear operator* $\mathcal{T}_{\Omega}$ *satisfies condition* (3) *for any integral function* f *with compact support. If* $\mathcal{T}_{\Omega,b}$ *is bounded on* $L_{p),\theta}(w)$ *with* $b\in BMO(\mathbb{R}^n)$, *then* $\mathcal{T}_{\Omega,b}$ *is bounded on* $M_{p),\theta,\lambda}(w)$.

Theorem 2.11. *Let* $p,r,q,t,\alpha,w,\lambda_i,\theta_i,s_i, i=1,2,$ *be in Theorem* 2.3 *and the sublinear operator* $\mathcal{T}_{\alpha,\Omega}$ *satisfy condition* (8) *for any integral function* f *with compact support. If* $\mathcal{T}_{\alpha,\Omega,b}$ *maps* $L^{s_2}(w^p)$ *into* $L^{s_1}(w^q)$ *with* $b\in BMO(\mathbb{R}^n)$, *then* $\mathcal{T}_{\alpha,\Omega,b}$ *is bounded from* $M_{p),\theta_2,\lambda_2}(w^p,w^q)$ *to* $M_{q),\theta_1,\lambda_1}(w^q)$.

Theorem 2.12. *Let* $p,r,q,t,\alpha,w,\lambda_i(i=1,2)$ *be in Theorem* 2.3, $0<\theta<\infty$ *and the sublinear operator* $\mathcal{T}_{\alpha,\Omega}$ *satisfy condition* (8) *for any integral function* f *with compact support. If* $\mathcal{T}_{\alpha,\Omega,b}$ *maps* $L_{p),\theta}(w^p)$ *into* $L_{q),\theta}(w^q)$ *with* $b\in BMO(\mathbb{R}^n)$, *then* $\mathcal{T}_{\alpha,\Omega,b}$ *is bounded from* $M_{p),\theta,\lambda_2}(w^p,w^q)$ *to* $M_{q),\theta,\lambda_1}(w^q)$.

The following lemmas about $BMO(\mathbb{R}^n)$ functions will help us to prove Theorem 2.9-Theorem 2.12.

Lemma 2.13. [3, Theorem 3.8] Let $1\le p<\infty$, $b\in BMO(\mathbb{R}^n)$. Then for any ball $B\subset\mathbb{R}^n$, the following statements are true

(a) There exist constants C_1, C_2 such that for all $\alpha>0$

$$\left|\{x\in B: |b(x)-b_B|>\alpha\}\right|\le C_1|B|e^{-C_2\alpha/\|b\|_{BMO(\mathbb{R}^n)}}. \quad (23)$$

Inequality (23) is called John-Nirenberg inequality.

(b)

$$|b_{2^{\lambda}B}-b_B|\le 2^n\lambda\|b\|_{BMO(\mathbb{R}^n)}. \quad (24)$$

Lemma 2.14. [4, Proposition 7.1.2] (see also [5, Theorem 5]) Let $w\in A_{\infty}$ and $1<p<\infty$. Then the following statements are equivalent

(a) $\|b\|_{BMO(\mathbb{R}^n)}\sim\sup_B\left(\frac{1}{|B|}\int_B|b(x)-b_B|^p\,dx\right)^{\frac{1}{p}}$;

(b) $\|b\|_{BMO(\mathbb{R}^n)}\sim\sup_B\inf_{a\in\mathbb{R}}\frac{1}{|B|}\int_B|b(x)-a|\,dx;$

(c) $\|b\|_{BMO(w)}=\sup_B\frac{1}{w(B)}\int_B|b(x)-b_{B,w}|\,w(x)dx,$ where $BMO(w)=\{b:\|b\|_{BMO(w)}<\infty\}$ and $b_{B,w}=\frac{1}{w(B)}\int_B b(y)w(y)dy.$

As in Section 2, we only need to give the proofs of Theorem 2.9 and Theorem 2.11.

Lemma 2.15. Let p,r,b,w,θ,t be in *Theorem* 3.1 and $B=B(x_0,1)$ be a generic fixed ball. Then for any $0<\varepsilon<p-r't$, the inequality

$$\left(\varepsilon^{\theta}\int_{|x_0-y|>2}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^n}|b_{B,w}-b(y)|\,dy\right)^{p-\varepsilon}\le C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)}w(B)^{\lambda-1} \quad (25)$$

holds for every $y\in(2B)^c$, where $(2B)^c=\mathbb{R}^n\setminus(2B)$.

Proof. We will consider two cases.

Case 1. $P-\varepsilon>r'$. In this case, $w\in A_{(p-\varepsilon)/r'}$. Using Hölder's inequality and Lemma 2.8 to the left-hand-side of (25), we have

$$\varepsilon^{\theta}\int_{|x_0-y|>2}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^n}|b_{B,w}-b(y)|\,dy$$

$$\le C\varepsilon^{\theta}\sum_{j=1}^{\infty}\int_{2^j<|x_0-y|<2^{j+1}}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^n}|b_{B,w}-b(y)|\,dy$$

$$\leq C\sum_{j=1}^{\infty}\frac{\varepsilon^{\theta}}{|2^{j}B|}\int_{A_j}|\Omega(x_0-y)f(y)||b_{B,w}-b(y)|dy$$

$$\leq C\|f\|_{M_{p),\theta,\lambda}(w)}\sum_{j=1}^{\infty}\frac{2^{nj/r}w(2^{j+1}B)^{\lambda/(p-\varepsilon)}}{|2^{j}B|}A,$$

where

$$A=\left(\int_{2^{j+1}B}|b_{B,w}-b(y)|^{\overline{p}'r'}\ w(y)^{1-\overline{p}'}dy\right)^{1/\overline{p}'r'},\quad \overline{p}=(p-\varepsilon)/r'>1.$$

Thus

$$A\leq\left(\int_{2^{j+1}B}(|b_{2^{j+1}B,w^{1-\overline{p}'}}-b(y)|+|b_{2^{j+1}B,w^{1-\overline{p}'}}-b_{B,w}|)^{\overline{p}'r'}w(y)^{1-\overline{p}'}dy\right)^{1/\overline{p}'r'}$$

$$\leq\left(\int_{2^{j+1}B}|b_{2^{j+1}B,w^{1-\overline{p}'}}-b(y)|^{\overline{p}'r'}\ w(y)^{1-\overline{p}'}dy\right)^{\frac{1}{\overline{p}'r'}}+|b_{2^{j+1}B,w^{1-\overline{p}'}}-b_{B,w}|\,w^{1-\overline{p}'}(2^{j+1}B)^{\frac{1}{\overline{p}'r'}}$$

$$:=A_1+A_2$$

Lemma 2.14 implies that

$$A_1\leq Cw^{1-\overline{p}'}(2^{j+1}B)^{1/\overline{p}'r'}.$$

We are now in a position to deal with A_2, by (24), we have

$$|b_{2^{j+1}B,w^{1-\overline{p}'}}-b_{B,w}|\leq|b_{2^{j+1}B,w^{1-\overline{p}'}}-b_{2^{j+1}B}|+|b_{2^{j+1}B}-b_{B}|+|b_{B}-b_{B,w}|$$

$$\leq\frac{1}{w^{1-\overline{p}'}(2^{j+1}B)}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|\,w(y)^{1-\overline{p}'}dy+2^{n}(j+1)\|b\|_{BMO(\mathbb{R}^n)}\ +\frac{1}{w(B)}\int_{B}|b(y)-b_{B}|\,w(y)dy$$

$$:=A_{21}+A_{22}+A_{23}$$

Combining (12) with (23), we have

$$A_{23}=\frac{1}{w(B)}\int_{0}^{\infty}w(\{x\in B:|b(y)-b_{B}|>\alpha\})d\alpha$$

$$\leq C\int_{0}^{\infty}e^{-C_2\alpha\delta/\|b\|_{BMO(\mathbb{R}^n)}}d\alpha$$

$$\leq C\ .$$

In the same manner we can see that

$$A_{21}\leq C.$$

It follows immediately that

$$A_2\leq C(2^{n}(j+1)+2)w^{1-\overline{p}'}(2^{j+1}B)^{1/\overline{p}'r'}.$$

Therefore

$$A\leq C(j+1)w^{1-\overline{p}'}(2^{j+1}B)^{1/\overline{p}'r'}.$$

A further use of (10) and $w\in(p-\varepsilon)/r'$ allow us to obtain

$$\sum_{j=1}^{\infty}\frac{2^{nj/r}w(2^{j+1}B)^{\lambda/(\bar{p}-\varepsilon)}}{|2^jB|}\left(\int_{2^{j+1}B}|b(y)-b_{B,w}|^{\bar{p}'r'}w(y)^{1-\bar{p}'}dy\right)^{1/\bar{p}'r'}$$

$$\leq\sum_{j=1}^{\infty}\frac{2^{nj/r}w(2^{j+1}B)^{\lambda/(p-\varepsilon)}}{|2^jB|}(j+1)w(y)^{1-\bar{p}'}(2^{j+1}B)^{1/(\bar{p}'r')}$$

$$\leq C\sum_{j=1}^{\infty}\frac{2^{nj/r}|2^{j+1}B|^{1/r'}(j+1)}{|2^jB|}\frac{w(B)^{(1-\lambda)/(p-\varepsilon)}}{w(2^{j+1}B)^{(1-\lambda)/(p-\varepsilon)}}w(B)^{(\lambda-1)/(p-\varepsilon)}$$

$$\leq C\sum_{j=1}^{\infty}\frac{j+1}{D^{(j+1)(1-\lambda)/(p-\varepsilon)}}w(B)^{(\lambda-1)/(p-\varepsilon)}\leq Cw(B)^{(\lambda-1)/(p-\varepsilon)},$$

where $D>1$ is a constant appeared in (10).

Case 2. $P-\varepsilon=r'$. In this case, $w\in A_1$. We can prove (25) by a similar analysis as in the proof of Theorem 2.1 (in the case $P-\varepsilon=r'$) and Case 1. □

Having disposed of the above preliminary step, we can now return to the proofs of Theorem 2.9 and Theorem 2.10.

Proof of Theorem 3.1. The task is now to find a constant C such that for fixed ball $B=B(x_0,1)$ and $0<\varepsilon<\min\{p-r't,p/r\}$, we can obtain

$$\frac{\varepsilon^\theta}{w(B)^\lambda}\int_B\left|T_{\Omega,b}f(x)\right|^{p-\varepsilon}w(x)dx\leq C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)}.$$

We decompose $f=f\chi_{2B}+f\chi_{(2B)^c}:=f_1+f_2$, and consider the corresponding splitting

$$\frac{\varepsilon^\theta}{w(B)^\lambda}\int_B\left|T_{\Omega,b}f(x)\right|^{p-\varepsilon}w(x)dx$$

$$\leq\frac{C\varepsilon^\theta}{w(B)^\lambda}\left(\int_B|T_{\Omega,b}f_1(x)|^{p-\varepsilon}w(x)dx+\int_B|T_{\Omega,b}f_2(x)|^{p-\varepsilon}w(x)dx\right)$$

$$=:K+KK.$$

It follows from the $L^{p-\varepsilon}(w)$ boundedness of $T_{\Omega,b}$ and $w\in A_{(p-\varepsilon)/r'}$ that

$$K\leq C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)}\tag{26}$$

Then a further use of (3) derives that

$$\left|T_{\Omega,b}f_2(x)\right|^{p-\varepsilon}\leq C\left(\int_{\mathbb{R}^n}\frac{|\Omega(x-y)f_2(y)||b(x)-b(y)|}{|x-y|^n}dy\right)^{p-\varepsilon}$$

$$\leq C\left(\int_{|x_0-y|>2}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^n}\{|b(x)-b_{B,w}|+|b_{B,w}-b(y)|\}dy\right)^{p-\varepsilon},$$

Then, we have

$$KK\leq\frac{C\varepsilon^\theta}{w(B)^\lambda}\left(\int_{|x_0-y|>2}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^n}dy\right)^{p-\varepsilon}\int_B|b(x)-b_{B,w}|^{p-\varepsilon}w(x)dx$$

$$+\frac{C\varepsilon^{\theta}}{w(B)^{\lambda}}\left(\int_{|x_0-y|>2}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^{n}}|b(y)-b_{B,w}|dy\right)^{p-\varepsilon}w(B)$$

$$:=KK_1+KK_2$$

Lemma 2.15 allows us to have

$$KK_2\le C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)}.$$

We proceed to estimate the term KK_1. Without loss of generality, we only need to consider the case $p-\varepsilon>r'$. Take into account (9), (11) and Lemma (28), we have

$$LL_1=\frac{C\varepsilon^{\theta}}{w(B)^{\lambda}}\left(\sum_{j=1}^{\infty}\int_{2^j<|x_0-y|<2^{j+1}}\frac{|\Omega(x_0-y)f(y)|}{|x_0-y|^{n}}dy\right)^{p-\varepsilon}\int_B|b(x)-b_{B,w}|^{p-\varepsilon}w(x)dx$$

$$\le\frac{C\varepsilon^{\theta}}{w(B)^{\lambda}}\left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{A_j}|\Omega(x_0-y)|^{r}dy\right)^{(p-\varepsilon)/r}$$

$$\times\left(\int_{2^{j+1}B}|f(y)|^{r'}dy\right)^{(p-\varepsilon)/r'}\int_B|b(x)-b_{B,w}|^{p-\varepsilon}w(x)dx$$

$$\le C\frac{C\varepsilon^{\theta}}{w(B)^{\lambda}}\Big[\sum_{j=1}^{\infty}\frac{2^{nj/r}}{|2^jB|}\left(\int_{2^{j+1}B}|f(y)|^{p-\varepsilon}w(y)dy\right)^{1/(p-\varepsilon)}$$

$$\times\left(\int_{2^{j+1}B}w(y)^{-r'/(p-\varepsilon-r')}dy\right)^{(p-\varepsilon-r')/(p-\varepsilon)r'}\Big]^{p-\varepsilon}\int_B|b(x)-b_{B,w}|^{p-\varepsilon}w(x)dx$$

$$\le C\|f\|^{p}_{M_{p),\theta,\lambda}(w)}\left(\sum_{j=1}^{\infty}\frac{w(B)^{(1-\lambda)/(p-\varepsilon)}}{w(2^{j+1}B)^{(1-\lambda)/(p-\varepsilon)}}\right)^{p-\varepsilon}w(B)^{\lambda}\le C\|f\|^{p}_{M_{p),\theta,\lambda}(w)}w(B)^{\lambda}.$$

Hence

$$KK\le C\|f\|^{p-\varepsilon}_{M_{p),\theta,\lambda}(w)}.\qquad(27)$$

Combining (26) with (27), we obtain the desired conclusion. □

Proof of Theorem 3.3. The proof of Theorem 2.11 is similar as that of Theorem 2.9, except using $w\in A_{(p,q)}$. □

3. Conclusion

This paper gives the boundedness of some sublinear operators with rough kernels on the generalized weighted grand Morrey spaces. An application of the boundedness of the sublinear operators with rough kernels to the corresponding commutators are also considered. We have some criterions to deduce the boundedness of the sublinear operators on certain spaces. Our theorems provide natural and intrinsic boundedness of operators on generalized weighted grand Morrey spaces and our viewpoints will shed some new lights on boundedness of other operators and their commuattores on generalized weighted grand Morrey space. Besides Euclidean space, boundedness of some sublinear operators with rough kernels on other spaces can similarly be considered, such as on homogeneous groups.

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