



A Study of Congruence on (n, m) -semigroup

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Abstract: Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures. The purpose of this paper is to study the quotient structure of (n, m) -semigroup by using the notion of congruence in (n, m) -semigroup. Firstly, the concept of homomorphism on (n, m) -semigroup is introduced. Then, the concept of congruence on (n, m) -semigroup is defined, and some basic properties are studied. Finally, it is proved that the set of congruences on an (n, m) -semigroup is a complete lattice. All these generalize the corresponding notions and results for usual binary and ternary semigroups.

Keywords: (n, m) -semigroup, Homomorphism, Congruence

1. Introduction

An (n, m) -semigroup $(S, [])$ is a nonempty set S with an associative mapping $[]: S^n \rightarrow S^m$. Thus, an $(n, 1)$ -semigroup is an n -ary semigroup, and a $(3, 1)$ -semigroup is a ternary semigroup, and a $(2, 1)$ -semigroup is a usual binary semigroup. In other words, the notion of an (n, m) -semigroup is a generalization of that of an n -ary semigroup, and the latter is a generalization of that of a semigroup.

An (n, m) -semigroup is also called a vector valued semigroup, which first studied in [4] by \acute{G} . Čupona. There are also several papers study the (n, m) -semigroup that can be found in [2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 17, 19, 20, 21, 22, 23].

Green's relations and congruences are two important tools for studying semigroup algebraic structure. In recent years, (n, m) -semigroups as a generalization of binary and ternary semigroups had been extensively studied. Many scholars have studied the Green's relations and the generalized Green's relations on (n, m) -semigroups. Several papers present the research results in this field that can be found in [2, 14, 19, 20, 22, 23]. As is known to all, congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures. Thus, it is meaningful to study the quotient structure of (n, m) -semigroup by using the notion of congruence in (n, m) -semigroup.

The aim of this paper is to generalize some definitions and

results of congruences on usual binary and ternary semigroups to (n, m) -semigroup. Firstly, the concept of congruence on an (n, m) -semigroup as congruence of the corresponding component algebra is introduced. Then some basic properties of congruences on (n, m) -semigroup are studied. Finally, it is proved that the set of congruences on an (n, m) -semigroup is a complete lattice.

2. Preliminaries

This section introduces some notions and notations, and gives some relevant basic results. All of these information concerning (n, m) -semigroup can be found in [3, 6, 21].

Throughout this paper, it is convenient to use the following simplified notation: the sequence

$x_i x_{i+1} \dots x_j$ is denoted by x_i^j . For $j < i$, x_i^j is the empty symbol. If $x_1 = x_2 = \dots x_j = a$, then x_1^j will be denoted by a^j . Under this convention the sequence

$$x_1 \dots x_i y_{i+1} \dots y_j z_{j+1} \dots z_n \quad (1)$$

will be written as $x_1^i y_{i+1}^j z_{j+1}^n$. Let $\underline{x} = x_1 \dots x_i$, $\underline{y} = y_{i+1} \dots y_j$ and $\underline{z} = z_{j+1} \dots z_n$. Then the sequence (1) can be also written as $\underline{x}\underline{y}\underline{z}$.

Let S be a nonempty set, n, m two positive integers and

$$[] : (x_1, x_2, \dots, x_n) \mapsto [x_1 x_2 \dots x_n]$$

a mapping from the n th Cartesian power S^n of S into S^m . Throughout the paper, it will be assumed that $\Delta = n - m > 0$. Then $(S, [])$ is an (n, m) -groupoid, and $[]$ is an (n, m) -operation on S . If for every $i, j \in \{1, 2, \dots, \Delta + 1\}$ and all $x_1^{n+\Delta} \in S^{n+\Delta}$, the following associative law holds:

$$[x_1^{i-1}[x_i^{i+n-1}]x_{i+n}^{n+\Delta}] = [x_1^{j-1}[x_j^{j+n-1}]x_{j+n}^{n+\Delta}],$$

then $(S, [])$ is called an (n, m) -semigroup, and $[]$ is an associative (n, m) -operation.

From [6, 21], there has such a conclusion, that is ever (n, m) -semigroup $(S, [])$ induces an $(m + k\Delta, m)$ -semigroup for each $k > 1$, which is still denoted by $(S, [])$. Now record it as the following lemma:

Lemma 2.1. ([21]). Let $(S, [])$ be an (n, m) -semigroup with $\Delta = n - m > 0$, and let $a_1^k \in S^k$ for some integer $k > 0$. Then $[a_1^k]$ is meaningful if and only if $k > m$ and $k \equiv m \pmod{\Delta}$.

Each (n, m) -groupoid can associate an algebra through with m n -ary operations defined by $[a_1^n]_i = b_i$ if and only if $[a_1^n] = b_1^m$, where $a_\nu, b_\lambda \in S$, $\nu \in \{1, 2, \dots, n\}$, and $\lambda, i \in \{1, 2, \dots, m\}$. Then $(S; []_1, \dots, []_m)$ is called a component algebra of S .

Definition 2.2. Let $(S, [])$ and $(T, [])$ be two (n, m) -semigroups and $f: S \rightarrow T$ be a mapping. Then the mapping f is called a homomorphism of S into T if

$$f([a_1^n]_j) = [f(a_1)f(a_2) \dots f(a_n)]_j$$

for $a_i \in S$ where $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$, i.e. if f is a homomorphism between their corresponding component algebras.

Moreover, a homomorphism $f: S \rightarrow T$ is called a monomorphism if it is one-one.

A homomorphism $f: S \rightarrow T$ is called an epimorphism if it is onto.

A homomorphism $f: S \rightarrow T$ is called an isomorphism if is both one-one and onto and in this case it says that the (n, m) -semigroup $(S, [])$ and $(T, [])$ are isomorphic and write $S \cong T$.

3. Congruences on (n, m) -semigroups

The aim of this section is to study congruence on (n, m) -semigroup. Next, record a definition from [7] as follow:

Definition 3.1. ([3]). A equivalence relation ρ on an (n, m) -semigroup $(S, [])$ is said to be a congruence if for $i \in \{1, 2, \dots, n\}$, have

$$a_i \rho b_i \Rightarrow [a_1^n]_j \rho [b_1^n]_j, j \in \{1, 2, \dots, m\},$$

i.e. if ρ is a congruence on the corresponding component algebra of S .

Definition 3.2. If ρ is a congruence on an (n, m) -semigroup $(S, [])$, then define an associative (n, m) -operation $[]$ on the quotient set S/ρ as follows:

$$[]: (S/\rho)^n \rightarrow (S/\rho)^m, ((a_1\rho), (a_2\rho), \dots, (a_n\rho),) \mapsto [a_1^n]_1\rho, [a_1^n]_2\rho \dots [a_1^n]_m\rho.$$

Clearly, this operation is well-defined because ρ is a congruence: for $i \in \{1, 2, \dots, n\}$ and $a_i, b_i \in S$, have $a_i \rho b_i \Rightarrow [a_1^n]_j \rho [b_1^n]_j, j \in \{1, 2, \dots, m\}$.

This (n, m) -operation $[]$ is also associative. Hence with this (n, m) -operation $[]$, $(S/\rho, [])$ forms an (n, m) -semigroup.

Let ρ be a congruence on an (n, m) -semigroup $(S, [])$. Then the mapping f^\natural from S onto S/ρ given by $f^\natural(x) = x\rho$ is a homomorphism which is called the natural homomorphism.

Theorem 3.3. Let $(S, [])$ and $(T, [])$ be two (n, m) -semigroups and $\phi: S \rightarrow T$ be a homomorphism. Then $\text{Ker}\phi = \phi \circ \phi^{-1} = \{(a, b) \in S \times S \mid \phi(a) = \phi(b)\}$ is a congruence on $(S, [])$ and there is a homomorphism $f: S/\text{Ker}\phi \rightarrow T$ such that $\text{Im}f = \text{Im}\phi$ and the diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & T \\ \downarrow (Ker\phi)^\natural & \nearrow f & \\ S/Ker\phi & & \end{array}$$

is commutative.

Proof. Clearly, $\text{Ker}\phi$ is an equivalence relation. To show that $\text{Ker}\phi$ is a congruence, suppose that for $i \in \{1, 2, \dots, n\}$, $a_i, b_i \in S$ and $a_i(Ker\phi)b_i$. Then $\phi(a_i) = \phi(b_i)$ for $i \in \{1, 2, \dots, n\}$. Since ϕ is a homomorphism, for $j \in \{1, 2, \dots, m\}$, have

$$\begin{aligned} \phi([a_1^n]_j) &= [\phi(a_1)\phi(a_2) \dots \phi(a_n)]_j \\ &= [\phi(b_1)\phi(b_2) \dots \phi(b_n)]_j \\ &= \phi([b_1^n]_j), \end{aligned}$$

that is $[a_1^n]_j(Ker\phi)[b_1^n]_j$. Hence $\text{Ker}\phi$ is a congruence on $(S, [])$.

Now define $f: S/\text{Ker}\phi \rightarrow T$ by $f(a\text{Ker}\phi) = \phi(a)$ for all $a \in S$. Then f is both well-defined and one-one, because

$$a\text{Ker}\phi = b\text{Ker}\phi \Leftrightarrow a(Ker\phi)b \Leftrightarrow \phi(a) = \phi(b).$$

Also f is a homomorphism, since for $j \in \{1, 2, \dots, m\}$, have

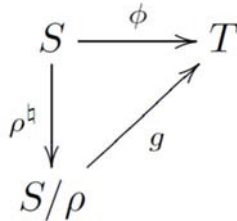
$$\begin{aligned} f([(a_1\text{Ker}\phi)(a_2\text{Ker}\phi) \dots (a_n\text{Ker}\phi)]_j) &= f([a_1^n]_j\text{Ker}\phi) \\ &= \phi([a_1^n]_j) \\ &= [\phi(a_1)\phi(a_2) \dots \phi(a_n)]_j \\ &= [f(a_1\text{Ker}\phi)f(a_2\text{Ker}\phi) \dots f(a_n\text{Ker}\phi)]_j. \end{aligned}$$

Clearly, $\text{Im}f = \text{Im}\phi$. Again, from the definition of f , it is clear that

$$[f \circ (Ker\phi)^\natural](a) = f[(Ker\phi)^\natural(a)] = f(a\text{Ker}\phi) = \phi(a), \text{ for all } a \in S.$$

Consequently, $f \circ (Ker\phi)^\natural = \phi$ and hence the above diagram is commutative. ■

Theorem 3.4. Let $(S, [])$ and $(T, [])$ be two (n, m) -semigroups. Let ρ be a congruence on $(S, [])$ and let $\phi : S \rightarrow T$ be a homomorphism such that $\rho \subseteq \text{Ker}\phi$. Then there is a unique homomorphism $g : S/\rho \rightarrow T$ such that $\text{Im}g = \text{Im}\phi$ and the following diagram



is commutative.

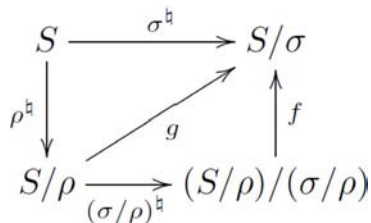
Proof. Now define $g : S/\rho \rightarrow T$ by $g(a\rho) = \phi(a)$ for all $a \in S$. Then g is well-defined, because for all $a, b \in S$, have $a\rho = b\rho \Rightarrow a\rho b \Rightarrow a(\text{Ker}\phi)b$ (since $\rho \subseteq \text{Ker}\phi$) $\Rightarrow \phi(a) = \phi(b)$.

It is easy to show that g is a homomorphism such that $\text{Im}g = \text{Im}\phi$. It can also easily to show that $g \circ \rho^\natural = \phi$ and hence the above diagram is commutative.

Now it is only to show that g is also unique. If possible, let $g' : S/\rho \rightarrow T$ by $g'(a\rho) = \phi(a)$ for all $a \in S$. Now $g'(a\rho) = \phi(a) = [g \circ \rho^\natural](a) = g(\rho^\natural(a)) = g(a\rho)$ for all $a\rho \in S/\rho$, which implies that $g' = g$. ■

Theorem 3.5. Let ρ and σ be two congruences on an (n, m) -semigroup $(S, [])$ such that $\rho \subseteq \sigma$. Then $\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}$ is a congruence on S/ρ and $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

Proof. Now consider the following figure:



Clearly, σ/ρ is an equivalence relation on S/ρ . Let $(a_i\rho)(\sigma/\rho)(b_i\rho)$, for $i \in \{1, 2, \dots, n\}$.

Then $a_i\sigma b_i$, $i \in \{1, 2, \dots, n\}$. Since σ is a congruence on $(S, [])$, have $[a_1^n]_j\sigma[b_1^n]_j$, $j \in \{1, 2, \dots, m\}$. This implies that $([a_1^n]_j\rho)(\sigma/\rho)([b_1^n]_j\rho)$, $j \in \{1, 2, \dots, m\}$ and hence σ/ρ is a congruence on S/ρ .

Note that σ/ρ is the kernel of g . From Theorem 3.3, it follows that there is an isomorphism $f : (S/\rho)/(\sigma/\rho) \rightarrow S/\sigma$ defined by $f((a\rho)(\sigma/\rho)) = a\sigma$ for all $a \in S$ and the above diagram is commutative. ■

Proposition 3.6. Let $(S, [])$ be an (n, m) -semigroup. If ρ_1 and ρ_2 are two congruences of S , then $\rho_1 \circ \rho_2$ is a congruence of S .

Proof. Let ρ_1 and ρ_2 be two congruences of S . Suppose that $a_i(\rho_1 \circ \rho_2)b_i$ hold for $i \in \{1, 2, \dots, n\}$. Then there exists $c_i \in S$ such that $a_i\rho_1 c_i$ and $c_i\rho_2 b_i$ hold for $i \in \{1, 2, \dots, n\}$. Since ρ_1 and ρ_2 are congruences of S , it follows that

$[a_1^n]_j\rho_1[c_1^n]_j$ and $[c_1^n]_j\rho_2[b_1^n]_j$ for $j \in \{1, 2, \dots, m\}$. This implies that $[a_1^n]_j(\rho_1 \circ \rho_2)[b_1^n]_j$ for $j \in \{1, 2, \dots, m\}$ and so $\rho_1 \circ \rho_2$ is a congruence of S . ■

From Proposition 3.6, it can be easily prove by induction the following result:

Corollary 3.7. Let $(S, [])$ be an (n, m) -semigroup. If $\rho_1, \rho_2, \dots, \rho_n$ are congruences of S , then $\rho_1 \circ \rho_2 \circ \dots \circ \rho_n$ is a congruence of S .

Proposition 3.8. The union of a non-empty family of congruences on an (n, m) -semigroup $(S, [])$ is a congruence of S .

Proposition 3.9. The intersection of a non-empty family of congruences on an (n, m) -semigroup $(S, [])$ is a congruence of S .

The set of congruences on an (n, m) -semigroup $(S, [])$ is denoted by $\mathcal{C}(S)$.

Theorem 3.10. Let $(S, [])$ be an (n, m) -semigroup. Then $(\mathcal{C}(S), \subseteq)$ is a complete lattice.

4. Conclusion

Green's relations and congruence are two important type of equivalence relations which used to study semigroup algebraic structures. In this paper, some definitions and results of congruences on usual binary and ternary semigroups are generalized to (n, m) -semigroup. The main results of this paper are the following:

Let $(S, [])$ and $(T, [])$ be two (n, m) -semigroups and $\phi : S \rightarrow T$ be a homomorphism. Then a congruence of $(S, [])$ can be structured by ϕ .

Let ρ and σ be two congruences on an (n, m) -semigroup $(S, [])$ such that $\rho \subseteq \sigma$. Then a congruence of S/ρ can be structured.

The set of congruences on an (n, m) -semigroup is a complete lattice.

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