Pure and Applied Mathematics Journal
2016; 5(6): 181-185
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20160506.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Bayesian Test for Lifetime Performance Index of Ailamujia Distribution Under Squared Error Loss Function

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To cite this article:
Lanping Li. Bayesian Test for Lifetime Performance Index of Ailamujia Distribution Under Squared Error Loss Function. *Pure and Applied Mathematics Journal.* Vol. 5, No. 6, 2016, pp. 181-185. doi: 10.11648/j.pamj.20160506.12

Received: October 5, 2016; **Accepted**: October 14, 2016; **Published**: November 7, 2016

Abstract: The aim of this paper is to estimate the lifetime performance index of Ailamujia distribution. A Bayesian test procedure is developed under squared error loss function. Firstly, Bayes estimation of life performance index is derived, then a Bayesian test procedure for lifetime performance index using posteriori probability ratio test method. Finally, an example is used to illustrate the effectiveness and feasibility of the method.

Keywords: Bayes Estimation, Lifetime Performance Index, Bayes Test, Ailamujia Distribution, Squared Error Loss Function

1. Introduction

For manufacture industry, process capability index is an effective and convenient tool for quality assessment. Many process capability indices have been put forward. For example, Juran [1] introduced the first process capability index C_P, Kane [2] proposed the process capability index C_{pk}, which are the two most commonly used indices, Boyles [3] and Pearn et al. [4] introduce two more-advanced indices Cpm and Cpmk respectively. Those four process capability indices have been defined explicitly as:

(i) $C_P = \dfrac{USL - LSL}{6\sigma}$,

where USL and LSL are the upper and the lower specification limits and σ is the process standard deviation.

(ii) $C_{pk} = \dfrac{d - |\mu - M|}{3\sigma} = \dfrac{\min\{USL - \mu, \mu - LSL\}}{3\sigma}$,

where μ is the process mean.

(iii) $C_{pm} = \dfrac{d}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \dfrac{d}{6\sqrt{E[(X-T)^2]}}$,

where T is the target value.

(iv) $C_{pmk} = \min\{\dfrac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \dfrac{\mu - LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\}$

The statistical inferences of process capability indices have drawn great attention by many authors. For example, Shiau et al. [5] applied Bayesian method to the estimation of Cpm and Cpk under the restriction that the process mean equals to the midpoint of the two specification limits. Pearn and Wu [6] studied the test of Cpk for general situation without restriction on the process mean based on Bayesian approach. Chen and Hsu [7] proposed a test for Cpk that is asymptotically equivalent to the likelihood ratio test. Baral and Anis [8] developed method of generalized confidence interval to measure the process capability index Cpm in presence of measurement errors. Macintyre[9] using Bayes approach to estimate the process capability indices and system availability for the inverse Rayleigh lifetime distribution.

To assess the products with the larger-the-better type of the quality characteristics, Montgomery [10] proposed a special unilateral specification process capability index, named as lifetime performance index C_L,

$$C_L = \frac{\mu - L}{\sigma} \tag{1}$$

where L is the lower bound of the specifications.

The statistical analysis of lifetime performance index for products whose lifetime distributed various distributions have been widely studied. For example, Wu et al. [11] discussed the maximum likelihood estimation, minimum variance unbiased estimation lifetime performance index of Rayleigh distribution product under progressively type II censored test. Lee et al. [12] derived maximum likelihood estimator and constructed a hypothesis testing procedure for lifetime performance index based on censored samples, which products' lifetime comes from the normal distribution but sample data modeled by fuzzy numbers. Lee et al. [13] studied the Bayesian estimation and Bayesian testing procedures for lifetime performance index under squared error loss function in which products' lifetime distributed with Rayleigh distribution. Liu and Ren [14] studied the Bayesian estimation and Bayesian test of lifetime performance index for exponential product under progressively type II censored samples.

The statistical inferences about various lifetime distributions, such as exponential distribution, Weibull distribution, normal distribution and Pareto distribution, etc. have been studied a lot [15-20]. Recent years, many new distributions are proposed for various engineer application. Ailamujia (Эрланга) distribution is one of these distribution proposed by Lv et al. [21]. Pan et al. [22] studied the interval estimation and hypothesis test of Ailamujia distribution based on small sample. Long [23] studied the Bayes estimation of Эрланга distribution under type-II censored samples based on three different prior distributions. Li [24] discussed the minimax estimation of the parameter ofЭрланга distribution under a non-informative prior distribution with three different loss functions.

Assume that X is the products's lifetime, and it distributed the Ailamujia distribution whose probability density function (pdf) and cumulative distribution function (cdf) respectively:

$$f(x;\theta)=4x\theta^2e^{-2\theta x},\quad x\geq 0,\theta>0 \tag{2}$$

$$F(x;\theta)=1-(1+2\theta x)e^{-2\theta x},\quad x\geq 0,\theta>0 \tag{3}$$

Here, θ is the unknown parameter.

In this paper, we study the estimation of lifetime performance index C_L using Bayesian approach, and the Bayes test procedure of C_L will also be constructed. Section 2 introduces some properties of the lifetime of product with the Ailamujia distribution. Moreover, the relationship between C_L and conforming rate is also discussed. Furthermore, the Bayesian estimator of C_L based on the conjugate Gamma prior distribution is also obtained under squared error loss function. A new Bayesian hypothesis testing procedure is developed in Section 3, and a practical example is given in Section 4. Finally, a conclusion is given in Section 5.

2. Lifetime Performance Index

Let X be the lifetime of such a product whose lifetime distribution is Ailamujia distribution with pdf (2).

It is easily to verify that the process mean $\mu=EX=1/\theta$ and the process standard deviation $\sigma=\sqrt{Var(X)}=1/\sqrt{2}\theta$.

Then the parameter θ^{-1} is often called the mean time. Then the lifetime performance index C_L of Ailamujia distribution can be rewritten as follows

$$C_L=\frac{\mu-L}{\sigma}=\frac{1/\theta-L}{1/\sqrt{2}\theta}=\sqrt{2}(1-\theta L) \tag{4}$$

The failure rate function $r(x)$ is defined by

$$r(x;\theta)=\frac{f(x\,|\,\theta)}{1-F(x\,|\,\theta)}=\frac{4x\theta^2}{1+2x\theta} \tag{5}$$

The derivative of the failure rate function $r(x)$ with respect to θ is

$$\frac{dr(x;\theta)}{d\theta}=\frac{8x\theta(1+x\theta)}{(1+2x\theta)^2} \tag{6}$$

From Eq. (6), we see that $\frac{dr(x;\theta)}{d\theta}$ is always strictly bigger than zero. Then the failure rate function $r(x)$ is strictly increasing function with respect to the parameter θ. Thus we can see that the smaller θ, i.e. the larger the mean $1/\theta$, the smaller the failure rate and larger the lifetime performance index C_L. Therefore, the lifetime performance index C_L is a reasonably and accurately index for assess the performance of products.

Moreover, conforming rate P_r of the product can be defined as a probability of the lifetime of a product X exceeding the lower specification limit L, that is

$$\begin{aligned}P_r&=P(X\geq L)=\int_L^\infty 4x\theta^2e^{-2\theta x}dx\\&=(1+2\theta L)e^{-2\theta L}\\&=(3-\sqrt{2}C_L)e^{\sqrt{2}C_L-2},-\infty<C_L<\sqrt{2}\end{aligned} \tag{7}$$

It is easily to verify that the conforming rate P_r and lifetime performance index C_L have a strictly increasing relationship. The relationships of P_r and C_L can be calculated using Matlab software for a list of various values. For the C_L values which are not listed in Table 1, the conforming rate P_r can be obtained through interpolation. Tong et al. [25] pointed out that the conforming rate P_r can be calculated by dividing the number of conforming products by total number of products sampled, while Montgomery in 1985 suggested that the sample size must be large to accurately estimation. However, a large sample size is usually not practical considering with the perspective of cost. Since there exist a one-to-one mathematical relationship between the conforming rate P_r and the lifetime performance index C_L. Then lifetime performance index can be a flexible and effective tool for estimating the conforming rate P_r.

3. Estimation

This section will discuss the maximul likelihood estimation and Bayes estimation of lifetime performance index C_L of Ailamujia distribution

3.1. Maximum Likelihood Estimation

Let $X_1, X_2, \cdots, X_n$ represent the lifetime of sample from the Ailamujia distribution with pdf (2), $x = (x_1, x_2, \cdots, x_n)$ is the observation of $X = (X_1, X_2, \cdots, X_n)$ and $t = 2\sum_{i=1}^{n} x_i$ is the observation of $T = 2\sum_{i=1}^{n} X_i$. Then the likelihood function corresponding to pdf (2) is given by

$$\begin{aligned} l(\theta; x) &= \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} 4\theta^2 x_i e^{-2\theta x_i} \\ &= \theta^{2n} \cdot (\prod_{i=1}^{n} 4x_i) \cdot e^{-2\theta\sum_{i=1}^{n} x_i} \\ &\propto \theta^{2n} \cdot e^{-\theta t} \end{aligned} \tag{8}$$

Then the log- likelihood function can be obtained as follows:

$$Ln[l(\theta; x)] = 2n\ln\theta + \sum_{i=1}^{n}\ln(4x_i) - 2\theta\sum_{i=1}^{n} x_i$$

The maximum likelihood estimator of θ can be easily derived from the log-likelihood equation

$$\frac{dLn[l(\theta; x)]}{d\theta} = 0 .$$

The maximum likelihood estimator of θ is

$$\hat{\theta}_{MLE} = \frac{2n}{T} \tag{9}$$

Where $T = 2\sum_{i=1}^{n} X_i$.

Then by invariance of maximum likelihood estimation, we can get the maximum likelihood estimator of lifetime performance index C_L as follows:

$$\hat{C}_{MLE} = \sqrt{2}(1 - \hat{\theta}_{MLE} L) = \sqrt{2}(1 - \frac{2nL}{T}) \tag{10}$$

We can also easily show that T is a random variable distributed with the Gamma distribution $\Gamma(2n, \theta)$, which has the following probability density function:

$$f_T(t;\theta) = \frac{\theta^n}{\Gamma(2n)} t^{2n-1} e^{-\theta t}, \quad t > 0, \theta > 0 \tag{11}$$

3.2. Bayes Estimation

In this section, we shall discuss the Bayes estimation of the lifetime performance index of Ailamujia distribution with pdf (2) using Bayesian approach. Squared error loss function is one of the most important loss functions in Bayesian statistical analysis, and the formula of it is

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \tag{12}$$

Assume that the conjugate prior distribution of θ is the Gamma prior distribution $\Gamma(\alpha, \beta)$, with pdf

$$\pi(\theta; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \alpha, \beta > 0, \theta > 0 \tag{13}$$

Where $\alpha, \beta > 0$ are two prior hyper parameters.

Combining the likelihood function (8) with the prior probability density function in Eq. (13), the posterior pdf of θ can be derived using Bayes Theorem as follows

$$\begin{aligned} h(\theta \mid x) &\propto l(\theta; x) \cdot \pi(\theta) \\ &\propto \theta^{2n} e^{-\theta t} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \\ &\propto \theta^{2n+\alpha-1} e^{-(\beta+t)\theta} \end{aligned} \tag{14}$$

That is

$$\theta \mid X \sim \Gamma(2n + \alpha, \beta + T) . \tag{15}$$

Then under the squared error loss function (12), Bayes estimator of θ is the posterior mean, i.e.

$$\hat{\theta}_B = E[\theta \mid x] = \frac{2n + \alpha}{\beta + T} \tag{16}$$

Further the Bayes estimator of C_L is

$$\begin{aligned} \hat{C}_{BL} &= E(C_L \mid X) = E(\sqrt{2}(1 - \theta L) \mid X) \\ &= \sqrt{2}[1 - L \cdot E(\theta \mid X)] \\ &= \sqrt{2}[1 - L \cdot \frac{2n + \alpha}{\beta + T}] \end{aligned} \tag{17}$$

4. Bayes Test of Lifetime Performance Index

This section will construct a Bayesian testing procedure to assess whether the lifetime performance index adheres to the required level. Assume that the required value of lifetime performance is larger than the target value. c . First, we establish the following hypothesis:

$$H_0 : C_L \le c \leftrightarrow H_1 : C_L > c . \tag{18}$$

Then the new proposed Bayesian testing procedure of C_L is as follows:

Step 1. Determine the lower lifetime limit L and sample size n .

Step 2. Calculate the Bayesian estimator

$$\hat{C}_{BL} = \sqrt{2}[1 - L \cdot \frac{2n+\alpha}{\beta+T}], \quad (19)$$

Where $T = 2\sum_{i=1}^{n} X_i$.

Step 3. Calculate the posterior probability odds ratio

$$BF = \frac{P(H_0 | X)}{P(H_1 | X)} = \frac{P(H_0 | X)}{1 - P(H_0 | X)}, \quad (20)$$

where $P(H_0 | X) = \int_{\frac{1-c}{L}}^{\infty} \pi(\theta | X) d\theta$, $\pi(\theta | x) = \frac{\beta^{2n+\alpha}}{\Gamma(2n+\alpha)} \theta^{2n+\alpha-1} e^{-(\beta+t)\theta}$ and $t = 2\sum_{i=1}^{n} x_i$ is the observation of $T = 2\sum_{i=1}^{n} X_i$.

Step 4. The decision rules are provided as follows:

If $\hat{C}_{BL} > c$ and $BF < 1$, we reject to the null hypothesis H_0, then it is concluded that the lifetime performance index or conforming rate of the products meets the required level;

If $\hat{C}_{BL} < c$ and $BF > 1$, we accept the null hypothesis H_0, then it is concluded that the lifetime performance index or conforming rate of the products does not meet the required level.

5. Numerical Example

To illustrate the practicability and feasibility of the proposed testing method, a Monte Carlo simulation is used to generate a sample of Ailamujia distribution with $\theta = 0.1$ and n=20. The data set is:

5.4833, 18.6857, 3.2724, 7.2842, 3.9008, 19.4737, 7.0814, 4.2670, 6.0806, 22.5402, 7.6014, 14.8026, 14.3133, 1.9649, 7.7563, 4.6101, 4.1231, 9.7974, 8.4994, 3.6257

Now we give the steps of the proposed Bayesian testing procedure about C_L as follows:

Step1. Calculate $t = 2\sum_{i=1}^{n} x_i = 350.3267$ and here we assume the lower lifetime limit L=4.078. To deal with the lifetime performances, the conforming rate P_r is required to exceed 0.80. According to Eq. (7), the value of C_L is required to exceed 0.8375. Thus, the target value of performance index is set at c=0.8375, then we establish the following testing hypothesis

$H_0 : C_L \le 0.8375 \leftrightarrow H_1 : C_L > 0.8375$.

Step 2. Under squared loss function, we get the Bayesian estimate $\hat{C}_{BL} = 0.7412$;

Step 3. Suppose that the prior parameter values $\alpha = 1.0$ and $\beta = 1.0$, then

$$P(H_0 | X) = \int_{\frac{1-c}{L}}^{\infty} \pi(\theta | X) d\theta \approx 1 .$$

Therefore, the posterior odds ratio

$$BF = \frac{P(H_0 | X)}{1 - P(H_0 | X)} > 1 .$$

Step 4. Obviously, $\hat{C}_{BL} = 0.7412 < c = 0.8375$ and $BF > 1$, then we can accept the null hypothesis $H_0 : C_L \le c$. That is, we conclude that the lifetime performance index does not meet the required level.

6. Conclusions

Process capability indices are well effective tools to assess the performance and potentiality of their process and widely employed by manufactures. Lifetime performance index is a the-larger-the-better index which is especially useful for non-normal distributions. This paper studied the Bayesian estimation and Bayesian test of life performance index under squared error loss function. The new proposed Bayesian testing method is easier than other classical approaches and the test process is easy to operate by using ordinary programming software as Matlab, Excel, etc. This testing method can be similar used to other life distributions. The testing procedure can provide reference for the enterprise engineers to assess whether the true lifetime performance of products meets the requirements.

Acknowledgement

This study is partially supported by Natural Science Foundation of Hunan Province (No. 2015JJ3030 and No. 2016JJ4012) and Foundation of Hunan Educational Committee (No.15C0228). The author also gratefully acknowledges the helpful comments and suggestions of the reviewers, which have improved the presentation.

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