

Exact and Explicit Solutions of Whitham-Broer-Kaup Equations in Shallow Water

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Abstract: In this paper, a simple direct method is presented to find equivalence transformation of a nonlinear Whitham-Broer-Kaup equations. Applying this equivalence transformation, we can obtain the symmetry group theorem of the Whitham-Broer-Kaup equations and then derive series of new exact and explicit solutions of the Whitham-Broer-Kaup equations according to solutions of the previous references.

Keywords: Whitham-Broer-Kaup Equations, Direct Method, Equivalence Transformation, Symmetry, Explicit Solutions

1. Introduction

Since the discovery of the soliton in 1965 by Zabusky and Kruskal, a large class of nonlinear evolution equations (NLEEs) have been derived and widely applied in various branches of physics and applied mathematics like condensed matter, nonlinear optics, fluid mechanics, plasma physics, theory of turbulence, ocean dynamics, biophysics and star formation. On the other hand, to better understand the nonlinear mechanisms in different physical contexts, many authors have diligently applied themselves to finding the exact analytical solutions for these NLEEs including the soliton solutions, periodic solutions and rational solutions.

In the present paper, we would like to consider the coupled Whitham-Broer-Kaup (WBK) equations which have been studied by Whitham [16], Broer [2] and Kaup [10]. The WBK equations are as follows,

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + (uv)_x - \beta v_{xx} + \alpha u_{xxx} = 0 \end{cases} \quad (1)$$

The WBK equations describe the propagation of shallow water waves, with different dispersion relations. Where $u = u(x, t)$ is the horizontal velocity, $v = v(x, t)$ is the height that deviates from equilibrium position of the liquid, and α, β are constants which are represented in different diffusion powers. WBK equations are very good models to

describe dispersive wave. For the background materials of model equation, we refer to the paper [2, 10, 16].

If $\alpha = 0$ and $\beta \neq 0$, Eq.(1) is classic long wave equations that describe shallow water wave with diffusion [17]. If $\alpha = 1$ and $\beta = 0$, Eq.(1) is modified Boussinesq equations [8]. Kaup [10] and Ablowitz [1] studied inverse transformation solution for the special case of Eq.(1), Kupershmidt discussed their symmetries and conservation laws. By using of Backlund transformation, Fan [8] found three pairs of solutions of WBK equation. Xie et al. [17] made use of hyperbolic function method and Wu elimination method obtained four pairs of solutions of WBK equation. Chen et al. [3] present a more general transform ation and applied it to WBK equation and obtained some traveling wave solutions. Xu et al. [20] obtained abundant solitary- wave solutions and reveal some novel solitary-wave structures under certain parametric conditions for the WBK equation by tanh-function method. Xu [18] presented an elliptic equation method for constructing new types of elliptic function solutions for the WBK equation. Sirendaoreji [15] suggested a new auxiliary ordinary differential equation method and constructed exact travelling wave solutions of the WBK equation in a unified way and so on.

However, there are still many researchers are discussing the exact and explicit solutions for this WBK equation [4, 5, 7, 9, 14, 19, 21] by many new methods, and all kinds of solutions, such as traveling wave solutions, tanh-function solutions, elliptic function solutions, Jacobi elliptic function

solutions, rational function solutions, exponential function solutions of the WBK equation are obtained in this references.

Naturally, an interesting questions is: Can we find a simple method to obtain a transformation between the present solutions and the new ones? If we find this transformation, we can obtain more and more all kinds of new solutions by the present references. As we know, one of the way to realize this aim is backland method of transformation and method of Lie-point symmetry group [13].

On the other hand, Clarkson and Kruskal [6] introduced a simple direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find all the possible similarity reductions. Recently, Lou and Ma [11, 12] generalized a new simple direct method basing on CK’s method, which is much simpler than the traditional Lie-point symmetry group.

Thus, in this paper, motivated by these ideas, the authors would like to obtain some new exact and explicit solutions by the simple direct method. And the organization of the paper is as follow. In the next section, we obtain the equivalence transformation of the WBK equation by the direct method. In section 3, we obtain series of new solutions of the WBK equation basing on the present references.

2. Symmetry Group Theorem of the WBK Equations

Following the idea of the direct method, we first suppose the WBK equations has the solutions in the following form

$$\begin{cases} u(x,t) = m(x,t) + n(x,t) \cdot U(\xi, \tau), \\ v(x,t) = p(x,t) + q(x,t) \cdot V(\xi, \tau) \end{cases} \tag{2}$$

where $\xi = \xi(x,t)$, $\tau = \tau(x,t)$ and $m(x,t), n(x,t), p(x,t), q(x,t)$ are functions to be determined later, and $U(\xi, \tau), V(\xi, \tau)$ is restricted to satisfy Eq.(1) as $u = u(x,t), v = v(x,t)$ under the transformation:

$$\{u, v, x, t\} \rightarrow \{U, V, \xi, \tau\},$$

that is,

$$\begin{cases} U_\tau = -UU_\xi - V_\xi - \beta U_{\xi\xi}, \\ V_\tau = -UV_\xi - VU_\xi + \beta V_{\xi\xi} - \alpha U_{\xi\xi\xi}. \end{cases} \tag{3}$$

Substituting Eq.(2) into Eq.(1) and using Eq.(3) one can get:

$$\beta^3 n(x,t) \tau_x^2 \cdot U_{\xi\xi\xi\xi} + \beta q(x,t) \tau_x \cdot V_{\xi\xi} + F(x,t, U, V, U_\xi, V_\xi, \dots) = 0, \tag{4}$$

$$-\alpha \beta^3 n(x,t) \tau_x^3 \cdot U_{\xi\xi\xi\xi\xi} - \beta^3 q(x,t) \tau_x^2 \cdot V_{\xi\xi\xi\xi} + G(x,t, U, V, U_\xi, V_\xi, \dots) = 0, \tag{5}$$

where the function $F(x,t, U, V, U_\xi, V_\xi, \dots)$ is independent of $U_{\xi\xi\xi\xi}$ and $V_{\xi\xi}$, while the function $G(x,t, U, V, U_\xi, V_\xi, \dots)$ is independent of $U_{\xi\xi\xi\xi\xi}$ and $V_{\xi\xi\xi\xi}$. So it is necessary to take the coefficient of $U_{\xi\xi\xi\xi}$ and $V_{\xi\xi}$ in Eq.(4), $U_{\xi\xi\xi\xi\xi}$ and $V_{\xi\xi\xi\xi}$ in Eq.(5) being zero. which leads to,

$$-\beta^3 n(x,t) \tau_x^2 = 0, \quad \beta q(x,t) \tau_x = 0, \quad \alpha \beta^3 n(x,t) \tau_x^3 = 0, \quad \beta^3 q(x,t) \tau_x^2 = 0. \tag{6}$$

Then we can get $\tau_x = 0$, that is, τ is dependent of the variable x , so

$$\tau = \tau(t) \tag{7}$$

At this time, if we substitute Eq.(7) into Eq.(4) and (5), then these two equations are reduced to the following

$$\beta n[\xi_x^2 - \tau'(t)] \cdot U_{\xi\xi} + n[n\xi_x - \tau'(t)]UU_\xi + nn_x U^2 + F_1(x,t, U, V, U_\xi, V_\xi) = 0 \tag{8}$$

$$\alpha[n(x,t)\xi_x^3 - q(x,t)\tau'(t)] \cdot U_{\xi\xi\xi} + \beta q(x,t)[\tau'(t) - \xi_x^2] \cdot V_{\xi\xi} + G_1(x,t, U, V, U_\xi, V_\xi, \dots) = 0 \tag{9}$$

where the function $F_1(x,t, U, V, U_\xi, V_\xi)$ is independent of $U_{\xi\xi}, UU_\xi$ and U^2 , while the function $G_1(x,t, U, V, U_\xi, V_\xi, \dots)$ is independent of $U_{\xi\xi\xi}$ and $V_{\xi\xi}$.

In the same way, it is necessary to take the coefficients of $U_{\xi\xi}, UU_\xi$ and U^2 in Eq.(8), $U_{\xi\xi\xi}$ and $V_{\xi\xi}$ in Eq.(9) being zero. That is,

$$\beta n[\xi_x^2 - \tau'(t)] = n[n\xi_x - \tau'(t)] = nn_x = 0, \tag{10}$$

$$\alpha[n(x,t)\xi_x^3 - q(x,t)\tau'(t)] = \beta q(x,t)[\tau'(t) - \xi_x^2] = 0, \tag{11}$$

Thus, it is easy to obtain

$$\tau'(t) = \xi_x^2, \quad n(x, t) = n(t) = \xi_x(x, t), \quad q(x, t) = q(t) = n^2(t), \tag{12}$$

Therefore, if we substitute Eq.(12) into Eq.(8) and (9), then these two equations can be reduced to the following

$$n[\xi_t + m\xi_x] \cdot U_\xi + [n'(t) + nm_x]U + [m_t + mm_x + p_x + \beta m_{xx}] = 0 \tag{13}$$

$$[n\xi_x - q\tau'(t)] \cdot UV_\xi + np\xi_x \cdot U_\xi + [\xi_t + m\xi_x]V_\xi + np_x \cdot U + [q'(t) + qm_x] \cdot V + [p_t + mp_x + pm_x - \beta p_{xx} + \alpha m_{xxx}] = 0 \tag{14}$$

Also, it is necessary to take all the coefficients of the Eq.(13) and Eq.(14), which leads to

$$\xi_t + m\xi_x = 0, \quad n'(t) + nm_x = 0, \quad m_t + mm_x + p_x + \beta m_{xx} = 0. \tag{15}$$

$$\begin{cases} n\xi_x - q\tau'(t) = 0, \\ np\xi_x = 0, \\ np_x = 0, \\ q'(t) + qm_x = 0, \\ p_t + mp_x + pm_x - \beta p_{xx} + \alpha m_{xxx} = 0 \end{cases} \tag{16}$$

From Eq.(12), Eq.(15) and Eq.(16) one can get

$$p(x, t) = 0, \quad q(x, t) = 1, \quad m(x, t) = c_1, \quad n(x, t) = \delta, \quad \xi(x, t) = \delta x - c_1 \delta t + c_2, \quad \tau(x, t) = t + c_3 \tag{17}$$

where c_1, c_2, c_3 are arbitrary constants and $\delta = \pm 1$.

Thus, we can easily get the following symmetry group theorem.

Theorem 1. If $U = U(x, t)$ and $V = V(x, t)$ is a given solution of the WBK equations, then

$$u(x, t) = c_1 + \delta U(\delta x - c_1 \delta t + c_2, t + c_3) \tag{18}$$

$$v(x, t) = V(\delta x - c_1 \delta t + c_2, t + c_3) \tag{19}$$

is also a solutions of the WBK equations.

According to the formula (18) and (19), we obtain the relationship between the new explicit solutions and the old ones of the WBK equations, which is the so-called

equivalence transformation of the Eq.(1). Also it is obvious to see that (18) and (19) is an auto-backlund transformation of the WBK equations (1).

3. Some New Exact and Explicit Solutions of the WBK Equations

Utilizing the conclusion of the Theorem 1, we can obtain following exact and explicit solutions of the WBK equations in this section.

Family 1. From reference [4] we know that WBK equations have the following solution:

$$U(\xi, \tau) = a_0 + \frac{a_1(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)))}{\mu_1(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)},$$

$$V(\xi, \tau) = A_0 + \frac{a_1(\pm \beta k \mu_2 - \beta k \mu_1 - \lambda - a_0)(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)))}{\mu_1(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)}$$

$$- \frac{a_1(\beta k \pm \beta k \mu_1 \mu_2 - \mu_1^2 \beta k + a_1)(\tanh(\zeta) \pm i \sec h(\zeta))^2}{2(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)^2}$$

$$- \frac{a_1 \mu_2(\pm \beta k \mu_1 \mu_2 - \mu_1^2 \beta k + a_1)(\tanh(\zeta) \pm i \sec h(\zeta))(\sec(\zeta) \pm \tan(\zeta))}{\mu_1(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)^2}$$

$$- \frac{a_1 \mu_2(\pm \beta k \mu_1 \mu_2 \pm \mu_1 \mu_2^2 \beta k - \mu_1^2 \mu_2 \beta k + a_1 \mu_2)(\sec(\zeta) \pm \tan(\zeta))^2}{2\mu_1^2(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)^2}.$$

where $\zeta = k(\xi + \lambda \tau)$, $a_0, A_0, a_1, \mu_1, \mu_2, k$ and λ are arbitrary constants.

Applying the equivalence transformation (18) and (19) we obtain following combining tanh, sech, sec and tan function solutions of the WBK equations:

$$u_1(x, t) = a_0 + \frac{a_1(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)))}{\mu_1(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)) + 1)}, \tag{20}$$

$$\begin{aligned} v_1(x, t) = & A_0 + \frac{a_1(\pm \beta k \mu_2 - \beta k \mu_1 - \lambda - a_0)(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)))}{\mu_1(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)) + 1)} \\ & - \frac{a_1(\beta k \pm \beta k \mu_1 \mu_2 - \mu_1^2 \beta k + a_1)(\tanh(\eta) \pm i \sec h(\eta))^2}{2(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)) + 1)^2} \\ & - \frac{a_1 \mu_2(\pm \beta k \mu_1 \mu_2 - \mu_1^2 \beta k + a_1)(\tanh(\eta) \pm i \sec h(\eta))(\sec(\eta) \pm \tan(\eta))}{\mu_1(\mu_1(\tanh(\eta) \pm i \sec h(\eta)) + \mu_2(\sec(\eta) \pm \tan(\eta)) + 1)^2} \\ & - \frac{a_1 \mu_2(\pm \beta k \mu_1 \mu_2 \pm \mu_1 \mu_2^2 \beta k - \mu_1^2 \mu_2 \beta k + a_1 \mu_2)(\sec(\zeta) \pm \tan(\zeta))^2}{2\mu_1^2(\mu_1(\tanh(\zeta) \pm i \sec h(\zeta)) + \mu_2(\sec(\zeta) \pm \tan(\zeta)) + 1)^2}, \end{aligned} \tag{21}$$

where $\eta = k(\delta x - c_1 \delta t + c_2) + k\lambda(t + c_3) = k\delta x + k(\lambda - c_1 \delta)t + k(c_2 + \lambda c_3)$, $a_0, A_0, a_1, \mu_1, \mu_2, k$ and λ are arbitrary constants, $\delta = \pm 1$.

Family 2. From reference [9] we know that WBK equations have the following solutions:

$$U(\xi, \tau) = a_0 + b_1 \sqrt{\frac{d_2}{d_0}} \csc h(\sqrt{d_2} \zeta) + k_1 \sqrt{d_2} \coth(\sqrt{d_2} \zeta),$$

$$\begin{aligned} V(\xi, \tau) = & A_0 + B_1 \sqrt{\frac{d_2}{d_0}} \csc h(\sqrt{d_2} \zeta) + \frac{B_2 d_2}{d_0} \csc h^2(\sqrt{d_2} \zeta) + \frac{P_1}{\sqrt{d_0}} \sinh(\sqrt{d_2} \zeta) \\ & + \frac{K_2 d_2}{\sqrt{d_0}} \coth(\sqrt{d_2} \zeta) \operatorname{csch}(\sqrt{d_2} \zeta) + \frac{P_2}{\sqrt{d_0}} \tanh(\sqrt{d_2} \zeta). \end{aligned}$$

where $\zeta = k(\xi + \lambda \tau)$, $\lambda, k_1, K_2, A_0, B_1, B_2, d_0, d_2$ and P_2 are mentioned in Eq.(3.5) in [9]. a_0, b_1 and P_1 are arbitrary constants.

So we obtain following hyperbolic function solutions of the WBK equations:

$$u_2(x, t) = a_0 + b_1 \sqrt{\frac{d_2}{d_0}} \csc h(\sqrt{d_2} \eta) + k_1 \sqrt{d_2} \coth(\sqrt{d_2} \eta), \tag{22}$$

$$\begin{aligned} v_2(x, t) = & A_0 + B_1 \sqrt{\frac{d_2}{d_0}} \csc h(\sqrt{d_2} \eta) + \frac{B_2 d_2}{d_0} \csc h^2(\sqrt{d_2} \eta) + \frac{P_1}{\sqrt{d_0}} \sinh(\sqrt{d_2} \eta) \\ & + \frac{K_2 d_2}{\sqrt{d_0}} \coth(\sqrt{d_2} \eta) \operatorname{csch}(\sqrt{d_2} \eta) + \frac{P_2}{\sqrt{d_0}} \tanh(\sqrt{d_2} \eta). \end{aligned} \tag{23}$$

where $\eta = k(\delta x - c_1 \delta t + c_2) + k\lambda(t + c_3) = k\delta x + k(\lambda - c_1 \delta)t + k(c_2 + \lambda c_3)$, $\lambda, k_1, K_2, A_0, B_1, B_2, d_0, d_2$ and P_2 are mentioned in Eq.(3.5) in [9], $\delta = \pm 1$.

Family 3. From reference [18] we know that WBK equations have the following solutions:

$$U(\xi, \tau) = c \pm k \sqrt{A(\alpha + \beta^2)} \left[\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} - \frac{\alpha_1 \alpha_2 \operatorname{sn}^2(B \zeta, m)}{\alpha_1 - \alpha_2 \operatorname{cn}^2(B \zeta, m)} \right],$$

$$\begin{aligned}
 V(\xi, \tau) = & \frac{k^2 A(\alpha + \beta^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3)}{4} \\
 & + \frac{k^2 A\alpha_1\alpha_2(\alpha + \beta^2)(\alpha_1 + \alpha_2 + \alpha_3)sn^2(B\zeta, m)}{\alpha_1 - \alpha_2cn^2(B\zeta, m)} - \frac{2k^2 A\alpha_1^2\alpha_2^2(\alpha + \beta^2)sn^4(B\zeta, m)}{(\alpha_1 - \alpha_2cn^2(B\zeta, m))^2} \\
 & \mp \frac{4k^2\beta^2\alpha_1\alpha_2(\alpha_1 - \alpha_2)\sqrt{A(\alpha + \beta^2)}sn(B\zeta, m)cn(B\zeta, m)dn(B\zeta, m)}{(\alpha_1 - \alpha_2cn^2(B\zeta, m))^2}.
 \end{aligned}$$

where $\zeta = k(\xi - c\tau + \xi_0)$, k, c are arbitrary constants and B, m refers to formula (4) in [18].

So we obtain following elliptic function solutions of the WBK equations:

$$u_3(x, t) = c \pm k\sqrt{A(\alpha + \beta^2)} \left[\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} - \frac{\alpha_1\alpha_2sn^2(B\eta, m)}{\alpha_1 - \alpha_2cn^2(B\eta, m)} \right], \tag{24}$$

$$\begin{aligned}
 v_4(x, t) = & \frac{k^2 A(\alpha + \beta^2)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3)}{4} \\
 & + \frac{k^2 A\alpha_1\alpha_2(\alpha + \beta^2)(\alpha_1 + \alpha_2 + \alpha_3)sn^2(B\eta, m)}{\alpha_1 - \alpha_2cn^2(B\eta, m)} - \frac{2k^2 A\alpha_1^2\alpha_2^2(\alpha + \beta^2)sn^4(B\eta, m)}{(\alpha_1 - \alpha_2cn^2(B\eta, m))^2} \\
 & \mp \frac{4k^2\beta^2\alpha_1\alpha_2(\alpha_1 - \alpha_2)\sqrt{A(\alpha + \beta^2)}sn(B\eta, m)cn(B\eta, m)dn(B\eta, m)}{(\alpha_1 - \alpha_2cn^2(B\eta, m))^2}.
 \end{aligned} \tag{25}$$

where $\eta = k(\delta x - c_1\delta t + c_2) - ck(t + c_3) + k\xi_0 = k\delta x - k(c + c_1\delta)t + k(c_2 - cc_3 + \xi_0)$, k, c are arbitrary constants and B, m refers to formula (4) in [18], $\delta = \pm 1$.

Family 4. From reference [18] we know that WBK equations have the following solutions:

$$\begin{aligned}
 U_1(\xi, \tau) &= \pm R\sqrt{\alpha + \beta^2} (\tan \zeta + \sec \zeta) - \lambda, \\
 V_1(\xi, \tau) &= \left[-R^2(\alpha + \beta^2) + R^2\beta\sqrt{\alpha + \beta^2} \right] (\tan^2 \zeta + m_2 \sec \zeta \tan \zeta + 1); \\
 U_2(\xi, \tau) &= \pm R\sqrt{\alpha + \beta^2} (-\cot \zeta + m_1 \csc \zeta) - \lambda, \\
 V_2(\xi, \tau) &= \left[R^2(\alpha + \beta^2) - R^2\beta\sqrt{\alpha + \beta^2} \right] (\cot^2 \zeta - m_2 \csc \zeta \cot \zeta + 1).
 \end{aligned}$$

where $\zeta = R(\xi + \lambda\tau + c) - \frac{1}{2} \ln \left| \frac{A}{B} \right|$, $m_1 = m_2 = \pm 1, A^2 + B^2 \neq 0$, A, B, R, λ, c are arbitrary constants.

Thus, we obtain following double periodic wave solutions of the WBK equations:

$$u_5(x, t) = \pm R\sqrt{\alpha + \beta^2} (\tan \eta + \sec \eta) - \lambda, \tag{26}$$

$$v_5(x, t) = \left[-R^2(\alpha + \beta^2) + R^2\beta\sqrt{\alpha + \beta^2} \right] (\tan^2 \eta + m_2 \sec \eta \tan \eta + 1); \tag{27}$$

$$u_6(x, t) = \pm R\sqrt{\alpha + \beta^2} (-\cot \eta + m_1 \csc \eta) - \lambda, \tag{28}$$

$$v_6(x, t) = \left[R^2(\alpha + \beta^2) - R^2\beta\sqrt{\alpha + \beta^2} \right] (\cot^2 \eta - m_2 \csc \eta \cot \eta + 1). \tag{29}$$

where $\eta = R(\delta x - c_1\delta t + c_2) + \lambda R(t + c_3) + Rc - \frac{1}{2} \ln \left| \frac{A}{B} \right| = R\delta x + R(\lambda - c_1\delta)t + R(c_2 + \lambda c_3 + c) - \frac{1}{2} \ln \left| \frac{A}{B} \right|$, $m_1 = m_2 = \pm 1, A^2 + B^2 \neq 0$,

A, B, R, λ, c are arbitrary constants, $\delta = \pm 1$.

4. Conclusions

From the theorem, it follows that the symmetry group is the product of the usual Lie point symmetry group (see [13]). Because if we take the constants in Eq.(18) and Eq.(19) as follows,

$$c_1 = \varepsilon C_1, c_2 = \varepsilon C_2, c_3 = \varepsilon C_3$$

where ε is an infinitesimal parameter and $C_i (i=1,2,3)$ are arbitrary constants. Then Eq. (18) and Eq.(19) can be written as,

$$u = U + \varepsilon\sigma(U), v = V + \varepsilon\sigma(V) \tag{30}$$

$$\sigma(U) = (\delta x - C_1\delta t + C_2)U_x + (t + C_3)U_t, \sigma(V) = (\delta x - C_1\delta t + C_2)V_x + (t + C_3)V_t \tag{31}$$

where $\sigma(U)$ and $\sigma(V)$ is the symmetry of the WBK equations.

In fact, finishing above discussion, we can see that the equivalence transformation obtained by the direct method is more extensive and simpler than that obtained by the Lie-point group, and we can obtain even more new exact and explicit solutions if we take above solutions (20)-(21) as seed solutions.

Moreover, by applying this direct method, we can find many new solutions of other nonlinear evolution equations with variant coefficients, and this left for the future work.

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