



# Fixed Point Theorems for Occasionally Weakly Compatible Maps in Intuitionistic Fuzzy Semi- Metric Space

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**Abstract:** In this paper, using the concept of occasionally weakly compatible maps, we prove common fixed point theorems for two maps and pairs of maps in intuitionistic fuzzy semi-metric space. Example is also given to prove the validity of proved results. Our results extends and generalizes various known fixed point theorems in the setting of metric, fuzzy, intuitionistic fuzzy and modified fuzzy metric spaces.

**Keywords:** Intuitionistic Fuzzy Semi- Metric Space, Occasionally Weakly Compatible Maps, Weakly Compatible Maps

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## 1. Introduction

Atanassove[2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park[13] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et al.[1] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t- conorms as a generalization of fuzzy metric space due to Kramosil and Michalek[6] . In 2006, Turkoglu[14] proved Jungck's[4] common fixed point theorem in the setting of intuitionistic fuzzy metric spaces for commuting mappings. Later, various authors (see, [7-12]) proved various fixed point theorems in the setting of intuitionistic fuzzy metric space. In this paper, using the concept of occasionally weakly compatible maps, we prove common fixed point theorems for two maps and pairs of maps in intuitionistic fuzzy semi-metric space. Example is also given to prove the validity of proved results. Our results extends and generalizes various known fixed point theorems (see, [3], [5]) in the setting of metric, fuzzy, intuitionistic fuzzy and modified fuzzy metric spaces.

## 2. Preliminaries

The concepts of triangular norms (t –norm) and triangular conorms (t- conorm) are known as the axiomatic skelton that

we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger in study of statistical metric spaces.

Definition 2.1:[1] A binary operation  $*$  :  $[0,1] \times [0,1] \rightarrow [0,1]$  is continuous t-norm if  $*$  is satisfies the following conditions: for all  $a, b, c, d \in [0, 1]$ ,

- (i)  $*$  is commutative and associative;
- (ii)  $*$  is continuous;
- (iii)  $a * 1 = a$ ;
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ .

Definition 2.2:[1] A binary operation  $\diamond$  :  $[0,1] \times [0,1] \rightarrow [0,1]$  is continuous t-conorm if  $\diamond$  is satisfies the following conditions: for all  $a, b, c, d \in [0, 1]$ ,

- (i)  $\diamond$  is commutative and associative;
- (ii)  $\diamond$  is continuous;
- (iii)  $a \diamond 0 = a$ ;
- (iv)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$ .

Alaca et al.[1] defined the notion of intuitionistic fuzzy metric space as follows :

Definition 2.3: A 5-tuple  $(X, M, N, *, \diamond)$  is said to be an intuitionistic fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous t-norm,  $\diamond$  is a continuous t-conorm and  $M, N$  are fuzzy sets on  $X^2 \times [0, \infty)$  satisfying the following conditions: for all  $x, y, z \in X$  and  $s, t > 0$ ,

- (i)  $M(x, y, t) + N(x, y, t) \leq 1$ ;
- (ii)  $M(x, y, 0) = 0$ ;

- (iii)  $M(x, y, t) = 1$  if and only if  $x = y$ ;
- (iv)  $M(x, y, t) = M(y, x, t)$ ;
- (v)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ;
- (vi)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous;
- (vii)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ ;
- (viii)  $N(x, y, 0) = 1$ ;
- (ix)  $N(x, y, t) = 0$  if and only if  $x = y$ ;
- (x)  $N(x, y, t) = N(y, x, t)$ ;
- (xi)  $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ ;
- (xii)  $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is right continuous;
- (xiii)  $\lim_{t \rightarrow \infty} N(x, y, t) = 0$ .

Then  $(M, N)$  is called an intuitionistic fuzzy metric space on  $X$ . The functions  $M(x, y, t)$  and  $N(x, y, t)$  denote the degree of nearness and the degree of non-nearness between  $x$  and  $y$  w.r.t.  $t$  respectively.

If only (i), (ii), (iii), (iv), (viii), (ix), (x) holds, then the 5-tuple  $(X, M, N, *, \diamond)$  is said to be fuzzy semi – metric (symmetric) space.

**Remark 2.1:[1]** Every fuzzy metric space  $(X, M, *)$  is an intuitionistic fuzzy metric space of the form  $(X, M, 1-M, *, \diamond)$  such that  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$  are associated as  $x \diamond y = 1 - ((1-x) * (1-y))$  for all  $x, y \in X$ .

**Lemma 2.1:[1]** In intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$ ,  $M(x, y, \cdot)$  is non-decreasing and  $N(x, y, \cdot)$  is non-increasing for all  $x, y \in X$ .

**Definition 2.4:[1]** Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy semi-metric space. Then

(a) a sequence  $\{x_n\}$  in  $X$  is said to be Cauchy sequence if, for all  $t > 0$  and  $p > 0$ ,

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence  $\{x_n\}$  in  $X$  is said to be convergent to a point  $x \in X$  if, for all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_n, x, t) = 0.$$

**Definition 2.5:[1]** An intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be complete if and only if every Cauchy sequence in  $X$  is convergent.

**Example 2.1:[1]** Let  $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$  and let  $*$  be the continuous  $t$ -norm and  $\diamond$  be the continuous  $t$ -conorm defined by  $a * b = ab$  and  $a \diamond b = \min\{1, a + b\}$  respectively, for all  $a, b$  in  $[0, 1]$ . For each  $t > 0$  and  $x, y \in X$ , define  $(M, N)$  by

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

and

$$N(x, y, t) = \begin{cases} \frac{|x-y|}{t + |x-y|}, & t > 0, \\ 1 & t = 0 \end{cases}$$

Clearly,  $(X, M, N, *, \diamond)$  is complete intuitionistic fuzzy metric space.

**Definition 2.6:[4]** A pair of self mappings  $(f, g)$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be commuting if

$$M(fgx, gfx, t) = 1 \text{ and}$$

$$N(fgx, gfx, t) = 0 \text{ for all } x \text{ in } X.$$

**Definition 2.7:[4]** A pair of self mappings  $(f, g)$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be commuting if

$$M(fgx, gfx, t) = 1 \text{ and } N(fgx, gfx, t) = 0 \text{ for all } x \text{ in } X.$$

**Definition 2.8:[4]** A pair of self mappings  $(f, g)$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be weakly commuting if

$$M(fgx, gfx, t) \geq M(fx, gx, t) \text{ and}$$

$$N(fgx, gfx, t) \leq N(fx, gx, t)$$

for all  $x$  in  $X$  and  $t > 0$ .

**Definition 2.9:[14]** A pair of self mappings  $(f, g)$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be compatible if

$$\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) = 1 \text{ and}$$

$$\lim_{n \rightarrow \infty} N(fgx_n, gfx_n, t) = 0 \text{ for all } t > 0,$$

whenever  $\{x_n\}$  is a sequence in  $X$  such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u \text{ for some } u \text{ in } X.$$

**Definition 2.10:[1]** Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy semi-metric space.  $f$  and  $g$  be self maps on  $X$ . A point  $x$  in  $X$  is called a coincidence point of  $f$  and  $g$  iff  $fx = gx$ . In this case,  $w = fx = gx$  is called a point of coincidence of  $f$  and  $g$ .

**Definition 2.11:[1]** A pair of self mappings  $(f, g)$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  is said to be weakly compatible if they commute at the coincidence points i.e., if  $fu = gu$  for some  $u$  in  $X$ , then  $fgu = gfu$ .

It is easy to see that two compatible maps are weakly compatible but converse is not true.

**Definition 2.12:[3]** Two self mappings  $f$  and  $g$  of a intuitionistic fuzzy semi-metric space  $(X, M, N, *, \diamond)$  are said to be occasionally weakly compatible (owc) iff there is a point  $x$  in  $X$  which is coincidence point of  $f$  and  $g$  at which  $f$  and  $g$  commute.

**Lemma 2.2[5]:** Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy semi-metric space.  $f$  and  $g$  be owc self maps on  $X$  and let  $f$  and  $g$  have a unique point of coincidence,  $w = fx = gx$ , then  $w$  is the unique common fixed point of  $f$  and  $g$ .

### 3. Main Results

Let a function  $\phi$  be defined by  $\phi : [0, 1] \rightarrow [0, 1]$  satisfying

the condition  $\phi(q) > q$  for all  $q < 1$  and a function  $\psi$  be defined by  $\psi : [0,1] \rightarrow [0,1]$  satisfying the condition  $\psi(q) < q$  for all  $q < 1$ .

Theorem 3.1: Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy semi-metric space with continuous t-norm and continuous t-conorm defined by  $a * b = \min \{a, b\}$  and  $a \diamond b = \max \{a, b\}$  for all  $a, b$  in  $[0, 1]$ . If  $f$  and  $g$  are owc self maps on  $X$  and

$$M(fx, fy, t) \geq \phi [ \min \{ M(gx, gy, t), M(gx, fy, t), M(gy, fx, t), M(gy, fy, t) \} ]$$

and

$$N(fx, fy, t) \leq \psi [ \max \{ N(gx, gy, t), N(gx, fy, t), N(gy, fx, t), N(gy, fy, t) \} ] \quad (1)$$

for all  $x, y$  in  $X$  and  $t > 0$ . Then  $f$  and  $g$  have a unique common fixed point.

Proof: Since  $f$  and  $g$  are owc, there exist a point  $u$  in  $X$  such that  $fu = gu$ ,  $fgu = gfu$ . We claim that  $fu$  is the unique common fixed point of  $f$  and  $g$ . We first assert that  $fu$  is a fixed point of  $f$ .

For, if  $ffu \neq fu$ , then from equation (1), we get

$$\begin{aligned} M(fu, ffu, t) &\geq \phi [ \min \{ M(gu, gfu, t), M(gu, ffu, t), M(gfu, fu, t), M(gy, ffu, t) \} ] \\ &= \phi [ \min \{ M(fu, ffu, t), M(fu, ffu, t), M(fu, ffu, t), M(gfu, gfu, t) \} ] = \phi [ M(fu, ffu, t) ] > M(fu, ffu, t) \end{aligned}$$

and

$$\begin{aligned} N(fu, ffu, t) &\leq \psi [ \max \{ N(gu, gfu, t), M(gu, ffu, t), N(gfu, fu, t), N(gy, ffu, t) \} ] \\ &= \psi [ \min \{ N(fu, ffu, t), N(fu, ffu, t), N(fu, ffu, t), N(gfu, gfu, t) \} ] = \psi [ N(fu, ffu, t) ] < N(fu, ffu, t) \end{aligned}$$

this a contradiction. So  $ffu = fu$  and  $ffu = fgu = gfu = fu$ . Hence  $fu$  is a common fixed point of  $f$  and  $g$ .

Now we prove uniqueness. Suppose that  $u, v$  in  $X$  such that  $fu = gu = u$  and  $fv = gv = v$  and  $u \neq v$ . Then from equation (1),

$$\begin{aligned} M(u, v, t) &= M(fu, fv, t) \geq \phi [ \min \{ M(gu, gv, t), M(gu, fv, t), M(gv, fu, t), M(gv, fv, t) \} ] = \phi [ \min \{ M(u, v, t), M(u, v, t), M(u, v, t), M(u, v, t) \} ] = \phi [ M(u, v, t) ] > M(u, v, t) \end{aligned}$$

and

$$\begin{aligned} N(u, v, t) &= N(fu, fv, t) \leq \psi [ \max \{ N(gu, gv, t), N(gu, fv, t), N(gv, fu, t), N(gv, fv, t) \} ] = \psi [ \max \{ N(u, v, t), N(u, v, t), N(u, v, t), N(u, v, t) \} ] = \psi [ N(u, v, t) ] < N(u, v, t) \end{aligned}$$

this is a contradiction. So  $u = v$ . Therefore, the common fixed point of  $f$  and  $g$  is unique.

Now, we give an example which satisfies the conditions of above theorem.

Example 3.1: Let  $X = [0, 1]$  and  $\phi : X \rightarrow X$ ,  $\psi : X \rightarrow X$  be

defined as  $\phi(q) = \frac{1+q}{2}$  and  $\psi(q) = \frac{q}{2}$ , then  $\phi(q) > q$

where  $0 \leq q < 1$ ,  $\psi(q) < q$  where  $0 < q \leq 1$ , and

$f(x) = \frac{1+2x}{2}$ ,  $g(x) = \frac{1+4x}{5}$ . Clearly,  $f$  and  $g$  satisfies all the

conditions w.r.t.  $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ ,  $M(x, y, 0) = 0$  and  $N(x, y, t) = \frac{e^{\left(\frac{|x-y|}{t}\right)-1}}{e^{\left(\frac{|x-y|}{t}\right)}}$ ,  $N(x, y, 0) = 1$ . In this example,  $f$  and  $g$

are owc maps and  $f$  and  $g$  satisfy the equation (1).  $f$  and  $g$  have a unique common fixed point  $x = 1$ .

Theorem 3.2: Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy semi-metric space with continuous t-norm and continuous t-conorm defined by  $a * b = \min \{a, b\}$  and  $a \diamond b = \max \{a, b\}$  for all  $a, b$  in  $[0, 1]$ . Suppose that  $f, g, S, T$  are self maps on  $X$  and the pairs  $\{f, S\}$  and  $\{g, T\}$  are owc. If

$$M(fx, gy, t) > \min \{ M(Sx, Ty, t), M(Sx, fx, t), M(Ty, gy, t), M(Sx, gy, t), M(Ty, fx, t) \} \text{ and}$$

$$N(fx, gy, t) < \max \{ N(Sx, Ty, t), N(Sx, fx, t), N(Ty, gy, t), N(Sx, gy, t), N(Ty, fx, t) \}, \quad (2)$$

for all  $x, y$  in  $X$  and  $t > 0$ . Then  $f, g, S$  and  $T$  have a unique common fixed point.

Proof: By hypothesis, there exists points  $x, y$  in  $X$  such that  $fx = Sx$  and  $gy = Ty$ . Suppose that  $M(fx, gy, t) \neq 1$  and  $N(fx, gy, t) \neq 0$  for all  $t > 0$ . Then from equation (2)

$$\begin{aligned} M(fx, gy, t) &> \min \{ M(Sx, Ty, t), M(Sx, fx, t), M(Ty, gy, t), M(Sx, gy, t), M(Ty, fx, t) \} = \min \{ M(fx, gy, t), M(fx, fx, t), M(gy, gy, t), M(fx, gy, t), M(gy, fx, t) \} = M(fx, gy, t) \end{aligned}$$

$$\begin{aligned} N(fx, gy, t) &< \max \{ N(Sx, Ty, t), N(Sx, fx, t), N(Ty, gy, t), N(Sx, gy, t), N(Ty, fx, t) \} = \max \{ N(fx, gy, t), N(fx, fx, t), N(gy, gy, t), N(fx, gy, t), N(gy, fx, t) \} = N(fx, gy, t) \end{aligned}$$

this is a contradiction. Hence  $M(fx, gy, t) = 1$  and  $N(fx, gy, t) = 0$  for all  $t > 0$ . This implies that  $fx = gy$ . So  $fx = Sx = gy = Ty$ . Moreover, if there is another point  $z$  such that  $fz = Sz$ , then, using (2) it follows that  $fz = Sz = gy = Ty$  or  $fx = fz$  and  $w = fx = Sx$  is the unique point of coincidence of  $f$  and  $S$ . Then by Lemma 2.2, it follows that  $w$  is the unique common fixed point of  $f$  and  $S$ . By symmetry, there is a unique common fixed point  $z$  in  $X$  such that  $z = gz = Tz$ . Suppose that  $w \neq z$ . Using (2),

$$M(w, z, t) = M(fw, gz, t) > \min \{ M(Sw, Tz, t), M(Sw, fw, t), M(Tz, gz, t), M(Sw, gz, t), M(Tz, fw, t) \}$$

$$M(w, z, t) > \min \{ M(w, z, t), M(w, w, t), M(z, z, t), M(w, z, t), M(z, w, t) \}$$

$$M(w, z, t) > M(w, z, t)$$

and

$$N(w, z, t) = N(fw, gz, t) < \max \{ N(Sw, Tz, t), N(Sw, fw, t), N(Tz, gz, t), N(Sw, gz, t), N(Tz, fw, t) \}$$

$$N(w, z, t) < \max \{ N(w, z, t), N(w, w, t), N(z, z, t), N(w, z, t), N(z, w, t) \}$$

$$N(w, z, t) < N(w, z, t)$$

This is a contradiction. Therefore  $w = z$  and  $w$  is a unique point of coincidence of  $f, g, S, T$ . By Lemma 2.2,  $w$  is the unique common fixed point of  $f, g, S, T$ .

## 4. Conclusion

In this paper, we proved two common fixed point theorems: one for two self maps and other for two pairs of self maps in the setting of intuitionistic fuzzy semi-metric space. Example is also given to prove the validity of main result. Proved results generalize and extend various known results (see, [3], [5]) in the literature.

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