Maximum Principle and the Applications of Mean-Field Backward Doubly Stochastic System

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Abstract: Since Pardoux and Peng firstly studied the following nonlinear backward stochastic differential equations in 1990. The theory of BSDE has been widely studied and applied, especially in the stochastic control, stochastic differential games, financial mathematics and partial differential equations. In 1994, Pardoux and Peng came up with backward doubly stochastic differential equations to give the probabilistic interpretation for stochastic partial differential equations. Backward doubly stochastic differential equations theory has been widely studied because of its importance in stochastic partial differential equations and stochastic control problems. In this article, we will study the theory of doubly stochastic systems and related topics further.

Keywords: Mean-Field Backward Doubly, Stochastic System, Stochastic Control

1. Introduction

Andersson and Djehiche, Buckdahn, Djehiche and Li, Meyer Brandis, ksendal and Zhou, and Lihave studied the optimal control problem about Mean-field s tochastic differential system .Inspired by the above problems, in the paper, we study the optimal control problem about Mean-field backward doubly stochastic system. In the situation that control field to the convex and coefficient contains control variable, Using convex variational and dual technology, we present the local and global stochastic maximum principle, proved a sufficient conditions of optimality (verification theorem) and a necessary condition[1-4].

2. The Control Problem of Mean-Field Backward Doubly Stochastic System

For simple marking, make $m = n = d = l = k_1 = k_2 = 1$. Given convex subset $U \subset \mathbb{R}^k$, allowing the control set is defined as

$$u_{ad} = \{\, \upsilon : [0, \ T\,] \times \Omega \to U \mid \upsilon \text{ is } F_t \text{ - measurable,}$$
$$\mathrm{E}\int_o^T |\upsilon(t)|^2\, dt < +\infty \}$$

For any $\upsilon \in u_{ad}, \xi \in L^2(\Omega, F_T, P; \mathrm{R})$, consider the following MF - BDSDE:

$$Y^{\upsilon}(t) = \xi + \int_t^T \Gamma^f(s, Y^{\upsilon}(s), Z^{\upsilon}(s), \upsilon(s))ds - \int_t^T Z^{\upsilon}(s)\, \overrightarrow{d\,W}(s) + \int_t^T \Gamma^g(s, Y^{\upsilon}(s), Z^{\upsilon}(s), \upsilon(s))\, \overleftarrow{d\,B}(s),$$

Where $i = f, g$

$$\Gamma^i(s, Y^{\upsilon}(s), Z^{\upsilon}(s), \upsilon(s)) = \int_\Omega \theta^i(s, \omega, \omega', Y^{\upsilon}(s, \omega), Z^{\upsilon}(s, \omega), \upsilon(s, \omega), Y^{\upsilon}(s, \omega'), Z^{\upsilon}(s, \omega'), \upsilon(s, \omega'))P(d\omega'),$$

And $\theta^f : \Omega^2 \times [0, T] \times \mathrm{R} \times \mathrm{R} \times U \times \mathrm{R} \times \mathrm{R} \times U \to \mathrm{R}$, $\theta^g : \Omega^2 \times [0, T] \times \mathrm{R} \times \mathrm{R} \times U \times \mathrm{R} \times \mathrm{R} \times U \to \mathrm{R}$

Performance indicators is

$$J(\upsilon(\cdot)) = E\int_0^T \Gamma^l(s, Y^\upsilon(s), Z^\upsilon(s), \upsilon(s))ds + E[E'h(Y_0^\upsilon(\omega), Y_0^\upsilon(\omega'))], \tag{1}$$

Where

$$\Gamma^l(s, Y^\upsilon(s), Z^\upsilon(s), \upsilon(s)) = \int_\Omega l(s, \omega, \omega', Y^\upsilon(s, \omega), Z^\upsilon(s, \omega), \upsilon(s, \omega), Y^\upsilon(s, \omega'), Z^\upsilon(s, \omega'), \upsilon(s, \omega'))P(d\omega')$$

$h : \Omega^2 \times R \times R \to R,$

$l : \Omega^2 \times [0,T] \times R \times R \times U \times R \times R \times U \to R,$

Control problem is looking for admission control to make performance indicators reaching the minimum value on the u_{ad}.
Supposing that [5-6]
(H1) (1) θ^f, θ^g, l, h is continuously differentiable about y, y', z, z', v, v', and the derivative of h and i is linear growth.
(2) θ^f, θ^g meet uniform Lipschitz condition about (y, z, y', z', v, v').
In other words there exist L_i, K_i, α_j, for $i = y, z, y', z', v, v', j = 3, 4$. making

$$\left| \theta^f(t, \omega, \omega', y_1, z_1, y_1', z_1', \upsilon_1, \upsilon_1') - \theta^f(t, \omega, \omega', y_2, z_2, y_2', z_2', \upsilon_2, \upsilon_2') \right|$$
$$\leq L_y |y_1 - y_2| + L_z |z_1 - z_2| + L_{y'} |y_1' - y_2'| + L_{z'} |z_1' - z_2'| + L_v |\upsilon_1 - \upsilon_2| + L_{v'} |\upsilon_1' - \upsilon_2'|,$$

$$\left| \theta^g(t, \omega, \omega', y_1, z_1, y_1', z_1', \upsilon_1, \upsilon_1') - \theta^g(t, \omega, \omega', y_2, z_2, y_2', z_2', \upsilon_2, \upsilon_2') \right|^2$$
$$\leq K_y^2 |y_1 - y_2|^2 + K_{y'}^2 |y_1' - y_2'|^2 + K_v^2 |\upsilon_1 - \upsilon_2|^2 + K_{v'}^2 |\upsilon_1' - \upsilon_2'|^2 + \alpha_3 |z_1 - z_2| + \alpha_4 |z_1' - z_2'|,$$

$$\forall (t, \omega, \omega') \in [0,T] \times \Omega^2, (y_i, z_i, y_i', z_i', \upsilon_i, \upsilon_i') \in R^6, i = 1,2$$

And $E\int_0^T \left| E'\theta_o^l(t, \omega, \omega') \right|^2 dt < \infty, \quad l = f, g$

Where $\theta_0^l(t, w, w') = \theta_0^l(t, w, w', 0, 0, 0, 0, 0, 0), \alpha_3 + \alpha_4 < 1$.

Under the above assumptions, for any $v(\bullet) \in u_{ad}$, there exists a unique solution $(Y^v, Z^v) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R})$ of the equation (1). And the performance index defined is reasonable.[7-8]

Assumed $\hat{u}(\cdot)$ is the optimal control. $(\hat{Y}(\cdot), \hat{Z}(\cdot))$ is the corresponding optimal trajectory. $v(\cdot)$ meet $\hat{u}(\cdot) + v(\cdot) \in u_{ad}$. because of the convexity of u_{ad}, for any

$0 \leq \varepsilon \leq 1, u^\varepsilon(\cdot) = \hat{u}(\cdot) + \varepsilon v(\cdot) \in u_{ad}$. there exists a unique solution $(Y^\varepsilon(\cdot), Z^\varepsilon(\cdot))$ of u^ε

Lemma 1. hypothesis (H1) is established, for any $t \in [0, T]$,

$$E\left| Y^\varepsilon(t) - \hat{Y}(t) \right|^2 \leq C\varepsilon^2, E\int_t^T \left| Z^\varepsilon(s) - \hat{Z}(s) \right|^2 ds \leq C\varepsilon^2.$$

Proof. Notice that $Y^\varepsilon(t) - \hat{Y}(t)$ to meet the following MF-BDSDE:

$$Y^\varepsilon(t) - \hat{Y}(t) = \int_t^T [\Gamma^f(s, Y^\varepsilon(s), Z^\varepsilon(s), u^\varepsilon(s)) - \Gamma^f(s, \hat{Y}(s), \hat{Z}(s), \hat{u}(s))]ds$$

$$+\int_t^T [\Gamma^g(s, Y^\varepsilon(s), Z^\varepsilon(s), u^\varepsilon(s)) - \Gamma^g(s, \hat{Y}(s), \hat{Z}(s), \hat{u}(s))]\overleftarrow{d}B(s) - \int_t^T (Z^\varepsilon(s) - \hat{Z}(s))\overrightarrow{d}W(s)$$

Applying $It\hat{o}$ formulas to $\left| Y^\varepsilon(t) - \hat{Y}(t) \right|^2$

$$E\left(\left| Y^\varepsilon(t) - \hat{Y}(t) \right|^2 + \int_t^T \left| Z^\varepsilon(s) - \hat{Z}(s) \right|^2 ds\right) = 2E\int_t^T \left\langle Y^\varepsilon(s) - \hat{Y}(s), \Gamma^f(s, Y^\varepsilon(s), Z^\varepsilon(s), u^\varepsilon(s)) - \Gamma^f(s, \hat{Y}(s), \hat{Z}(s), \hat{u}(s)) \right\rangle ds$$

$$+ E\int_t^T \left| \Gamma^g(s, Y^\varepsilon(s), Z^\varepsilon(s), u^\varepsilon(s)) - \Gamma^g(s, \hat{Y}(s), \hat{Z}(s), \hat{u}(s)) \right|^2 ds$$

According to (H1), there is

$$\mathrm{E}\left|Y^{\mathcal{E}}(t)-\hat{Y}(t)\right|^2+\mathrm{E}\int_t^T\left|Z^{\mathcal{E}}(s)-\hat{Z}(s)\right|^2 ds \le k_1\mathrm{E}\int_t^T\left|Y^{\mathcal{E}}(t)-\hat{Y}(t)\right|^2 ds + k_2\varepsilon^2\mathrm{E}\int_t^T\left|\upsilon(s)\right|^2 ds$$

Where $k_i(i=1,2)$ is constant rely on (H1). According Gronwall Inequality and Burkholder-Davis-Gundy Inequality, results are verified.

For simple marking, make

$$\hat{\alpha}(\cdot)=\alpha(\cdot,\omega,\omega',\hat{Y}(\cdot,\omega),\hat{Z}(\cdot,\omega),\hat{u}(\cdot,\omega),\hat{Y}(\cdot,\omega'),\hat{Z}(\cdot,\omega'),\hat{u}(\cdot,\omega')),$$

$$\alpha^\varepsilon(\cdot)=\alpha(\cdot,\omega,\omega',Y^\varepsilon(\cdot,\omega),Z^\varepsilon(\cdot,\omega),u^\varepsilon(\cdot,\omega),Y^\varepsilon(\cdot,\omega'),Z^\varepsilon(\cdot,\omega'),u^\varepsilon(\cdot,\omega')),$$

$$\xi(t)=\psi(t)+\int_t^T F_1\left(s,\xi(s),\eta(s)\right)ds+\int_t^T G_1\left(s,\xi(s),\eta(s)\right)\overline{d}B(s)-\int_t^T \eta(s)\overrightarrow{d}W(s), \tag{2}$$

Where

$$F_1\left(s,\xi(s),\eta(s)\right)=\mathrm{E}'\left[\widehat{\theta^f}_y(s)\xi(s)+\widehat{\theta^f}_z(s)\eta(s)+\widehat{\theta^f}_{y'}(s)\xi'(s)+\widehat{\theta^f}_{z'}(s)\eta'(s)\right],$$

$$G_1\left(s,\xi(s),\eta(s)\right)=\mathrm{E}'\left[\widehat{\theta^g}_y(s)\xi(s)+\widehat{\theta^g}_z(s)\eta(s)+\widehat{\theta^g}_{y'}(s)\xi'(s)+\widehat{\theta^g}_{z'}(s)\eta'(s)\right],$$

And

$$\psi(t)=\int_t^T \mathrm{E}'\left[\widehat{\theta^f}_\upsilon(s)\upsilon(s)+\widehat{\theta^f}_{\upsilon'}(s)\upsilon'(s)\right]ds+\int_t^T \mathrm{E}'\left[\widehat{\theta^g}_\upsilon(s)\upsilon(s)+\widehat{\theta^g}_{\upsilon'}(s)\upsilon'(s)\right]\overline{d}B(s).$$

Marked

$$\mathrm{E}'\left[\widehat{\theta^f}_y(s)\xi(s)\right]=\int_\Omega \widehat{\theta^f}_y(s,\omega,\omega')\xi(s,\omega)\mathrm{P}(d\omega'),$$

$$\mathrm{E}'\left[\widehat{\theta^f}_{y'}(s)\xi'(s)\right]=\int_\Omega \widehat{\theta^f}_{y'}(s,\omega,\omega')\xi(s,\omega')\mathrm{P}(d\omega').$$

Under the above assumptions, for any $v(\bullet)\in u_{ad}$, there exists a unique solution $\left(\xi(t),\eta(t)\right)\in S^2\left([0,T];\mathrm{R}\right)\times M^2\left(0,T;\mathbb{R}\right)$ of the equation (2).

Lemma 2. Marked

$$y^\varepsilon(t)=\frac{Y^\varepsilon(t)-\hat{Y}(t)}{\varepsilon}-\xi(t)\quad,\quad z^\varepsilon(t)=\frac{Z^\varepsilon(t)-\hat{Z}(t)}{\varepsilon}-\eta(t).$$

$$\lim_{\varepsilon\to0}\sup_{t\in[0,T]}\mathrm{E}\left|y^\varepsilon(t)\right|^2=0,\quad \lim_{\varepsilon\to0}\mathrm{E}\int_0^T\left|z^\varepsilon(t)\right|^2 dt=0. \tag{3}$$

$\left(y^\varepsilon,z^\varepsilon\right)$ is the solution of the equation as follows,

$$\begin{cases}-dy^\varepsilon(t)=\\ \qquad \mathrm{E}'\left[f_y^\varepsilon(t)y^\varepsilon(t)+f_z^\varepsilon(t)z^\varepsilon(t)+f_{y'}^\varepsilon(t)y'^\varepsilon(t)+f_{z'}^\varepsilon(t)z'^\varepsilon(t)+f_1^\varepsilon(t)\right]dt\\ \qquad +\mathrm{E}'\left[g_y^\varepsilon(t)y^\varepsilon(t)+g_z^\varepsilon(t)z^\varepsilon(t)+g_{y'}^\varepsilon(t)y'^\varepsilon(t)+g_{z'}^\varepsilon(t)z'^\varepsilon(t)+g_1^\varepsilon(t)\right]\overline{d}B(t)\\ \qquad -z^\varepsilon(t)\overrightarrow{d}W(t),\\ y^\varepsilon(T)=0,\end{cases}$$

Where $\delta=f,g$, $\overline{Y}_{t,\omega}=\hat{Y}_{t,\omega}+\lambda\left(Y_{t,\omega}^\varepsilon-\hat{Y}_{t,\omega}\right)$, $\overline{u}_{t,\omega}=\hat{u}_{t,\omega}+\lambda\left(u_{t,\omega}^\varepsilon-\hat{u}_{t,\omega}\right)$

$$\delta_y^\varepsilon(\cdot)=\int_0^1\theta_y^\delta\left(\cdot,\overline{Y}_{\cdot,\omega},\overline{Z}_{\cdot,\omega},\overline{u}_{\cdot,\omega},\overline{Y}_{\cdot,\omega'},\overline{Z}_{\cdot,\omega'},\overline{u}_{\cdot,\omega'}\right)d\lambda,$$

and

$$\delta_1^\varepsilon(\cdot) = \left[\delta_y^\varepsilon(\cdot) - \widehat{\theta^\delta}_y(\cdot)\right]\xi_{\cdot,\omega} + \left[\delta_z^\varepsilon(\cdot) - \widehat{\theta^\delta}_z(\cdot)\right]\eta_{\cdot,\omega} + \left[\delta_{y'}^\varepsilon(\cdot) - \widehat{\theta^\delta}_{y'}(\cdot)\right]\xi_{\cdot,\omega'}$$

$$+ \left[\delta_{z'}^\varepsilon(\cdot) - \widehat{\theta^\delta}_{z'}(\cdot)\right]\eta_{\cdot,\omega'} + \left[\delta_\upsilon^\varepsilon(\cdot) - \widehat{\theta^\delta}_\upsilon(\cdot)\right]\upsilon_{\cdot,\omega} + \left[\delta_{\upsilon'}^\varepsilon(\cdot) - \widehat{\theta^\delta}_{\upsilon'}(\cdot)\right]\upsilon_{\cdot,\omega'}.$$

Applying $It\widehat{o}$ formulas to $\left|y^\varepsilon(t)\right|^2$ on $[t,T]$

$$\mathrm{E}\left|y^\varepsilon(t)\right|^2 + \mathrm{E}\int_t^T\left|z^\varepsilon(s)\right|^2 ds = 2\mathrm{E}\int_t^T\left\langle y^\varepsilon(s), \mathrm{E}'\left[f_y^\varepsilon(s)y^\varepsilon(s) + f_z^\varepsilon(s)z^\varepsilon(s) + f_{y'}^\varepsilon(s)y'^\varepsilon(s) + f_{z'}^\varepsilon(s)z'^\varepsilon(s) + f_1^\varepsilon(s)\right]\right\rangle ds$$

$$+ \mathrm{E}\int_t^T\left|\mathrm{E}'\left[g_y^\varepsilon(s)y^\varepsilon(s) + g_z^\varepsilon(s)z^\varepsilon(s) + g_{y'}^\varepsilon(s)y'^\varepsilon(s) + g_{z'}^\varepsilon(s)z'^\varepsilon(s) + g_1^\varepsilon(s)\right]\right|^2 ds.$$

According to (H1), there is

$$\mathrm{E}\left|y^\varepsilon(t)\right|^2 + \mathrm{E}\int_t^T\left|z^\varepsilon(s)\right|^2 ds \le k\mathrm{E}\int_t^T\left|y^\varepsilon(s)\right|^2 ds + C_\varepsilon,$$

Where k is constant, when $\varepsilon \to 0$ 时, $C_\varepsilon \to 0$. According Gronwall Inequality, results are verified. Because of $\hat{u}(\cdot)$ is the optimal control,

$$\varepsilon^{-1}\left[J\left(u^\varepsilon(\cdot)\right) - J\left(\hat{u}(\cdot)\right)\right] \ge 0. \tag{4}$$

According to lemma 2, there is
lemma 3. Hypothesis (H1) was established, then the following variation inequality is established[9-10]:

$$\mathrm{E}\int_0^T\mathrm{E}'\left[\hat{l}_y(s)\xi(s) + \hat{l}_z(s)\eta(s) + \hat{l}_{y'}(s)\xi'(s) + \hat{l}_{z'}(s)\eta'(s) + \hat{l}_\upsilon(s)\upsilon(s) + \hat{l}_{\upsilon'}(s)\upsilon'(s)\right] ds$$

$$+ \mathrm{E}\mathrm{E}'\left[h_y\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right)\xi_{0,\omega} + h_{y'}\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right)\xi_{0,\omega'}\right] \ge 0. \tag{5}$$

Where

$$\mathrm{E}'\left[\hat{l}_{y'}(s)\xi'(s)\right] = \int_\Omega \hat{l}_{y'}(s,\omega,\omega')\xi(s,\omega')\mathrm{P}(d\omega').$$

Proof.

$$\mathrm{E}\mathrm{E}'\varepsilon^{-1}\left[h\left(Y_{0,\omega}^\varepsilon, Y_{0,\omega'}^\varepsilon\right)\right] - h\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right) = \mathrm{E}\mathrm{E}'\varepsilon^{-1}\int_0^1 h_y\left(\overline{Y}_{0,\omega}, \overline{Y}_{0,\omega'}\right)\left(Y_{0,\omega}^\varepsilon - \hat{Y}_{0,\omega}\right)d\lambda + \mathrm{E}\mathrm{E}'\varepsilon^{-1}\int_0^1 h_{y'}\left(\overline{Y}_{0,\omega}, \overline{Y}_{0,\omega'}\right)\left(Y_{0,\omega'}^\varepsilon - \hat{Y}_{0,\omega'}\right)d\lambda$$

$$\to \mathrm{E}\mathrm{E}'\left[h_y\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right)\xi_{0,\omega} + h_{y'}\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right)\xi_{0,\omega'}\right], \varepsilon \to 0$$

Where $\overline{Y}_{0,\omega} = \hat{Y}_{0,\omega} + \lambda\left(Y_{0,\omega}^\varepsilon - \hat{Y}_{0,\omega}\right)$.

$$\varepsilon^{-1}\left\{\mathrm{E}\int_0^T\mathrm{E}'\left[l^\varepsilon(t) - \hat{l}(t)\right]dt\right\} \to \mathrm{E}\int_0^T\mathrm{E}'\left[\hat{l}_y(s)\xi(s) + \hat{l}_z(s)\eta(s) + \hat{l}_{y'}(s)\xi'(s) + \hat{l}_{z'}(s)\eta'(s) + \hat{l}_u(s)\upsilon(s) + \hat{l}_{\upsilon'}(s)\upsilon'(s)\right] ds$$

so (5) is verified.
Considering the adjoin equation:

$$p(t) = \mathrm{E}'h_y\left(\hat{Y}_{0,\omega}, \hat{Y}_{0,\omega'}\right) + \mathrm{E}^*h_{y'}\left(\hat{Y}_{0,\omega^*}, \hat{Y}_{0,\omega}\right) + \int_0^t F_2\left(s, p(s), q(s)\right)ds + \int_0^t G_2\left(s, p(s), q(s)\right)\vec{d}W(s) - \int_0^t q(s)\vec{d}B(s), \tag{6}$$

Where

$$F_2\left(s, p(s), q(s)\right) = \mathrm{E}'\left[\widehat{\theta^f}_y(s)p(s) + \widehat{\theta^g}_y(s)q(s) + \hat{l}_y(s)\right] + \mathrm{E}^*\left[\widehat{\theta^f}_{y'}(s)p^*(s) + \widehat{\theta^g}_{y'}(s)q^*(s) + \hat{l}_{y'}(s)\right],$$

$$G_2\left(s,p(s),q(s)\right) = \mathrm{E}'\left[\widehat{\theta^f}_z(s)p(s)+\widehat{\theta^g}_z(s)q(s)+\hat{l}_z(s)\right]+\mathrm{E}^*\left[\widehat{\theta^f}_{z'}(s)p^*(s)+\widehat{\theta^g}_{z'}(s)q^*(s)+\hat{l}_{z'}(s)\right].$$

$$\mathrm{E}^*\hat{l}_{y'}(s) = \int_\Omega \hat{l}_{y'}\left(s,\omega^*,\omega\right)\mathrm{P}\left(d\omega^*\right),$$

$$\mathrm{E}^*\left[\widehat{\theta^f}_{y'}(s)p^*(s)\right] = \int_\Omega \widehat{\theta^f}_{y'}\left(s,\omega^*,\omega\right)p\left(s,\omega^*\right)\mathrm{P}\left(d\omega^*\right).$$

Define the Hamiltonian function $H:[0,T]\times R\times R\times R\times R\times R\times R\times R\times R \to R$ as follows

$$H\left(t,y_1,z_1,\upsilon_1,y_2,z_2,\upsilon_2,p,q\right) = \theta^f\left(t,\omega,\omega',y_1,z_1,\upsilon_1,y_2,z_2,\upsilon_2\right)p + \theta^g\left(t,\omega,\omega',y_1,z_1,\upsilon_1,y_2,z_2,\upsilon_2\right)q$$

$$+l\left(t,\omega,\omega',y_1,z_1,\upsilon_1,y_2,z_2,\upsilon_2\right) \tag{7}$$

By the variational inequality (7), we present MF - BDSDEs stochastic control problem of stochastic maximum principle.

Theorem 1,(stochastic maximum principle) Assumed $\left(\hat{Y}(\cdot),\hat{Z}(\cdot),\hat{u}(\cdot)\right)$ is the optimal trajectory of the control problem$\{(1),(2)\}$, $\forall v \in U, a.e.$ $t \in [0,T], a.s.$

$$\left[\mathrm{E}'\widehat{H}_\upsilon\left(t,\omega,\omega'\right)+\mathrm{E}^*H_{\upsilon'}\left(t,\omega^*,\omega\right)\right]\cdot\left(v-\hat{u}(t)\right)\geq 0 \tag{8}$$

where

$$\widehat{H}\left(t,\omega,\omega'\right) = H\left(t,\omega,\omega',\hat{Y}(t,\omega),\hat{Z}(t,\omega),\hat{u}(t,\omega),\hat{Y}(t,\omega'),\hat{Z}(t,\omega'),\hat{u}(t,\omega'),p(t,\omega),q(t,\omega)\right) \tag{9}$$

Proof. Applying *Itô* formulas to $\langle \xi(t),p(t)\rangle$,we can get

$$-\mathrm{E}\xi_0 p_0 = \mathrm{E}\int_0^T \mathrm{E}'\left[l_y(s)\xi(s)+l_z(s)\eta(s)+l_{y'}(s)\xi'(s)+l_{z'}(s)\eta'(s)\right]ds -\mathrm{E}\int_0^T \mathrm{E}'\left[\widehat{\theta^f}_{\upsilon'}(s)\upsilon'(s)p(s)+\widehat{\theta^g}_{\upsilon'}(s)\upsilon'(s)q(s)\right]ds$$

$$-\mathrm{E}\int_0^T \mathrm{E}'\left[\widehat{\theta^f}_\upsilon(s)\upsilon(s)p(s)+\widehat{\theta^g}_\upsilon(s)\upsilon(s)q(s)\right]ds$$

According (5) ,we can get

$$\mathrm{E}\int_0^T \mathrm{E}'\left[\widehat{\theta^f}_{\upsilon'}(s)\upsilon'(s)p(s)+\widehat{\theta^g}_{\upsilon'}(s)\upsilon'(s)q(s)+\hat{l}_{\upsilon'}(s)\upsilon'(s)\right]ds$$

$$+\mathrm{E}\int_0^T \mathrm{E}'\left[\widehat{\theta^f}_\upsilon(s)\upsilon(s)p(s)+\widehat{\theta^g}_\upsilon(s)\upsilon(s)q(s)+\hat{l}_\upsilon(s)\upsilon(s)\right]ds \geq 0$$

According Hamiltonian function, we can get

$$\mathrm{E}\int_0^T\left[\mathrm{E}^*H_{\upsilon'}\left(t,\hat{Y}(t)(\omega^*),\hat{Z}(t)(\omega^*),\hat{u}(t)(\omega^*),\hat{Y}(t,\omega),\hat{Z}(t,\omega),\hat{u}(t,\omega),\right.\right.$$

$$\left.\left. p(t)(\omega^*),q(t)(\omega^*)\right)+\mathrm{E}'H_\upsilon\left(t,\hat{Y}(t,\omega),\hat{Z}(t,\omega),\hat{u}(t,\omega),\hat{Y}(t)(\omega'),\hat{Z}(t)(\omega'),\hat{u}(t)(\omega'), p(t,\omega),q(t,\omega)\right)\right]\cdot\upsilon(t)dt \geq 0$$

For any $v \in U, F$ is the any element of $\sigma-A\lg ebra\left(F_t\right)$,setting

$$\bar{\upsilon}(s) = \begin{cases} \hat{u}(s), & s \in [0,t), \\ \upsilon, & s \in [t,,t+\varepsilon),\omega \in F, \\ \hat{u}(s), & s \in [t,,t+\varepsilon),\omega \in \Omega-F, \\ \hat{u}(s), & s \in [t+\varepsilon,T], \end{cases}$$

We can know $v(s) \in u_{ad}$, because $\upsilon(t)$ meet $\hat{u}(t)+\upsilon(t)\in u_{ad}$, setting $\upsilon(t)=\bar{\upsilon}(t)-\hat{u}(t)$, The above inequalities can be rewritten as

$$\mathrm{El}_F \int_t^{t+\varepsilon} \left[\mathrm{E}' \widehat{H}_\upsilon \left(s, \omega, \omega' \right) + \mathrm{E}^* \widehat{H}_{\upsilon'} \left(s, \omega^*, \omega \right) \right] \cdot \left(\upsilon - \hat{u}(s) \right) ds \geq 0$$

Differential on a variable ε at $\varepsilon = 0$, we can get

$$\mathrm{El}_F \left[\mathrm{E}' \widehat{H}_\upsilon \left(t, \omega, \omega' \right) + \mathrm{E}^* \widehat{H}_{\upsilon'} \left(t, \omega^*, \omega \right) \right] \cdot \left(\upsilon - \hat{u}(t) \right) \geq 0$$

So (8) is verified.

3. Mean-Field Backward Doubly Stochastic LQ Problem

This section, we apply the maximum value principle to Mean-field backward doubly stochastic LQ problem.

$$q\left(Y_{0,\omega}, Y_{0,\omega'} \right) = \frac{1}{2} Q_0^1 Y_{0,\omega}^2 + \frac{1}{2} Q_0^2 Y_{0,\omega'}^2$$

And

$$f\left(s, \omega, \omega', Y_{s,\omega}, Z_{s,\omega}, \upsilon_{s,\omega}, Y_{s,\omega'}, Z_{s,\omega'}, \upsilon_{s,\omega'} \right) = A^1(s) Y_{s,\omega} + B^1(s) Z_{s,\omega} + C^1(s) \upsilon_{s,\omega} + A^2(s) Y_{s,\omega'} + B^2(s) Z_{s,\omega'} + C^2(s) \upsilon_{s,\omega'}$$

$$g\left(s, \omega, \omega', Y_{s,\omega}, Z_{s,\omega}, \upsilon_{s,\omega}, Y_{s,\omega'}, Z_{s,\omega'}, \upsilon_{s,\omega'} \right) = D^1(s) Y_{s,\omega} + E^1(s) Z_{s,\omega} + F^1(s) \upsilon_{s,\omega} + D^2(s) Y_{s,\omega'} + E^2(s) Z_{s,\omega'} + F^2(s) \upsilon_{s,\omega'}$$

$$l\left(s, \omega, \omega', Y_{s,\omega}, Z_{s,\omega}, \upsilon_{s,\omega}, Y_{s,\omega'}, Z_{s,\omega'}, \upsilon_{s,\omega'} \right) = \frac{1}{2} \left[M^1(s) Y_{s,\omega}^2 + N^1(s) Z_{s,\omega}^2 + W^1(s) \upsilon_{s,\omega}^2 + M^2(s) Y_{s,\omega'}^2 + N^2(s) Z_{s,\omega'}^2 + W^2(s) \upsilon_{s,\omega'}^2 \right]$$

Where $A^i : [0,T] \times \Omega^2 \to \mathbb{R}$ is bounded. $(s, w, w') \mapsto A^i(s, w, w')$ is $\mathfrak{M}_s^2$ measurable (similarly, other coefficient satisfies the hypothesis). M^i, N^i are nonnegative, R^i is positive. The state equation is

$$Y_{t,\omega}^\upsilon = \xi + \int_t^T \left\{ \left[\mathrm{E}' A^1(s) \right] Y_{s,\omega}^\upsilon + \left[\mathrm{E}' B^1(s) \right] Z_{s,\omega}^\upsilon + \left[\mathrm{E}' C^1(s) \right] \upsilon_{s,\omega} \right\} ds + \int_t^T \mathrm{E}' \left[A^2(s) Y_{s,\omega'}^\upsilon + B^2(s) Z_{s,\omega'}^\upsilon + C^2(s) \upsilon_{s,\omega'} \right] ds$$

$$+ \int_t^T \left\{ \left[\mathrm{E}' D^1(s) \right] Y_{s,\omega}^\upsilon + \left[\mathrm{E}' E^1(s) \right] Z_{s,\omega}^\upsilon + \left[\mathrm{E}' F^1(s) \right] \upsilon_{s,\omega} \right\} \overline{d} B_{s,\omega} + \int_t^T \left\{ \mathrm{E}' \left[D^2(s) Y_{s,\omega'}^\upsilon + E^2(s) Z_{s,\omega'}^\upsilon + F^2(s) \upsilon_{s,\omega'} \right] \right\} \overline{d} B_{s,\omega}$$

$$- \int_t^T Z_{s,\omega}^\upsilon \overline{d} W_{s,\omega}, \tag{10}$$

Performance indicators is

$$J\left(\upsilon(\cdot) \right) = \frac{1}{2} \mathrm{E} \left(\int_0^T \mathrm{E}' \left[M^1(s) \left| Y_{s,\omega}^\upsilon \right|^2 + N^1(s) \left| Z_{s,\omega}^\upsilon \right|^2 + W^1(s) \upsilon_{s,\omega}^2 \right] ds + \mathrm{E}' \left[Q_0^1 \left| Y_{0,\omega}^\upsilon \right|^2 \right] \right)$$

$$\frac{1}{2} \mathrm{E} \left(\int_0^T \mathrm{E}' \left[M^2(s) \left| Y_{s,\omega'}^\upsilon \right|^2 + N^2(s) \left| Z_{s,\omega'}^\upsilon \right|^2 + W^2(s) \left| \upsilon_{s,\omega'} \right|^2 \right] ds + \mathrm{E}' \left[Q_0^2 \left| Y_{0,\omega'}^\upsilon \right|^2 \right] \right).$$

In order to mark is simple, put $A^i(s.\omega, \omega')$ for $A^i(s)$. Hamiltonian function is

$$H\left(s, \omega, \omega', y_1, z_1, v_1, y_2, z_2, v_2, p, q \right) = \left[A^1(s) y_1 + B^1(s) z_1 + C^1(s) v_1 + A^2(s) y_2 + B^2(s) z_2 + C^2(s) v_2 \right] p$$

$$+ \left[D^1(s) y_1 + E^1(s) z_1 + F^1(s) v_1 + D^2(s) y_2 + E^2(s) z_2 + F^2(s) v_2 \right] q$$

$$+ \frac{1}{2} \left[M^1(s) y_1^2 + N^1(s) z_1^2 + W^1(s) v_1^2 + M^2(s) y_2^2 + N^2(s) z_2^2 + W^2(s) v_2^2 \right].$$

According Theorem 1, we can get

$$0 = \mathrm{E}' \left[C^1(s) p_s + F^1(s) q_s + W^1(s) \hat{u}_s \right] + \mathrm{E}^* \left[C^2(s) p_s^* + F^2(s) q_s^* + W^2(s) \hat{u}_s \right], \tag{11}$$

Where

$$\mathrm{E}^*\left[C^1(s)p_s\right]=\int_\Omega C^1(s,\omega,\omega')p(s,\omega)P(d\omega'),$$

$$\mathrm{E}^*\left[W^2(s)\hat{u}(s)\right]=\int_\Omega W^2(s,\omega^*,\omega)u(s,\omega)P(d\omega^*),$$

$$\mathrm{E}^*\left[C^2(s)p_s^*\right]=\int_\Omega C^2(s,\omega^*,\omega)p(s,\omega^*)P(d\omega^*),$$

And

$$p(t)=\mathrm{E}'\left[Q_0^1\hat{Y}_{0,\omega}\right]+\mathrm{E}^*\left[Q_0^2\hat{Y}_{0,\omega}\right]+\int_0^t F_2(s,p(s),q(s))ds +\int_0^t G_2(s,p(s),q(s))\overleftarrow{d}W(s)-\int_0^t q(s)\overleftarrow{d}B(s), \tag{12}$$

And

$$F_2(s,p(s),q(s))=\mathrm{E}'\left[A^1(s)p(s)+D^1(s)q(s)+M^1(s)\right]+\mathrm{E}^*\left[A^2(s)p^*(s)+D^2(s)q^*(s)+M^2(s)\right]$$

$$G_2(s,p(s),q(s))=\mathrm{E}'\left[B^1(s)p(s)+E^1(s)q(s)+N^1(s)\right]+\mathrm{E}^*\left[B^2(s)p^*(s)+E^2(s)q^*(s)+N^2(s)\right]$$

Theorem 2. Assumes $\hat{u}$ that satisfy (9), the (p, g) is the solution of equation (10), the above LQ problem have unique solution Proof.

$$J(v)-J(\hat{u})=\frac{1}{2}\mathrm{E}\int_0^T\mathrm{E}'\left[M^1(s)\left(\left|Y_{s,\omega}^v\right|^2-\left|\hat{Y}_{s,\omega}\right|^2\right)+N^1(s)\left(\left|Z_{s,\omega}^v\right|^2-\left|\hat{Z}_{s,\omega}\right|^2\right)\right]ds$$

$$+\frac{1}{2}\mathrm{E}\int_0^T\mathrm{E}'\left[W^1(s)\left(\left|v_{s,\omega}\right|^2-\left|\hat{u}_{s,\omega}\right|^2\right)+M^2(s)\left(\left|Y_{s,\omega'}^v\right|^2-\left|\hat{Y}_{s,\omega'}\right|^2\right)\right]ds$$

$$+\frac{1}{2}\mathrm{E}\int_0^T\mathrm{E}'\left[N^2(s)\left(\left|Z_{s,\omega'}^v\right|^2-\left|\hat{Z}_{s,\omega'}\right|^2\right)+W^2(s)\left(\left|v_{s,\omega'}\right|^2-\left|\hat{u}_{s,\omega'}\right|^2\right)\right]ds +\frac{1}{2}\mathrm{E}\mathrm{E}'\left[Q_0^1\left(\left|Y_{0,\omega}^v\right|^2-\left|\hat{Y}_{0,\omega}\right|^2\right)+Q_0^2\left(\left|Y_{0,\omega'}^v\right|^2-\left|\hat{Y}_{0,\omega'}\right|^2\right)\right]$$

$$\geq\mathrm{E}\int_0^T\mathrm{E}'\left[M^1(s)\hat{Y}_{s,\omega}\left(Y_{s,\omega}^v-\hat{Y}_{s,\omega}\right)+N^1(s)\hat{Z}_{s,\omega}\left(Z_{s,\omega}^v-\hat{Z}_{s,\omega}\right)\right]ds +\mathrm{E}\int_0^T\mathrm{E}'\left[W^1(s)\hat{u}_{s,\omega}\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)+M^2(s)\hat{Y}_{s,\omega'}\left(Y_{s,\omega'}^v-\hat{Y}_{s,\omega'}\right)\right]ds$$

$$+\mathrm{E}\int_0^T\mathrm{E}'\left[N^2(s)\hat{Z}_{s,\omega'}\left(Z_{s,\omega'}^v-\hat{Z}_{s,\omega'}\right)+W^2(s)\hat{u}_{s,\omega'}\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)\right]ds +\mathrm{E}\mathrm{E}'\left[Q_0^1\hat{Y}_{0,\omega}\left(Y_{0,\omega}^v-\hat{Y}_{0,\omega}\right)+Q_0^2\hat{Y}_{0,\omega'}\left(Y_{0,\omega'}^v-\hat{Y}_{0,\omega'}\right)\right]$$

Applying $It\hat{o}$ formulas to $p_{s,w}\left(Y_{s,\omega}^v-\hat{Y}_{s,\omega}\right)$ on $[0,T]$

$$\mathrm{E}\mathrm{E}'\left[Q_0^1\hat{Y}_{0,\omega}\left(Y_{0,\omega}^v-\hat{Y}_{0,\omega}\right)+Q_0^2\hat{Y}_{0,\omega'}\left(Y_{0,\omega'}^v-\hat{Y}_{0,\omega'}\right)\right]=\mathrm{E}\int_0^T\mathrm{E}'\left[C^1(s)p_{s,\omega}+F^1(s)q_{s,\omega}\right]\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)ds$$

$$-\mathrm{E}\int_0^T\mathrm{E}'\left[M^1(s)\hat{Y}_{s,\omega}\left(Y_{s,\omega}^v-\hat{Y}_{s,\omega}\right)+N^1(s)\hat{Z}_{s,\omega}\left(Z_{s,\omega}^v-\hat{Z}_{s,\omega}\right)\right]ds +\mathrm{E}\int_0^T\mathrm{E}'\left[C^2(s)\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)p_{s,\omega}+F^2(s)\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)q_{s,\omega}\right]ds$$

$$-\mathrm{E}\int_0^T\mathrm{E}'\left[M^2(s)\hat{Y}_{s,\omega'}\left(Y_{s,\omega'}^v-\hat{Y}_{s,\omega'}\right)+N^2(s)\hat{Z}_{s,\omega'}\left(Z_{s,\omega'}^v-\hat{Z}_{s,\omega'}\right)\right]ds.$$

We can get

$$J(v)-J(\hat{u})\geq\mathrm{E}\int_0^T\mathrm{E}'\left[C^1(s)p_{s,\omega}+F^1(s)q_{s,\omega}\right]\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)ds$$

$$+\mathrm{E}\int_0^T\mathrm{E}'\left[W^1(s)u_{s,\omega}\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)+W^2(s)u_{s,\omega'}\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)\right]ds+\mathrm{E}\int_0^T\mathrm{E}'\left[C^2(s)\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)p_{s,\omega}+F^2(s)\left(v_{s,\omega'}-\hat{u}_{s,\omega'}\right)q_{s,\omega}\right]ds$$

$$=\mathrm{E}\int_0^T\mathrm{E}'\left[C^1(s)p_{s,\omega}+F^1(s)q_{s,\omega}\right]\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)ds+\mathrm{E}\int_0^T\left[\left[\mathrm{E}'W^1(s)\right]v_{s,\omega}\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)+\mathrm{E}^*\left[W^2(s)\hat{u}_S\right]\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)\right]ds$$

$$+\mathrm{E}\int_0^T\mathrm{E}^*\left[C^2(s)p_s^*+F^2(s)q_s^*\right]\left(v_{s,\omega}-\hat{u}_{s,\omega}\right)ds$$

Theorem 2 is verified.

4. Summary

Theorem 1,(stochastic maximum principle) Assumed $\left(\widehat{Y}(\cdot), \widehat{Z}(\cdot), \widehat{u}(\cdot)\right)$ is the optimal trajectory of the control problem{(1),(2)}, $\forall v \in U, a.e. \quad t \in [0, T], a.s.$

$$\left[\mathrm{E}'\widehat{H}_v\left(t, \omega, \omega'\right) + \mathrm{E}^*H_{v'}\left(t, \omega^*, \omega\right)\right] \cdot \left(v - \widehat{u}(t)\right) \geq 0 \qquad (13)$$

where

$$\widehat{H}\left(t, \omega, \omega'\right) = H\left(t, \omega, \omega', \widehat{Y}(t, \omega), \widehat{Z}(t, \omega), \widehat{u}(t, \omega), \widehat{Y}(t, \omega'), \widehat{Z}(t, \omega'), \widehat{u}(t, \omega'), p(t, \omega), q(t, \omega)\right) \qquad (14)$$

Theorem 2. Assumes $\widehat{u}$ that satisfy (9), the (p, g) is the solution of equation (10), the above LQ problem have unique solution.

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