

# Oscillation of Second Order Nonlinear Neutral Differential Equations

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**Abstract:** The oscillation criteria are investigated for all solutions of second order nonlinear neutral delay differential equations. Our results extend and improve some results well known in the literature see ([14] theorem 3.2.1 and theorem 3.2.2 pp.385-388). Some examples are given to illustrate our main results.

**Keywords:** Oscillation, Neutral Differential Equations

## 1. Introduction

In this paper we consider the second order neutral differential equations of the form

$$[y(t) + \delta p(t)y(\tau(t))]' + q(t)f(y(\sigma(t))) = 0, t \geq t_0, \delta = \pm 1 \quad (1.1)$$

Under the following assumptions:

(A<sub>1</sub>)  $p(t), q(t) \in C([t_0, \infty); R^+), p(t) > 0$ .

(A<sub>2</sub>)  $\tau(t), \sigma(t) \in C([t_0, \infty); R), \lim_{t \rightarrow \infty} \tau(t) = \infty, \lim_{t \rightarrow \infty} \sigma(t) = \infty, \tau(t) < t$  is increasing function. and  $\tau^k(t) = \tau^{k-1}(\tau(t)), k = 1, 2, \dots, n, \tau^{k-1}(t) \geq t_0$ .

(A<sub>3</sub>)  $f(\omega) \in C(R; R), \omega f(\omega) > 0$  for  $\omega \neq 0$ ,  $f(uv) \geq f(u)f(v)$ ,  $uv > 0$  and  $|f(u)| \geq \beta|u|, \beta > 0$ .

By a solution of (1.1), we mean a function  $y(t)$  such that  $y(t) + \delta p(t)y(\tau(t))$  is twice continuously differentiable and  $y(t)$  satisfies (1.1) on  $[t_0, \infty)$ .

A solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory, that is a solution  $y(t)$  is nonoscillatory if and only if  $y(t)$  is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory.

There has been a considerable investigation of the oscillations of second order neutral differential equations. For typical results we refer to the papers [1,3,5-7,10,12-14]. The results in this paper extend and improve some results well known in ([14], theorem 3.2.1 and theorem 3.2.2 pp.385-388). Also, we show that (A<sub>3</sub>) implies that  $f(t)$  is nondecreasing while in [13] they assume (A<sub>3</sub>) holds in addition to  $f(t)$  is nondecreasing. Some examples are given to illustrate the

obtained results.

## 2. Main Results

In this section we present our main results, the following lemma shows that (A<sub>3</sub>) implies  $f(u)$  is nondecreasing.

**Lemma 2.1** Suppose that  $f \in C(R, R), uf(u) \geq 0$  for  $u \neq 0, \frac{f(u)}{u} \geq \beta > 0$  and  $f(uv) \geq f(u)f(v)$  for  $uv > 0$ . Then  $f$  is nondecreasing function.

**Proof:** Assume that  $u, v > 0$  then  $f(u), f(v) > 0$ . We have  $f(uv) \geq f(u)f(v), f(u) \geq \beta u$ . Then

$$f(uv) \geq \beta uf(v)$$

$$f(uv) \geq \beta vf(u)$$

$$0 \geq \beta[uf(v) - vf(u)]$$

$$vf(u) \geq uf(v)$$

$$\frac{vf(u)}{uf(u)} \geq \frac{uf(v)}{uf(u)}$$

$$\frac{f(v)}{f(u)} \leq \frac{v}{u} \quad (2.1)$$

If  $v \leq u$  then (2.1) implies that  $f(v) \leq f(u)$

Hence  $f$  is nondecreasing function.  $\square$

**Lemma 2.2.** [7] Suppose that  $f(t); g(t) : R^+ \rightarrow R^+; g(t) < t, \lim_{t \rightarrow \infty} g(t) = \infty$  and

$$\liminf_{t \rightarrow \infty} \int_{g(t)}^t f(s) ds > \frac{1}{e}$$

Then the inequality  $y'(t) + f(t)y(g(t)) \leq 0$  cannot have eventually positive solution, and the inequality  $y'(t) + f(t)y(g(t)) \geq 0$  cannot have eventually negative solution.

In the following theorem we investigate eq. (1.1) when  $\delta = 1$ .

Theorem 2.3 Assume that  $(A_1) - (A_3)$  hold,  $\sigma(t) \geq t$ , and

$$\limsup_{t \rightarrow \infty} \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t))) < 1 \quad (2.2)$$

For some positive odd integer  $n$  and there exists a function  $h(t) \in C^1[t_0, \infty), R^+]$  such that

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ \beta h(s) q(s) f(\gamma(s)) - \frac{[h'(s)]^2}{4h(s)} \right] ds = \infty, \quad T \geq t_0 \quad (2.3)$$

where  $\gamma(t) = 1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t)))$ . Then every solution of (1.1) oscillates.

Proof. Assume for the sake of contradiction that (1.1) has a nonoscillatory solution, let  $y(t)$  be eventually positive solution of (1.1) and there exists a positive odd integer  $n$  such that  $y(\tau^i(t)) > 0, i = 0, 1, 2, \dots, n, y(\sigma(t)) > 0$ , for  $t \geq t_1 \geq t_0$ . Let

$$z(t) = y(t) + p(t)y(\tau(t)) \quad (2.4)$$

From (1.1) and (2.4) it follows that

$$z''(t) = -q(t)f(y(\sigma(t))) \leq 0, t \geq t_1, \quad (2.5)$$

Since  $z(t) > 0$  then it follows that  $z'(t) > 0, t \geq t_2 \geq t_1$ . From (2.4) we obtain

$$\begin{aligned} y(t) &= z(t) - p(t)y(\tau(t)) \\ &= z(t) - p(t)[z(\tau(t)) - p(\tau(t))y(\tau^2(t))] \\ &= z(t) - p(t)[z(\tau(t)) - p(\tau(t))[z(\tau^2(t)) - p(\tau^2(t))y(\tau^3(t))]] \\ &= \\ &= z(t) + \prod_{i=0}^n p(\tau^i(t))y(\tau^{n+1}(t)) + \\ &\quad \sum_{i=1}^{\frac{n-1}{2}} \prod_{k=0}^{2i-1} p(\tau^k(t))z(\tau^{2i}(t)) - \\ &\quad \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(t))z(\tau^{2i-1}(t)) \\ y(t) &\geq z(t) - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(t))z(\tau^{2i-1}(t)) \\ &\geq \left[ 1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(t)) \right] z(t) \\ y(\sigma(t)) &\geq \left[ 1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t))) \right] z(\sigma(t)) \end{aligned} \quad (2.6)$$

By  $(A_3)$  and Lemma 2.1 we get

$$\begin{aligned} f(y(\sigma(t))) &\geq f\left(1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t))) z(\sigma(t))\right) \\ &\geq f\left(1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t)))\right) f(z(\sigma(t))) \end{aligned}$$

$$f(y(\sigma(t))) \geq f\left(1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t)))\right) \beta z(\sigma(t)) \quad (2.7)$$

Substituting (2.7) into (2.5) we obtain

$$z''(t) + \beta q(t) \left[ f\left(1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t)))\right) \right] z(\sigma(t)) \leq 0 \quad (2.8)$$

Define

$$w(t) = \frac{h(t)z'(t)}{z(\sigma(t))} \quad (2.9)$$

It is obvious that  $w(t) > 0$ . Let  $\gamma(t) = 1 - \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t)))$  then differentiate (2.9) and using (2.8) it follows that

$$\frac{h(t)w'(t) - w(t)h'(t)}{(h(t))^2} z(t) + \frac{w(t)}{h(t)} z'(t) + \beta q(t)f(\gamma(t))z(\sigma(t)) \leq 0 \quad (2.8)$$

$$w'(t) - \frac{h'(t)}{h(t)} w(t) + \frac{z'(t)}{z(t)} w(t) + \beta h(t)q(t)f(\gamma(t)) \leq 0$$

Then for  $t \geq t_3 (= T) \geq t_2$

$$\begin{aligned} w'(t) &\leq \frac{h'(t)}{h(t)} w(t) - \frac{z'(t)}{z(t)} w(t) - \beta h(t)q(t)f(\gamma(t)) \\ &= \frac{h'(t)}{h(t)} w(t) - \frac{1}{h(t)} w^2(t) - \beta h(t)q(t)f(\gamma(t)) \\ &= -\beta h(t)q(t)f(\gamma(t)) - \left[ \frac{1}{\sqrt{h(t)}} w(t) - \frac{h'(t)}{2\sqrt{h(t)}} \right]^2 + \frac{(h'(t))^2}{4h(t)} \\ &\leq -\beta h(t)q(t)f(\gamma(t)) + \frac{(h'(t))^2}{4h(t)} \\ w'(t) &\leq -\left[ \beta h(t)q(t)f(\gamma(t)) - \frac{(h'(t))^2}{4h(t)} \right] \end{aligned}$$

Integrating the last inequality from  $T$  to  $t$ , we get

$$w(t) \leq w(T) - \int_T^t \left[ \beta h(s)q(s)f(\gamma(s)) - \frac{(h'(s))^2}{4h(s)} \right] ds,$$

as  $t \rightarrow \infty$  the last inequality leads to  $w(t) \rightarrow -\infty$ , which is a contradiction.  $\square$

Example 1 Consider the neutral differential equation

$$\left[ y(t) + e^{-t}y\left(t - \frac{3\pi}{2}\right) \right]'' + e^{-\frac{\pi}{2}}f(y(t + 2\pi)) = 0, t \geq 1, \quad (E1)$$

$$\tau(t) = t - \frac{3\pi}{2}, \sigma(t) = t + 2\pi, p(t) = e^{-t},$$

$$f(y(t + 2\pi)) = y(t + 2\pi) \left[ e^{\frac{\pi}{2}} - 2e^{-t+\frac{\pi}{2}} \right], q(t) = e^{-\frac{\pi}{2}}$$

one can find that all conditions of theorem 2.3 are hold

Let  $n = 3, \beta = 3, h(t) = c > 0$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{i=1}^{\frac{n+1}{2}} \prod_{k=1}^{2i-1} p(\tau^{k-1}(\sigma(t))) &= \\ = \lim_{t \rightarrow \infty} [e^{-t-2\pi} + e^{-3t-2\pi}] &= 0 \end{aligned}$$

$$\limsup_{t \rightarrow \infty} \int_T^t \left[ \beta h(s) q(s) f(\gamma(s)) - \frac{[h'(s)]^2}{4ah(s)} \right] ds =$$

$$= \lim_{t \rightarrow \infty} \int_T^t \frac{3c}{2} e^{-\frac{\pi}{2}} f(\gamma(s)) ds = \infty$$

so every solution of eq.(E1) oscillates, for instance  $y(t) = \sin t$  is such solution.

The following theorem investigate equation (1.1) when  $\delta = -1$ .

**Theorem 2.4** Assume  $(A_1) - (A_3)$  hold,  $p(t) \leq 1$ ,  $\tau^n(\sigma(t)) \geq t$ ,  $\tau^{-n}(\sigma(\alpha(t))) \leq t$ ,  $\alpha(t) > t$  and there exists a function  $h(t) \in C^1[t_0, \infty), R^+$  such that

$$\int_T^\infty [\beta h(s) q(s) f(\gamma_1(s)) - \frac{(h'(s))^2}{4h(s)}] ds = \infty, T \geq t_0, \quad (2.10)$$

$$\liminf_{t \rightarrow \infty} \int_{\tau^{-n}(\sigma(\alpha(t)))}^t q(v) f(\theta(v)) dv ds > \frac{1}{\beta e} \quad (2.11)$$

where  $\gamma_1(t) = 1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))$ ,  $\theta(t) = \frac{1}{\sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^{-k}(\sigma(t)))}$ . Then every solution of (1.1) is oscillatory.

**Proof :** Assume for the sake of contradiction that (1.1) has a eventually positive solution  $y(t)$ , then there exists  $t_1 \geq t_0$  such that for  $t \geq t_1$ ,  $y(\tau^i(t)) > 0, i = 1, 2, \dots, n$ , where  $n$  is positive integer and  $y(\sigma(t)) > 0$ . Let

$$z(t) = y(t) - p(t)y(\tau(t)) \quad (2.13)$$

From equation (1.1) and (2.13) we get

$$z''(t) = -q(t)f(y(\sigma(t))) \leq 0, t \geq t_1 \quad (2.14)$$

hence  $z'(t), z(t)$  are monotone functions, we claim that  $z'(t) > 0$ , for  $t \geq t_2 \geq t_1$  otherwise  $z'(t) < 0$  leads to  $z(t) < 0$  for  $t \geq t_2 \geq t_1$  and  $\lim_{t \rightarrow \infty} z(t) = -\infty$  it follows that  $y(t) < p(t)y(\tau(t)) \leq y(\tau(t))$  then  $y(t)$  is bounded.

On the other hand  $z(t) > -p(t)y(\tau(t)) \geq -y(\tau(t))$

Then  $\lim_{t \rightarrow \infty} y(t) = \infty$ , leads to a contradiction. Therefore  $z'(t) > 0$  for  $t \geq t_2 \geq t_1$ , there are two cases for  $z(t)$ :

(a)  $z(t) > 0$  for  $t \geq t_2 \geq t_1$ ; (b)  $z(t) < 0$  for  $t \geq t_2 \geq t_1$

Case (a)  $z(t) > 0$ ,  $z'(t) > 0$  and  $z''(t) \leq 0$

From (2.13) it follows that

$$y(t) = z(t) + p(t)y(\tau(t)) =$$

$$= z(t) + p(t)[z(\tau(t)) + p(\tau(t))y(\tau^2(t))]$$

$$= z(t) + p(t)z(\tau(t)) + p(t)p(\tau(t))[z(\tau^2(t))$$

$$+ p(\tau^2(t))y(\tau^3(t))]$$

$$y(t) =$$

$$z(t) + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(t)) z(\tau^i(t)) + \prod_{i=0}^n p(\tau^i(t)) y(\tau^{n+1}(t))$$

$$\geq z(t) + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(t)) z(\tau^i(t))$$

$$\geq [1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(t))] z(\tau^n(t))$$

Then there exists  $t_3 (= T) \geq t_2$  such that for  $t \geq T$

$$y(\sigma(t)) \geq [1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))] z(\tau^n(\sigma(t))) \quad (2.15)$$

Then by using  $(A_3)$  and Lemma 2.1 we have

$$f(y(\sigma(t))) \geq f[1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))] f(z(\tau^n(\sigma(t))))$$

$$\geq \beta f(1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))) z(\tau^n(\sigma(t))) \quad (2.16)$$

By substituting (2.16) in (1.1) we obtain

$$z''(t) + \beta q(t) f(1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))) z(\tau^n(\sigma(t))) \leq 0 \quad (2.17)$$

Define

$$w(t) = \frac{h(t)z'(t)}{z(t)}$$

Then  $w(t) > 0$ , moreover

$$w'(t) + \frac{z'(t)}{z(t)} w(t) - \frac{h'(t)}{h(t)} w(t) +$$

$$\beta h(t) q(t) f(\gamma_1(t)) \frac{z(\tau^n(\sigma(t)))}{z(t)} \leq 0$$

Where  $\gamma_1(t) = 1 + \sum_{i=1}^n \prod_{k=0}^{i-1} p(\tau^k(\sigma(t)))$ .

Since  $z(t) \leq z(\tau^n(\sigma(t)))$  then

$$w'(t) \leq -\frac{z'(t)}{z(t)} w(t) + \frac{h'(t)}{h(t)} w(t) - \beta h(t) q(t) f(\gamma_1(t))$$

$$= -\frac{1}{h(t)} w^2(t) + \frac{h'(t)}{h(t)} w(t) - \beta h(t) q(t) f(\gamma_1(t))$$

$$= -\left[ \frac{1}{\sqrt{h(t)}} w(t) - \frac{h'(t)}{2\sqrt{h(t)}} \right]^2 + \frac{(h'(t))^2}{4h(t)}$$

$$- \beta h(t) q(t) f(\gamma_1(t))$$

Thus

$$w'(t) \leq \frac{(h'(t))^2}{4h(t)} - \beta h(t) q(t) f(\gamma_1(t))$$

Integrating the last inequality from  $T$  to  $t$  we get

$$w(t) \leq w(T) - \int_T^t [\beta h(s) q(s) f(\gamma_1(s)) - \frac{(h'(s))^2}{4h(s)}] ds$$

as  $t \rightarrow \infty$ , and by (2.11) we deduce that  $w(t) \rightarrow -\infty$  this is a contradiction.

Case (b)  $z(t) < 0$ ,  $z'(t) > 0$  and  $z''(t) \leq 0$

From (2.13) it follows that  $y(\tau(t)) = \frac{1}{p(t)} (y(t) - z(t))$

Then  $y(t) = \frac{1}{p(\tau^{-1}(t))} y(\tau^{-1}(t)) - \frac{1}{p(\tau^{-1}(t))} z(\tau^{-1}(t))$

$$y(t) = \frac{1}{p(\tau^{-1}(t))} \left[ \frac{1}{p(\tau^{-2}(t))} y(\tau^{-2}(t)) \right.$$

$$\left. - \frac{1}{p(\tau^{-2}(t))} z(\tau^{-2}(t)) \right] - \frac{1}{p(\tau^{-1}(t))} z(\tau^{-1}(t))$$

$$y(t) = \frac{1}{\prod_{i=1}^n p(\tau^{-i}(t))} y(\tau^{-n}(t)) - \sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(t))} z(\tau^{-i}(t))$$

$$\geq - \sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(t))} z(\tau^{-i}(t))$$

$$\geq -\sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(t))} z(\tau^{-n}(t))$$

Then there exists  $t_3(=T) \geq t_2$  such that for  $t \geq T$

$$y(\sigma(t)) \geq -\sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(\sigma(t)))} z(\tau^{-n}(\sigma(t))) \quad (2.18)$$

Then by using  $(A_3)$  and Lemma 2.1 we have

$$\begin{aligned} f(y(\sigma(t))) &\geq f\left(\sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(\sigma(t)))}\right) f(-z(\tau^{-n}(\sigma(t)))) \\ &\geq -\beta f\left(\sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(\sigma(t)))}\right) z(\tau^{-n}(\sigma(t))) \end{aligned} \quad (2.19)$$

By substituting (2.19) in (2.14) we obtain

$$z''(t) - \beta q(t) f\left(\sum_{i=1}^n \prod_{k=1}^i \frac{1}{p(\tau^{-k}(\sigma(t)))}\right) z(\tau^{-n}(\sigma(t))) \leq 0 \quad (2.20)$$

Integrating (2.20) from  $t$  to  $\alpha(t)$  where  $\alpha(t) > t$  is continuous function, we get

$$z'(t) + \beta z(\tau^{-n}(\sigma(\alpha(t)))) \int_t^{\alpha(t)} q(s) f(\theta(s)) ds \geq 0 \quad (2.21)$$

where  $\theta(t) = \sum_{i=1}^n \prod_{k=0}^{i-1} \frac{1}{p(\tau^{-k}(\sigma(t)))}$ . By lemma 2.2 and condition (2.13) it follows that eq(2.21) cannot has eventually negative solution, which is a contradiction.

Then every solutions of (1.1) is oscillatory.  $\square$

Example 2 Consider the neutral differential equation

$$\begin{aligned} [y(t) - e^{-2\pi}(3 - \sin t) y(t - 2\pi)]'' \\ + 2e^{-\frac{9\pi}{2}}(\cos t + 2) y\left(t - \frac{9\pi}{2}\right) = 0, t \geq T \end{aligned} \quad (E2)$$

one can find that all conditions of theorem 2.4(or corollary 2.5) are hold, to see that:

$$\text{Let } n = 2, \beta = \frac{1}{2}, h(t) = c > 0,$$

$$f(\gamma_1(t)) = 1 + e^{-2\pi}(3 + \cos t) + e^{-2\pi}(3 + \cos t)^2$$

$$f(\theta(t)) = \frac{1}{e^{-2\pi}(3 + \cos t)} + \frac{1}{e^{-4\pi}(3 + \cos t)^2}$$

$$\lim_{t \rightarrow \infty} \int_T^t c e^{-\frac{9\pi}{2}}(\cos s + 2) f(\gamma_1(s)) ds = \infty$$

$$\lim_{t \rightarrow \infty} \int_T^t c e^{-\frac{9\pi}{2}}(\cos s + 2) f(\theta(s)) ds = \infty$$

so every solution of eq.(E2) oscillate for instance  $y(t) = e^t \cos t$  is such a solution.

### 3. Conclusions

As mention in the abstract the results in this paper extended and improved theorem 3.2.1 and theorem 3.2.2 [4]

The authors in[13] assume  $(A_3)$  holds in addition to  $f(u)$  is nondecreasing while we show that  $(A_3)$  implies  $f(u)$  is nondecreasing. Moreover we can establish other results by assuming  $f(uv) \leq f(u)f(v)$  and  $f(u) \leq \beta u$  also implies

that  $f(u)$  is nondecreasing in similar way.

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