

Certain ranks of the quotient semigroup S/ρ and prime subsets of a semigroup

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Abstract: In this paper we find a relationship between $r_2(S)$ and $r_2(S/\rho)$ where S is a finite semigroup, ρ is a congruence on S and S/ρ is the quotient semigroup ($|S| \geq 2$ and $|S/\rho| \geq 2$). We also determine $r_5(S/\rho)$ under certain conditions. Moreover we find prime subsets of S/ρ .

Keywords: Rank, Prime Subset, Congruence, Independent Set

1. Introduction

In algebra many studies have been constructed based upon the notion "rank". Problems about rank of a group, rank of an algebra and rank of a semigroup have been widely studied in many articles for years.

In [3] the notion of rank of a semigroup and ranks of some certain semigroups are determined. In [1] and [2] the small rank, lower rank, intermediate rank, upper rank and large rank (i.e. $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$) are defined. In the following definitions these ranks are given:

$r_1(S) = \max \{k: \text{every subset of cardinality } k \text{ is independent}\}$
 $r_2(S) = \min \{k: \text{there exists a subset } U \text{ of cardinality } k \text{ such that } U \text{ generates } S\}$

$r_3(S) = \max \{k: \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ which is independent and which generates } S\}$

$r_4(S) = \max \{k: \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ which is independent}\}$

$r_5(S) = \min \{k: \text{every subset } U \text{ of cardinality } k \text{ generates } S\}$

One can see that $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$. Let S be a finite semigroup. Let $U \subseteq S$. If for all $x \in U$, $x \notin U / \{x\}$ then U is an independent set. In [4] ranks of certain semigroups are determined.

If $x, y \in S$ and $(s, t) \in \rho$ and $(xs, yt) \in \rho$ then ρ is a congruence on S . If ρ is a congruence on S then S/ρ is a subsemigroup of $S \times S$. $S/\rho = \{x\rho: x \in S\}$. If ρ is a congruence on S then S/ρ is a semigroup with the operation $(x\rho) \cdot (y\rho) = (xy)\rho$. S/ρ is the quotient semigroup.

In [5] it is shown that if $T=S$ or $T= S/\rho$ then $r_i(T) \leq r_i(\rho)$ for $i=2, 3, 4, 5$. In general it is not equal.

In this paper in Section 2 we determine the relationship between $r_2(S)$ and $r_2(S/\rho)$. We also obtain some results about $r_3(S)$ and $r_5(S/\rho)$.

Let S be a finite multiplicative semigroup. Let $U \subseteq S$. If for all $a, b \in S$, $ab \in U$ implies $a \in U$ or $b \in U$ then U is called a prime subset of S . In Section 3 we obtain some results about the prime subsets of the quotient semigroup S/ρ . We also determine the union and intersection of two prime subsets of a finite semigroup S .

2. Some Ranks of Quotient Semigroup

We find a relationship between $r_2(S)$ and $r_2(S/\rho)$ where S is a finite semigroup and ρ is a congruence on S ($|S| \geq 2$ and $|S/\rho| \geq 2$). We also show that $r_5\left(\frac{S}{\rho}\right) = \left\lfloor \frac{|S|}{|\rho|} \right\rfloor$ when $r_5(S) = |S|$.

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Lemma 2.1: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S . Assume that $|S| \geq 2$ and $|S/\rho| \geq 2$. Then $r_2(S/\rho) \leq r_2(S)$.

Proof: Let S be a finite semigroup and $|S| \geq 2$. Let ρ be a congruence on S . Let $|\rho| \geq 2$ and $|S/\rho| \geq 2$. Let $r_2(S) = m$. Then S has a minimal generating set $A = \{x_1, x_2, \dots, x_m\}$. Assume that $r_2(S/\rho) = m'$. Assume that $A' = \{x_1\rho, x_2\rho, \dots, x_m\rho\}$. Let $s\rho \in S/\rho$. Since $s \in S$ and S is a generating set $s = x_{i_1}x_{i_2} \dots x_{i_k}$ ($1 \leq i_1, i_2, \dots, i_k \leq m$). Then $s\rho = (x_{i_1}\rho)(x_{i_2}\rho) \dots (x_{i_k}\rho)$. Thus $A' = \{x_1\rho, x_2\rho, \dots, x_m\rho\}$ is a generating set for S/ρ . Since $r_2(S/\rho) = m'$ we have $m' \leq m$.

By the help of the next theorem we give a relationship

between irreducible elements of S and irreducible elements of S/ρ .

Theorem 2.2: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S . Assume that $|\mathcal{I}_S| \geq 2$ and $|\mathcal{I}_{S/\rho}| \geq 2$. If there is an irreducible element in S , there is also an irreducible element in S/ρ .

Proof: Let $x \in S$ be an irreducible element. We determine whether $x\rho \in S/\rho$ is irreducible or not. Let $x\rho = y\rho \cdot z\rho$. Let $y\rho = \{y_1, y_2, \dots, y_n\}$ and $z\rho = \{z_1, z_2, \dots, z_m\}$. Let $y\rho \cdot z\rho = yz\rho = \{y_1z_1, y_1z_2, \dots, y_1z_m, y_2z_1, y_2z_2, \dots, y_2z_m, \dots, y_nz_1, y_nz_2, \dots, y_nz_m\}$. Since $x\rho = yz\rho$, $x\rho = yz\rho$ and $x \in x\rho$ there is $\exists k, l$ such that $x = y_kz_l$ ($1 \leq k \leq n, 1 \leq l \leq m$). Since x is an irreducible element we have $x = y_k$ or $x = z_l$. Thus $x \in y\rho$ or $x \in z\rho$. If $x \in y\rho$ then since $x \in x\rho$ we have $x\rho \cap y\rho \neq \emptyset$. We obtain $x\rho = y\rho$. By the same way, if $x \in z\rho$ and $x \in x\rho$ we have $x\rho \cap z\rho \neq \emptyset$. Then $x\rho = z\rho$. As a result $x\rho = y\rho$ or $x\rho = z\rho$. As a result $x\rho \in S/\rho$ is an irreducible element.

In the following Theorem we give the relationship between $r_3(S)$ and an irreducible element of a finite semigroup.

Theorem 2.3: (see [2], Corollary 4) Let S be a finite semigroup. S contains an irreducible element if and only if $r_3(S) = |S|$.

Lemma 2.4: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S . Assume that $|\mathcal{I}_S| \geq 2$ and $|\mathcal{I}_{S/\rho}| \geq 2$. If $r_3(S) = |\mathcal{I}_S|$ then $r_3(S/\rho) = |\mathcal{I}_{S/\rho}|$.

Proof: By the previous theorem we have S has an irreducible element. So $r_3(S) = |S|$. (see [2]). Then by the previous theorem S/ρ has an irreducible element. We obtain

$$r_3\left(\frac{S}{\rho}\right) = \left|\frac{S}{\rho}\right|.$$

3. Prime Subsets of a Semigroup

In [6] Kumar J. and Krishna, K.V. have defined the prime subsets of a semigroup as follows.

Definition 3.1: Let S be a finite multiplicative semigroup. Let $U \subseteq S$. If for all $a, b \in S, ab \in U$ implies $a \in U$ or $b \in U$ then U is called a prime subset of S .

In the following Theorem we determine the prime subsets of S/ρ .

Theorem 3.2: Let $U \subseteq S$ be a prime subset of S . Let $U' = \{u\rho : u \in U\} \subseteq S/\rho$. Then U' is a prime subset of S/ρ .

Proof: Let $U \subseteq S$ be a prime subset of S . Then for all $a, b \in S, ab \in U$ implies $a \in U$ or $b \in U$. Let $x\rho = \{s \in S : x\rho s\}$ ($x \in S$). Let $S/\rho = \{x\rho : x \in S\}$ and $U' = \{u\rho : u \in U\}$. Assume that for $x\rho, y\rho \in S/\rho, x\rho \cdot y\rho \in U'$. Then $x\rho \cdot y\rho = xy\rho \in U'$. Thus there is some $u \in U$ such that $xy = u$. Since U is a prime subset of S we have $x \in U$ or $y \in U$. We obtain $x\rho \in U'$ or $y\rho \in U'$. So U' is a prime subset of S/ρ .

Corollary 3.3: For every prime subset $U \subseteq S$, there exists a prime subset $U' \subseteq S/\rho$.

We try to determine the union of prime subsets of a finite semigroup S . In the next theorem we examine the union of prime subsets of S .

Lemma 3.4: Let S be a finite semigroup and U_1 and U_2 be prime subsets of S . Then $U_1 \cup U_2$ is a prime subset of S .

Proof: Let $xy \in U_1 \cup U_2$. Then $xy \in U_1$ or $xy \in U_2$. Since U_1 is a prime subset of S then $x \in U_1$ or $y \in U_1$. In each cases $x \in U_1 \cup U_2$ or $y \in U_1 \cup U_2$. We obtain $U_1 \cup U_2$ is a prime subset of S .

We also determine the intersection of prime subsets of S in the following Lemma.

Lemma 3.5: Let S be a finite semigroup and U_1 and U_2 be prime subsets of S . Then $U_1 \cap U_2$ is a prime subset of S .

Proof: Let $xy \in U_1 \cap U_2$. Then $xy \in U_1$ and $xy \in U_2$. Since U_1 is a prime subset of S then $x \in U_1$ and $y \in U_1$. Also U_2 is a prime subset of S . Then we have $x \in U_2$ and $y \in U_2$. We obtain $x \in U_1 \cap U_2$ and $y \in U_1 \cap U_2$. So $U_1 \cap U_2$ is a prime subset of S .

4. Conclusion

In [5] Theorem 1 it is shown that $r_1(S) = r_1(\rho) = r_1(S/\rho)$ when ρ is not a royal semigroup. ($|\mathcal{I}_S| \geq 2$ and $|\mathcal{I}_{S/\rho}| \geq 2$) If ρ is a royal semigroup then $r_1(S) = |\mathcal{I}_S|$ and $r_1(S/\rho) = |\mathcal{I}_{S/\rho}|$. In Section 2 we show that $r_2(S/\rho) \leq r_2(S)$. Moreover we show that

$$r_3\left(\frac{S}{\rho}\right) = \left|\frac{S}{\rho}\right| \text{ when } r_3(S) = |S|.$$

Let $U \subseteq S$ be a prime subset of S . In Section 3 we determine $U' \subseteq S/\rho$ which is a prime subset of S/ρ . As a result for each prime subset of S one can find a prime subset of the quotient semigroup. So each prime subset of a semigroup S coincides with a prime subset of S/ρ .

Moreover we show that union of two prime subsets of a finite semigroup S and intersection of two prime subsets is also a prime subset of S .

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References

- [1] John Macintosh H. and Maria Isabel R. "Rank Properties in Finite Semigroups", Communications in Algebra, Vol 27 (11), pp. 5333-5347, 1999.
- [2] John Macintosh H. and Maria Isabel R. "Rank Properties in Finite Semigroups II: The Small Rank and The Large Rank", Southeast Asian Bulletin of Math., Vol 24, pp. 231-237, 2000.
- [3] John Macintosh H. Foundations of Semigroup Theory, Clarendon Press, Oxford, 1995.
- [4] Melis M. "Ranks of Certain Semigroups", Semigroup Forum, Vol 28, pp. 99-105, 2009.
- [5] Fikret K. "Relations Between Ranks of Certain Semigroups", Selçuk J. Of Applied Math., Vol. 12, No 1, pp. 123-126, 2011.
- [6] Jitender K. and Krishna K.V. "The Large Rank of a Finite Semigroup Using Prime Subsets", Semigroup Forum, DOI 10.1007/s00233-014-9577-0.