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On the symmetry classes of tensors associated with certain frobenius groups

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Abstract: In this paper, we study the symmetry classes of tensors associated with some Frobenius groups of order pq, where $q|p-1$as a subgroups of the full symmetric group on p letters. We calculate the dimension of the symmetry classes of tensor associated with some Frobenius groups and some irreducible complex characters and we obtain two useful corollary with an example.

Keywords: Symmetry Classes of Tensors, Frobenius Groups

1. Introduction

Let $n \geq 2$ and $m \geq 2$ be positive integer numbers. Denote by S_n the full symmetric group on $\{1,2,\ldots,n\}$. Let V be a unitary complex vector space of dimension m over the complex numbers field. Let $\otimes^n V$ be the n-th tensor power of V, and write

$$v^{\otimes} := v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

for the decomposable tensor product of the indicated vectors.

For $\sigma \in S_n$, there is a unique linear operator $P(\sigma^{-1})$ on $\otimes^n V$ which has the effect

$$P(\sigma^{-1})(v_1 \otimes v_2 \otimes \cdots \otimes v_n) :=$$

$$v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)},$$

for all $v_1, v_2, \ldots, v_n \in V$.

Let G be a subgroup of S_n and χ be an irreducible complex character of G and $I(G)$ be the set of all the irreducible complex characters of G. We define $T(G,\chi)$ as a linear operator on $\otimes^n V$ with the following definition

$$T(G,\chi) := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma).$$

With respect to the induced inner product in $\otimes^n V$, $T(G,\chi)$ is an orthogonal projection onto its range $V_\chi^n(G)$, which is called the symmetry class of tensors associated with G and χ, and the dimension of $V_\chi^n(G)$ is (see [3], [7])

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) m^{c(\sigma)}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ, (see [6]). It follows from the orthogonality relations for characters that $\{ T(G,\chi) \mid \chi \text{ in } I(G) \}$ is a set of annihilating idempotents which sum to the identity. With respect to the induced inner product in $\otimes^n V$, and the orthogonal relations for characters we have:

$$\otimes^n V = \sum_{\chi \in I(G)} V_\chi^n(G)$$

which is an orthogonal direct sum. Let Γ_m^n be the set of all sequences

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $1 \leq \alpha_i \leq m$, $i = 1,2,\ldots,n$.

Then the group G acts on Γ_m^n by

$$\sigma \cdot \alpha := \alpha \circ \sigma^{-1} = (\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \ldots, \alpha_{\sigma^{-1}(n)})$$

which is a composition of two functions σ^{-1} and α.

Let Δ be a system of distinct representatives of the orbits of

G acting on Γ_m^n and define:

$$\overline{\Delta}=\left\{\alpha\in\Delta \mid \sum_{\sigma\in G_\alpha}\chi(\sigma)\neq 0\right\},$$

Which $G_\alpha := \{\sigma\in G \mid \sigma\cdot\alpha=\alpha\}$ is the stabilizer of α. Let $\{e_1,e_2,...,e_m\}$ be an orthonormal basis of V and $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$, the image of $e_\alpha^{\otimes}=e_{\alpha_1}\otimes e_{\alpha_2}\otimes\cdots\otimes e_{\alpha_n}$ under $T(G,\chi)$ is denoted by

$$e_\alpha^\chi = e_{\alpha_1} * e_{\alpha_2} * \cdots * e_{\alpha_n}.$$

For $\gamma\in\overline{\Delta}, V_\gamma^\chi=\langle e_{\sigma\cdot\gamma}^\chi \mid \sigma\in G\rangle$ is called the orbital subspace of $V_\chi^n(G)$. In [3], prove that

$$\dim V_\gamma^\chi=\frac{\chi(1)}{|G_\gamma|}\sum_{\sigma\in G_\gamma}\chi(\sigma).$$

Definition 1.1.

An orthogonal basis of the form $\{e_\gamma^\chi : \gamma\in B\}$, where B is a subset of Γ_m^n, is called an orthogonal basis of decomposable symmetrized tensor for $V_\chi^n(G)$, in this case we say that $V_\chi^n(G)$ has an O-basis.

Since $V_\chi^n(G)=\sum_{\gamma\in\overline{\Delta}} V_\gamma^\chi$, so $V\chi n(G)$ has an O-basis if and only if V_γ^χ has an O-basis. Several papers are devoted in investigation of the existence of an O-basis for $V\chi n(G)$ for example [9, 1, 2, 4]. Some parts of this paper was presented as a talk in the 43rd Annual Iranian Mathematics Conference [8]. In this paper we study the symmetry classes of tensors associated with some Frobenius groups of order pq, where q|p-1. We calculate the dimension of the symmetry classes of tensor associated with some groups and we obtain two usefule corollary with an example.

2. The Group $F_{p,q}$

Let G be a group and let T be a faithful transitive G-set with each nontrivial element $g\in G$ having at most one fixed point. If no such g has a fixed point, then T is a regular G-set; if some g does have a fixed point, then G is called a Frobenius group.

Definition 2.1. [5]

If p is a prime and q|p-1, then we write Fp,q for the group of order pq with presentation:

$$F_{p,q}=\langle a,b:\ a^p=b^q=1, b^{-1}ab=a^u\rangle$$

where u is an element of order q in the product group the integer numbers modulo p, i.e., Z_p^*.

The groups $F_{p,q}$ belong to a wider class of Frobenius groups. This group is of order pq and its elements are of the form

$$F_{p,q}=\{a^ib^j : 0\le i\le p-1, 0\le j\le q-1\}.$$

It is not hard to see that $F_{p,q}$ has $\frac{p-1}{q}+q$ conjugacy classes which are

$$\{1\}, \{a^{v_i s} \mid s\in S\},\quad i=1,2,\ldots,\frac{p-1}{q},$$
$$\{a^h b^l \mid h=0,1,\ldots,p-1\},\ l=1,2,\ldots,q-1,$$

where S is the subgroup of Z_p^* generated by u, an element of order q in Z_p^* and $\left\{v_1,v_2,\ldots,v_{\frac{p-1}{q}}\right\}$ is a transversal set of S in Z_p^*, and the irreducible complex characters table of $F_{p,q}$ is:

σ	1	a^{v_i}	b^l
χ_k	1	1	ω^{lk}
φ_j	q	$\sum_{s\in S}\omega^{\frac{q}{p}sv_iv_j}$	0

where $\omega=\exp\left(\frac{2\pi i}{q}\right), 0\le k\le q-1, 1\le l\le q-1$ $1\le i,j\le\frac{p-1}{q}$.

From the above table we see that $F_{p,q}$ has q linear characters $\chi_k, 0\le k\le q-1$, and $\frac{p-1}{q}$ non-linear $\phi_j, 1\le j\le\frac{p-1}{q}$ of degree q.

Now we will embed this group in a suitable full symmetric group. We can consider

$$(1\quad 2\quad 3\quad \cdots\quad p)$$

is a cyclic permutation in S_p, it can be verified that the mapping

$$a\mapsto(1\quad 2\quad 3\quad \cdots\quad p),$$

since $b^{-1}ab=a^u\mapsto(1\quad 1+u\quad 1+2u\quad \ldots\quad 1+(p-1)u)$, so b can be map to the following permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ 1 & \overline{1+u} & \overline{1+2u} & \cdots & \overline{1+(p-1)u} \end{pmatrix}$$

where for $k=1,2,\dots,p-1$,

$$\overline{1+ku} \equiv 1+ku \pmod{p},\ 1 \leq \overline{1+ku} \leq p,$$

in S_p. Note that

$$\{1,\overline{1+u},\overline{1+2u},\dots,\overline{1+(p-1)}\}=\{1,2,\dots,p\}.$$

So $F_{p,q}$ can be consider as a permutation subgroup in S_p, i.e., $F_{p,q} \leq S_p$.

Theorem 2.2.

Let p be a prime number and $G = F_{p,p-1}$. We have only one nonlinear irreducible character ϕ, where $\phi(1)=p-1$, $\phi(a)=-1$ and $\phi(b^l)=0$, $l=1,2,\dots,p-2$. Let V be an m-dimensional inner product space, then the dimension of the symmetry classes of tensor associated with G and ϕ is

$$\dim V_\phi^p(G)=\frac{(p-1)}{p}(m^p-m).$$

Proof. We know

$$\dim V_\phi^p(F_{p,p-1})=\frac{\phi(1)}{|G|}\sum_{\sigma\in F_{p,p-1}}\phi(\sigma)m^{c(\sigma)}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ. Therefore:

$$\dim V_\phi^p(F_{p,p-1})$$

$$=\frac{p-1}{p(p-1)}\left[\phi(1)m^p+\sum_{1\neq\sigma\in\langle(1\,2\cdots p)\rangle}-1\times m^1\right]$$

$$=\frac{1}{p}\left[(p-1)m^p-(p-1)m\right]$$

$$=\frac{(p-1)}{p}(m^p-m).$$

The proof is complete.

Corollary 2.3. (little Fermat theorem)

Let p be a prime and m be a positive integer number, then $m^p \equiv m \pmod{p}$.

Proof. By using the above Theorem 2.2, since $\frac{(p-1)}{p}(m^p-m)$ is an integer number and the greatest common divided of p and $(p-1)$ is 1, so p must divided m^p-m, then $m^p \equiv m \pmod{p}$.

In general, the mapping from $F_{p,q}$ to S_P is not known, therefore we can't calculate the dimension of the symmetry classes of tensor associated with $F_{p,q}$ and linear irreducible character, in the next example we calculate these for $F_{5,4}$.

Example 2.4.

If (1 2 3 4 5) and (2 3 5 4) are cyclic permutations of order 5 and 4, respectively in S_5, then it can be verified that the mapping

$$a \mapsto (1\,2\,3\,4\,5),$$
$$b \mapsto (2\,3\,5\,4),$$

embeds $F_{5,4}$ in S_5. Now considering $F_{5,4}$ as a subgroup of S_5. By use the character table of $F_{5,4}$, we find the dimensions of the symmetry classes of tensors associated with this group as follows:

$$\dim \mathrm{V}_{\chi_0}^5=\frac{1}{20}\left[m^5+5m^3+10m^2+4m\right]$$

$$\dim \mathrm{V}_{\chi_1}^5=\frac{1}{20}\left[m^5-5m^3+4m\right]$$

$$\dim \mathrm{V}_{\chi_2}^5=\frac{1}{20}\left[m^5+5m^3-10m^2+4m\right]$$

$$\dim \mathrm{V}_{\chi_3}^5=\frac{1}{20}\left[m^5-5m^3+4m\right]$$

$$\dim \mathrm{V}_{\phi}^5=\frac{4}{5}\left[m^5-m\right].$$

Corollary 2.5.

Let m be a positive integer number, then the number 20 divided the numbers [m5 +5m3 +10m2 +4m], [m5 +5m3 -10m2 +4m] and [m5-5m3 +4m].

Proof. By using Example 2.3., since the dimension of a vector space is an integer number, so the number 20 must br divide the numbers [m5 +5m3 +10m2 +4m], [m5 +5m3 -10m2 +4m] and [m5-5m3 +4m] and the proof is complete.

In continue we try to calculate the dimension of $V_{\phi_j}^p(F_{p,q})$. By use conjugacy classes of $F_{p,q} \leq S_p$, we have $c(1)=p$ and $c(a^{v_i s})=1,\ i=1,\dots,\frac{p}{q-1},\ s\in S.$

If we consider $\omega=\exp\left(\frac{2\pi i}{q}\right)$, we have:

$$\dim V_{\phi_j}^{p}(F_{p,q}) = \frac{\phi_j(1)}{pq}\left[qm^p + \sum_{i=1}^{\frac{p-1}{q}} \sum_{s\in S} \omega^{\frac{p}{q}sv_iv_j} m^1 \right]$$

On the other hand, we know the set

$$\left\{ Sv_i : i = 1,\ldots, \frac{p-1}{q} \right\}$$

and for $j = 1,\ldots,\frac{p}{q-1}$ the set

$$\left\{ Sv_iv_j : i = 1,\ldots, \frac{p-1}{q} \right\}$$

are partitions of Z_p^*, therefore

$$\sum_{i=1}^{\frac{p-1}{q}} \sum_{s\in S} \omega^{\frac{p}{q}sv_iv_j} = \sum_{r\in Z_p^*} \omega^{\frac{p}{q}r} = \left(\omega^{\frac{p}{q}} + \omega^{2\frac{p}{q}} + \cdots + \omega^{(p-1)\frac{p}{q}} \right)$$

$$= \frac{1-\omega^{\frac{p^2}{q}}}{1-\omega^{\frac{p}{q}}} - 1.$$

Therefore

$$\dim V_{\phi_j}^{p}(F_{p,q}) = \frac{1}{p}\left[qm^p + \frac{1-\omega^{\frac{p^2}{q}}}{1-\omega^{\frac{p}{q}}} m - m \right].$$

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References

[1] M.R. Darafsheh, N.S. Poursalavati, Orthogonal basis of the symmetry classes of tensors associated with the direct product of permutation groups, Pure Mathematics and Applicans, 10 (1999), pp. 241–248.

[2] M.R. Darafsheh, N.S. Poursalavati, On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups, SUT Journal of Mathematics, 37 (2001), pp. 1–17.

[3] R. Freese, Inequalities for generalized matrix functions based on arbitrary characters, Linear Algebra, 7 (1973), pp. 337–345.

[4] R.R. Holmes, Orthogonal bases of symmetrized tensor spaces, Linear and Multilinear Algebra, 39 (1995), pp. 241–243.

[5] G. James, M. Liebeck, Representations and characters of groups, Cambridge University Press, 1993.

[6] M. Marcus, Finite Dimensional Multilinear Algebra, part I, Marcel Dekker, 1973.

[7] R. Merris, Multilinear Algebra, Gordon and Breach science publishers, 1997.

[8] N. Shajareh Poursalavati, On the symmetry classes of tensors associated with certain Frobenius groups, the 43rd Annual Iranian Mathematics Conference, 27-30 August 2012, University of Tabriz, Tabriz, Iran, (2012), pp 149-152.

[9] B.Y. Wang, M.P. Gonge, The subspace and orthonormal bases of symmetry classes of tensors, Linear and Multilinear Algebra, 30 (1991), pp. 195–204.