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On some properties of hollow and hollow dimension modules

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Abstract: No doubt, a notion of the hollow dimension modules can constitute a very important situation in the module theory. Therefore, our work presents a key role mainly in some properties and characterizations of hollow and hollow dimension module. We prove that if R be a V-ring and M is semisimple with indecomposable property, then M is hollow module. Also we study characterization the relation between lifting property and hollow-lifting module. We prove that if M is a nonzero indecomposable and lifting module over a commutative noetherian ring R then M is hollow module. Let M be an R-module and N be a submodule of M if hdim(M) = hdim($\frac{M}{N}$) + hdim(N), then M is supplemented module.

Keywords: Hollow Module, Indecomposable Module, Hollow Dimension Module, Hollow-Lifting Module

1. Definitions and Notations

Throughout this paper, all rings will have identities and all modules will be unital right modules. Let M be a module. Any small submodule K of M is denoted by ($K \ll M$). A submodule K of M is small in M if for every submodule L of $M \ni K + L = M$ then $K = M$. A module M is called lifting if, for all N submodule of M, there is a decomposition $M = H \oplus G \ni H$ submodule of N and $(N \cap H) \ll M$. We call a non-zero R-module M hollow if every proper submodule is superfluous in M. Or a module H is said to be hollow if it is an indecomposable lifting module. Therefore we can say any factor modules of hollow modules are again hollow. If M has a largest submodule, i.e. a proper submodule which contains all other proper submodules, then M is called a local module therefore it is obvious that a largest submodule has to be equal to the radical of M and in this case $R\mathrm{ad}(M) \lll M$. Let M be an R-module and N, L are submodules of M then L is a radical supplement (Rad–supplement) of N in M if $N + L = M$ and $(N \cap L) \subseteq R\mathrm{ad}(L)$. Therefore M is Rad–supplemented if every submodule in M has a Rad-supplement. A module M is called (D_1)-module if for every submodule A of M, there is a decomposition $M{=}M_1 \oplus M_2$ such that $M_1 \leq A$ and $(M_2 \cap A) \ll M$ and a module M has (D_2) property if $N \leq M \ni (\frac{M}{N})$ is isomorphic to a direct summand of M, then N is a direct summand of M. A module M is called (D_3) if for every direct summands K and L of M with $M = K + L$ and $(K \cap L)$ is a direct summand of M. Recall that a lifting module M is called quasi-discrete module if $M = M_1 + M_2 \ni M_1$, M_2 are direct summands of M and so $(M_1 \cap M_2)$ also is a direct summand of M. Therefore if we have M has (D_1)-property and (D_3)-property imples M is quasi-discrete module. We recall that a ring R is a right V-ring if and only if every simple R-module is injective, if and only if $R\mathrm{ad}(M) = 0$. Let $N \leq M$ such that N is proper and in a maximal submodule of M, then M is coatomic module; or equivalently, let N submodule of M, and radical of $(\frac{M}{N})$ equal $(\frac{M}{N})$ implies that M equal N and if every submodule of M is a direct summand in M then we can say that M is semisimple. See the following examples:

*) semisimple modules, are coatomic.
*) finitely generated modules, are coatomic module.
*) hollow modules and local modules are coatomic modules.

This paper is divided into four different sections. In section 2 we study the relations between hollow and indecomposable module and we prove that if R be a V-ring and if M is semisimple and indecomposable, then M is hollow module. In section 3 we conclude some properties of hollow module and the relation between hollow and hollow-lifting module. Finally in section 4 we conclude some properties of hollow dimension and the relation between dimension property and hollow module.

2. Hollow and Indecomposable Module

In this section, we study hollow module and we give some properties to explaine the relation between indecomposable and hollow module. A module M is called indecomposable if $M \neq 0$ and it is not a direct sum of two nonzero submodules. Let R be a Dedekind domain. Then every torsion-free divisible left R-module is Rad-supplemented.

(P_1) Let R be Neotherian ring. If M lifting module and have indecomposable property, then M is hollow module.

Also, an R-module M is called c-f-lifting if every submodule of M which contained coessentialy in a finitely generated submodule lies a bove a direct summand. Therefore if M have D_1 property (M is lifting module) then it is c-f-lifting and so:

(P_2) For every coessentialy finitely generated submodule N of $M \ni M_1 \subseteq N$, $(N \cap M_2) \ll M$, then we can get a decomposition $M = M_1 \oplus M_2$.

Theorem 2.1.

Let R be a Dedekind domain and commutative noetherian ring and let M satisfies the following statements:

1. M is hereditary module with small radical.
2. M is torsion-free divisible left R-module.
3. M is indecomposable module.

Then M is hollow module.

Proof: From condition 2 if K be the field of quotient of R. Then radical of K equal K implies K is Rad-supplemented and since M is a torsion-free divisible module, then it is a direct sum of copies of K. This means M is a Rad-supplemented module. Let B any supplement submodule of M. Thus, B is a direct summand of M. Also let A be any submodule of M and since M is a Rad-supplemented module, then there exists a submodule B of M such that $A + B = M$ and $(A \cap B)$ subset of radical of B. Now $(A \cap B)$ subset of radical of B and so subset of radical of M and it small of M therefore $(A \cap B)$ small in M:

Thus $(A \cap B)$ small in B and so M is supplemented module. Hence M is an amply supplemented module, and so is lifting and hence] M is hollow module.

Theorem 2.2.

Let R be a commutative noetherian ring and let $M = \oplus N_i$, $i= 1,..., n$, such that M satisfy the following conditions:

1- $N_1, N_2,.., N_n$, are hollow modules.
2- M has (D_3) property.
3- M is quasi-discrete module.
4- M is indecomposable module.
Then M is hollow module.

Theorem 2.3.

Let M_1 and M_2 be hollow modules with local endomorphism rings. Assume that there is no epimorphism between M_1 and M_2 such that $M = M_1 \oplus M_2$. If M is quasi–discrete; then M is hollow module.

Proof: By [10, Theorem 4.5 and Corollary 2.8].

Lemma 2.4.

[16, Lemma 3.16] Let R be a V-ring. An R-module M is lifting if and only if it is semisimple.

Theorem 2.5.

Let R be a commutative noetherian ring and let M be an R-module satisfying the following:

1- Every simple left R-module is injective.
2- M is semisimple R-module.
3- M is indecomposable R-module.
Then M is hollow module.

Proof: Since every simple module M over any ring R is injective then R is V-ring, but M is semisimple then by [Lemma 2.4] M is lifting module therefore for all N submodule of M, there is a decomposition $M = H \oplus G \ni H \leq N$ and $(N \cap H) \ll M$, but R is commutative noetherian ring with indecomposable property implies M is hollow module [Property P_1].

Theorem 2.6.

Let M be an R-module. If M is local, then M is hollow module.

Proof: By definition of local module we can say that every proper submodule N of M implies N subset of radical of M and radical of M is small in M (i.e. M has a largest proper submodule). Then N is small in M and by definition of hollow module we get the proof.

Theorem 2.7.

Let M be a nonzero module such that it satisfying the following:

1- $M \neq R\mathrm{ad}(M)$.
2- Every submodule of M lies over a summand of M.
3- M is indecomposable module.
Then M is hollow.

Proof: We must prove that M is local module. Let M be a nonzero module such that M satisfying conditions 2 and 3. Let N be a proper submodule of M. Then $M = L \oplus K$, L submodule of N and $(K \cap N)$ is small in M. Since $L \neq M$, $K=M$ and $N=(K \cap N) \subset R\mathrm{ad}(M)$ then radical of M is the unique maximal submodule of M. Thus M is local and by [Theorem 2.6] M is hollow module.

Theorem 2.8.

Let M be an R-module. If M is hollow module, then M is supplemented module .

Proof: Let M be an R-module and K be a submodule of M. Then $K+M = M$. By hypothesis, $(K \cap M)=K \ll M$. Hence M is supplemented module.

Remark 2.9.

If M has no maximal submodule ($R\mathrm{ad}(M) = M$), then all finitely generated submodules of M are small in M and this means M be finitely supplemented but not hollow.

Theorem 2.10.

Let M be (D_1)- module and let M_1 and M_2 are hollow modules, then M is a direct summand of hollow modules.

Proof: Since M is (D_1)- module then M is lifting module and so M is c-f-lifting. Now by (P_2) property M is a direct summand of hollow modules ($M = M_1 \oplus M_2$)

Corollary 2.11.

Let R be a Neotherian and right perfect ring then every indecomposable projective right R-module is hollow.

Corollary 2.12.

Let R be a ring such that it is right perfect commutative noetherian. If M is indecomposable R-module then M is hollow module.

Corollary 2.13.

Let M be projective and indecomposable module such that every factor module of M has a projective cover, then M is hollow module.

Corollary 2.14.

Let M be an R-module. If M is hollow and radical of M not equal M ($R\mathrm{ad}(M) \neq M$), then M is local.

3. Hollow and Hollow-Lifting Module

In this Section we conclude some properties to study the relation between hollow and hollow-lifting. There is an important question which is how we can use the indecomposablity with hollow-lifting to get lifting module?. Let M be a direct summand of hollow modulesH_i, i =1,...,n, and let M is (D_3)-module, then a module M is called hollow-lifting if every submodule N of M with (M / N) hollow has a coessential submodule in M that is a direct summand of M then the following are equivalent:

(1) M is hollow-lifting.

(2) M is lifting.

(3) M is quasi-discrete;.

Let $M = \sum M_i$, i =1,., n such that all M_i are hollow module and let M has (D_2) property with hollow-lifting, then M is lifting module. For an indecomposable module M, the module M is hollow-lifting if and only if M is hollow, or else M has no hollow factor modules and also semisimple modules are hollow-lifting.

Theorem 3.1.

Let $M = (M_1 \cap M_2)$ be an indecomposable module. If M is hollow-lifting then it is hollow module.

Proof: Suppose that $(M_1 \cap M_2)$ has a hollow factor module. Then there exists a proper submodule N of $(M_1 \cap M_2) \ni M_1 \cap (M_2 / N)$ is hollow. Since $(M_1 \cap M_2)$ is hollow-lifting, then there exists K a direct summand of $(M_1 \cap M_2) \ni (N / K) \ll M_1 \cap (M_2 / K)$ and since $(M_1 \cap M_2)$ is indecomposable module then $(M_1 \cap M_2)$ not equal zero and it is not a direct sum of two nonzero submodules and this means $K = 0$ and N small in $(M_1 \cap M_2)$. Therefore $(M_1 \cap M_2) = M$ itself is a hollow module.

Theorem 3.2.

Let M_1, M_2 be modules having no hollow factor modules. Then $M = M_1 \oplus M_2$ is hollow-lifting.

Proof: Suppose that M has a submodule N such that $(\frac{M}{N})$ is hollow. Since $\frac{M1+N}{N} + \frac{M2+N}{N} = \frac{M}{N}$, there exists $i = 1,\ldots, n \ni \frac{Mi+N}{N} = \frac{M}{N}$ is hollow. So M_i has a hollow factor module, but M_1 and M_2 are having no hollow factor modules (contradiction). Therefore $(M_1 + M_2)$ is hollow-lifting.

Example 3.3.

Let p be any prime integer then $(Z / p^2Z \oplus Z / p^3Z)$ is lifting, therefore it is hollow-lifting. Also $(Z / pZ) \oplus (Z / p^3Z)$ it is not lifting and this implies it is not hollow-lifting (see [13]).

Proposition 3.4.

Let $M = H_1 \oplus H_2 \oplus \ldots \oplus H_n$ such that $H_1,\ldots,H_n$ be hollow modules. Then if M is hollow-lifting module then M is lifting module.

Proof: Let $N \leq M$. If we have the projections $\pi_1 : M \to H_1$, π_2: $M \to H_2,\ldots,$ and π_n: $M \to H_n$. If $\pi_1(N) \neq H_1$, $\pi_2(N) \neq H_2,\ldots,$ and $\pi_n(N) \neq H_n$ then $N \ll M$. Suppose that $\pi_1(N) = H_1$. Then $M = N + H_2$. Therefore, (M / N) is hollow. Hence there exists a direct summand K of $M \ni K \leq N$ and $(N / K) \ll (M / K)$. Thus M is lifting.

Let M be an R-module. Then the direct summand of two hollow-lifting modules not hollow-lifting. For example, let M be the Z-module $(Z / 2Z) \oplus (Z / 8Z)$. We know that $(Z / 2Z)$ and $(Z / 8Z)$ are hollow-lifting but M not hollow-lifting. Also, when $M = M_1 \oplus M_2$ be duo module and if $M_1$and M_2 are hollow-lifting, then M is also hollow-lifting module.

Theorem 3.5.

Let M be an R-module with small radical. Let N be submodule of M such that if $N + K=M$ and N is minimal with respect to this property then M is hollow-lifting.

Proof: Firstly, we must prove that $R\mathrm{ad}(M) = 0$. Suppose $R\mathrm{ad}(M) \neq 0$, there exists an element r belong to $R\mathrm{ad}(M)$. We have R_r supplement then $R_r + H = M$ and $R_r \cap H$ is small in $R_r \ni H \leq M$. Since $r \in (M)$, $R_r \ll M$ and $H = M$ and hence $Rr \ll Rr$ but this impossible. Then $R\mathrm{ad}(M) = 0$, also $M = N + N_1$ and $(N \cap N_1) \ll N \ni N_1 \leq M$, therefore $(N \cap N_1) \subseteq R\mathrm{ad}(M) = 0$, then $(N \cap N_1) = 0$ and hence $M = N_1 \oplus N$, then M is semisimple module hence M is hollow-lifting.

Corollary 3.6.

Let M be an R-module satisfying the following conditions:

1- M coatomic module.

2- M supplemented module.

3- If N submodule of M such that N is supplement in M.

Then M is hollow-lifting.

Corolary 3.7.

Let M be an R-module. If M satisfy the following conditions:

1- M is local module.

2- M is coatomic module.

3- $Rad(M) = 0$.

Then M is hollow-lifting.

4. Finite Hollow Dimension Property

A module M is called to have finite hollow dimension if M does not contain an infinite coindependent family of submodule. The module M is said to have finite dual Goldie dimension if every coindependent family of submodules of M is finite. Not that a module M with dual Goldie dimension 1 is said to be hollow, and a cyclic hollow module is said to be local [6]. Let $N \leq R$-module M. If M has finite hollow dimension then M is weakly supplemented, therefore an R-module M with $Rad\,(M) = 0$ is weakly supplemented if and only if M is semisimple, in this case $\mathrm{hdim}(M) = \mathrm{length}(M)$ holds. If R is semilocal ring and M be finitely generated then M has finitely hollow dimension. Also if M is artinian module then any submodule N of M is semiartinian and so finitely hollow dimension. Let M be an R-module with finite hollow dimension then M is hollow-lifting if and only if M is lifting, and so M is c-f-lifting module [16]. A Z-module has finite hollow dimension if and only if it is artinian. Since every artinian module is amply supplemented and so supplemented then Z-module $M = Z\,/\,pZ \oplus Z\,/\,p^3Z$ is an amply supplemented module, where p is any prime integer. Therefore Z-module is an amply supplemented module.

Definition 4.1.

A module M has hollow dimension n, if there exists a small epimorphism from M to a direct sum of n hollow modules.

Or: a module M is said to have hollow dimension (or finite dual Goldi dimension) if there exists an exact sequence:

g: $M \rightarrow \oplus H_i$, $i = 1,..., n$, $\ni H_i$ hollow and kernel of g small in M.

Remarks 4.2.

1. We called n hollow dimension (dual Goldi dimension) of M and we write $\mathrm{hdim}(M) = n$.

2. If $M = 0$, then hollow dimension of $M = 0$, but if M does not have finite hollow dimension, then hollow dimension of $M = \infty$.

3. If we have descending chain $H_1 \supseteq H_2 \supseteq ...$ of submodules of M there exists $i \ni H_i$ lies above H_k in M, $\forall$ $k \geq i$.

4. Any module is hollow if and only if it has hollow dimension 1.

Recall that every artinian module is amply supplemented and so supplemented then Z-module $M = Z\,/\,pZ \oplus Z\,/\,p^3Z$ is an amply supplemented module, where p is any prime integer. Therefore Z-module is an amply supplemented module.

Proposition 4.3.

Let M be a Z-module. If M has finite hollow dimension then it is artinian module.

Proof: Let M has finite hollow dimension. We must prove that M is a finite direct sum of hollow modules. Suppose that $t(M)$ be the torsion submodule of M. Then, if $t(M) \neq M$, then we have hollow dimension of $(M\,/\,t(M)) = \infty$ and this implies hollow dimension of $M = \infty$, but this contradiction. Therefore $M = t(M)$, i.e., M is torsion module. By induction on $n = \mathrm{hdim}(M)$ we show that M is a direct sum of n hollow Z-modules. If $n = 1$ then M is hollow module. Let $n \in N$, all Z-modules such that hollow dimension of $M = k$, and $1 \leq k \leq n$, are a direct sum of k hollow modules. Let M be an abelian torsion group with hollow dimension of $M = n + 1$. Therefore M is not indecomposable module, since $M \simeq Z_{pk}$ and so of hollow dimension 1 then there exists a proper decomposition $M = M_1 \oplus M_2$, let $(n + 1)$ = hollow dimension of $M = n_1 + n_2$ such that n_1 = hollow dimension of M_1 not equal zero and n_2 = hollow dimension of M_2 not equal zero. Then, we use mathematical induction , to obtain M_1 and M_2 are direct sums of n_1 and n_2 hollow modules respectively. Thus M is a direct sum of $(n + 1)$ hollow modules. Therefore every Z-module M with finite hollow dimension is a finite direct sum of hollow modules. Let M be hollow and let $(M\,/\,t(M))$ is hollow and torsion free, then we have $M = t(M)$. Hence M is an indecomposable torsion abelian group and so, is isomorphic to Z_{pk} for some prime p and $1 \leq k \leq \infty$ then these summands is isomorphic to an artinian module of the form Z_{pk} with p a prime and $1 \leq k \leq \infty$. Hence M is artinian module.

Since every artinian module is amply supplemented and so supplemented then Z-module $M = Z\,/\,pZ \oplus Z\,/\,p^3Z$ is an amply supplemented module, where p is any prime integer. Therefore Z-module is an amply supplemented module.

Theorem 4.4.

Let $H = H_1 \oplus H_2 \oplus ... \oplus H_n$, such that H_i are submodules of H, then $\mathrm{hdim}(H) = \mathrm{hdim}(H_1) + \mathrm{hdim}(H_2) + ... + \mathrm{hdim}(H_n)$.

Proof: Since each H_i is a factor of H, $\mathrm{hdim}(H) \geq \mathrm{hdim}(H_i)$ because: let N and K be submodules of the R-module H. If $\{P_1\,/\,N,\ldots,P_k\,/\,N\}$ is a coindependent family of submodules of $(H\,/\,N)$ then $\{P_1,..., P_k\}$ is a coindependent family of submodules of H. Hence $\mathrm{hdim}(H\,/\,N) \leq \mathrm{hdim}(H)$ and if $\mathrm{hdim}(H_i) = \infty$ for any direct summand H_i, implies $\mathrm{hdim}(H) = \infty$. Now for all $i \in \{1,\ldots,k\}$, $\mathrm{hdim}(H_i) = n_i$ less than ∞ then, for each $1 \leq i \leq k$, there is a coindependent family $\{H_{i1},...,H_{ini} \leq H_i\} \ni \cap H_j$, j=1, … , n_i small in M_i and $(H_i\,/\,H_{ij})$ is a hollow module $\forall\, j = 1,...,n_i$. For each i_0 belong to $\{1\ldots,k\}$ and j_0 belong to $\{1,..,n_{i0}\}$, now define H_{1i0j0} to be the submodule of $H = \oplus H_i$, i=1…k, given by $\oplus A_i$, i=1,…,k, $\ni A_{i0} = H_{i0j0}$ and $A_j = H_j$ for $j \neq i_0$. Then it is straightforward to show that $\{ H_{1i0j0} : 1 \leq i_0 \leq k, 1 \leq j_0 \leq n_{i0}\}$ is coindependent family of $\sum ni$ i=1…k $\leq H$ whose intersection is small in H and $(H / H_{1i0j0}) \simeq (H_{i0} / H_{i0j0})$ is a hollow module $\forall$ i_0, j_0. Consequently, hdim(H) $= \sum ni$, i=1…k.

The module M is said to have finite dual Goldie dimension if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel. In this case n is the dual Goldie dimension of M and we denote n by codim(M). Let d: R-Mod $\rightarrow N \cup \{\infty\}$ a rank function R-Mod if for all $M, N \in R$-Mod = 0 $\Leftrightarrow d(M) = 0$ and $d(M \oplus N) = d(M) + d(N)$ holds. Note that if d is a rank funcrion and M a module with $d(M)$ =1, then M is indecomposable. Therefore, dim(M) and codim(M) are rank functions.

Theorem 4.5.

Let M be a lifting right R-module. If M has finite dual Goldie dimension, then M is a direct sum of hollow modules.

Proof: Suppose M a lifting module such that there exists a rank function d and $d(M)$ =1. Then M is indecomposable and hence hollow. Assume now $n \geq 1$ and assume that for every lifting module N such that there exists a rank function d with $d(N) < n$, N is direct sum of hollow modules. Let M be a lifting module and d a rank function with $d(N) = n$. Let M is indecomposable, then it is hollow. Otherwise M has a decomposition $N \oplus L \ni N$ and L are nonzerod. Therefore since we have $d(M \oplus N) = d(M) + d(N)$ and $n = d(M) + d(N)$ then $d(N)$ and $d(L) < n$ and N, L are lifting modules and by hypothesis they are finite direct sums of hollows and so is M.

Theorem 4.6.

An R-module M with Rad (M) = 0 is weakly supplemented if and only if M is semisimple, and in this case hollow dimension of M = length(M).

Theorem 4.7.

Let M be an R-module with finite hollow dimension and N be a submodule of M. If hollow dimension of M = hollow dimension of (M / N) + hollow dimension of N, then M is supplemented module.

Proof : Suppose that hdim(M) = hollow dimension of (M / N) + hollow dimension of N. We must prove N is a supplement of K in M for all N ana K are submodules of M. Since M has finite hollow dimension, then M is a small cover of a finite direct sum of hollow modules. Since hollow modules are weakly supplemented, so too is this direct sum, for every submodule $N \subseteq M$, $M_1 + (M_2 + N)$ has a trivial weak supplement. Also $(M_2 + N)$ also has a weak supplement in M. Also we get a weak supplement for N. Then M is weakly supplemented by. Then N has a weak supplement K. By assumption, $K + N = M$ and $(K \cap N) \ll M$. Thus hdim(M) = hdim($M / (K \cap N)$) = hdim($K / (K \cap N) \oplus N / (K \cap N)$) = hdim($K / (K \cap N)$) + hdim($N / (K \cap N)$), = hdim($M / N$) + hdim($M / K$) and so we have hdim($M$) = hdim($M / N$) + hdim($M / K$). Thus, from our assumption, hdim(N) = hdim(M / K) = hdim($N / (N \cap K)$) and, in particular, hdim(N) is finite. we get $(N \cap K) \ll N$ and so N is a supplement of K in M. Hence M is supplemented module.

Theorem 4.8.

Let M be an R-module having finite hollow dimension and let K, L be submodules of M with $M = K + L$. If K and L are supplements of each other in M then hdim(M) = hdim(K) + hdim(L).

Proof: suppose that K and L are supplements of each other in M. Since $K \cap L << K$, hdim(K) = hdim($K / K \cap L$). If M has finite hollow dimension, then for any submodule N of M, $N << M \Leftrightarrow$ hdim(M) = hdim(M / N). Also since L is a supplement submodule of M, if M has finite hollow dimension and N a submodule of M, then N is a supplement submodule of $M \Leftrightarrow$ hdim(M) = hdim(M / N) + hdim(N). Hence hdim(M) = hdim(M / L) + hdim(L). Therefore $h(M) = h(K) + h(L)$.

Theorem 4.9.

Let M be an amply supplemented module M with finite hollow dimension. If M is hollow-lifting then it is lifting module.

Proof: Suppose that M is hollow-lifting and let K be a coclosed submodule of M. Since M has finite hollow dimension, (M / K) has finite hollow dimension by induction on hollow dimension of (M / K). If hollow dimension of (M / K) is 1, then K is a direct summand of M, since M is hollow-lifting therefore we assume that hollow dimension of (M / K) is n and for every coclosed submodule T of M such that (M / T) has hollow dimension less than n, then T is a direct summand of M. Let (H / K) be coclosed in (M / K) $\ni$ (M / K) / (H / K) is hollow. Therefore, H is coclosed in M. Hence $M = H \oplus H_1$ for some submodule H_1 of M as M is hollow-lifting. Then $K = H \cap (K \oplus H_1)$ and $(M / K) = (H / K) \oplus (K \oplus H_1) / K$. Therefore $(K \oplus H_1) / K$ is coclosed in (M / K). Also, by [5], $(K \oplus H_1)$ is coclosed in M. Also by mathematical induction, we get $(K \oplus H_1)$ is a direct summand of M, and so K is a direct summand of M. Therefore K is a direct summand of M. Hence M is lifting.

Theorem 4.10.

Let M be a hollow-lifting module and let M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $(M_1 \cap M_2)$ is also a direct summand of M. If M has finite hollow dimension, then M is lifting and it is a finite direct sum of hollow modules.

Proof: Suppose M is hollow-lifting module and if M_1 and M_2 are direct summands of $M \ni M = M_1 + M_2$, then $(M_1 \cap M_2)$ is also a direct summand of M. We must prove that M is a finite direct sum of hollow modules. If we use again Mathematical induction on hdim(M). Suppose hdim(M) = 1, then by [Remark 4.2(4)] M is hollow module. Suppose r greater than 1 and suppose that for every hollow-lifting module A with $(D_3) \ni$ hdim(A) less than r, A is a finite direct sum of hollow modules. Let M be a hollow-lifting module with hdim(M) = r. Suppose that M is indecomposable. Since M has finite hollow dimension, there exists a proper submodule B of $M \ni (M / B)$ is hollow. As M is hollow-lifting, there exists a direct summand C of $M \ni C \leq B$ and $(B / C) \ll (M / C)$. Then M is hollow, a contradiction. Therefore we can assume that M is not indecomposable. So M has a decomposition $M = A \oplus D$ with A and D are nonzero submodules of M. Since hdim(M) = hdim(A) + hdim(D), implies hdim(A) and hdim(D) are less than r. Therefore, by [10], A and D are hollow-lifting modules. By hypothesis they are finite direct sum of hollow modules and so M is lifting.

Corollary 4.11.
For any submodule N of an R-module M, we have hdim(M / N) $\leq$ hdim(M).

Corollary 4.12.
For any submodule N of R-module M, such that N is small in M, then hdim(M) = hdim(M / N).

Corollary 4.13.
Let M be an R-module and N, L submodules of M. Then hdim($N \oplus L$) = hdim(N) + hdim(L).

Corollary 4.14.
Let M be an R-module. If M has finite hollow dimension; then M is semilocal.

Corollary 4.15.
Let A be a Z-module. If A is nonzero and torsion free then hdim(A) = ∞.

References

[1] A. Idelhadj, R. Tribak, On Some Properties of ⊕-Supplemented Modules, IJMMS :69, 4373–4387, 2003.

[2] C. Lomp, On Dual Goldie Dimension, Diplomarbeit (M. Sc. Thesis), Doesseldorf (1996).

[3] C. Lomp, On semilocal modules and rings, Comm. Algebra 27(4), (1999), PP. 1921-1935.

[4] D. Keskinl and C. Lomp, On Lifting LE-Modules, Vietnam Journal ol Mathematics 3O:2, 167-176, 2002.

[5] D. Keskin, On lifting modules, Comm. Algebra, 28(7) , 3427-3440, 2000.

[6] D. Herbera and A. Shamsuddin, Moduleswith Semi-local Endomorphism Ring, proceedings of the american mathematical society, Vol 123, No 12, 1995.

[7] D. K. Tutuncu and S. H. Mohamed, Weak Lifting Modules with Small Radical, Ring and Module Theory Trends in Mathematics, 129–134, 2010 Springer Basel AG.

[8] E. Turkmen and A. Pancar, On Radical Supplemented Modules, International Journal of Computational Cognition, Vol. 7, No. 1, MARCH 2009.

[9] J. Clark, C. Lomp, N. Vajana, and R. Wisbauer. Lifting Modules. Birkhauser Verlag Basel, Boston-Berlin, 2006. 22:3

[10] N. Orhan, D. K.Tutuncu and R.Tribak, On Hollow-Lifting Modules,Taiwanese Journal of Mathematics,Vol. 11, No. 2, 545-568, June 2007.

[11] O. Nil, K. T. Derya, Hollow dimension of modules, J Zhejiang Univ SCI 2, 6A(10):1055-1057, 2005.

[12] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, Philadelphia, (1991).

[13] S. H. Mohamed and B. J. Muller, Continuous and discrete modules, London Math. Soc. LNS 147 Cambridge Univ. Press, Cambridge, 1990., Proposition A.7

[14] S. Top, Totally weak Supplemented Modules, Thesis Master, July (2007), Izmir.

[15] T. Amoozegar, On Generalized Lifting Modules,Tarbiat Moallem University, 20th Seminar on Algebra,2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 21-23.

[16] Y. Talebi, A Generalization of Lifting Modules, Int. J. Contemp. Math. Sciences, Vol. 2, no. 22, 1069 – 1075, 2007.