

# On Existence and Uniqueness of Syphilis Model

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**Abstract:** A good understanding of the transmission dynamics of disease is necessary to proffer solution(s) to syphilis problem. The aim of this research was to use mathematical modelling to understand the intricacies and different approaches to Syphilis screening on epidemic dynamics and the health of MSM. A non-linear mathematical model for the transmission dynamics of syphilis in an heterogeneous setting with complications is developed and analysed. The existence and uniqueness of the system of equations is examined. We use the concept of Lipchitz criteria to analyse the model.

**Keywords:** Syphilis, Existence, Lipchitz, Uniqueness

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## 1. Introduction

The CDC launched the National Plan to Eliminate Syphilis from the USA in October 1999. [12] In order to reach this goal, a good understanding of the transmission dynamics of the disease is necessary. According to Sir, William Osler, Syphilis is a great imitator. In fact, it was only identified as a distinct infection from gonorrhoea by Phillippe Ricordin 1838. Syphilis is a human infectious disease, transmitted almost always through sexual contact, caused by the spirochete *Treponema pallidum* sub-species *pallidum*, and first recognized in Western Europe following Columbus's return from the Americas in 1500.

Syphilis is resurgent in many high-income countries, disproportionately affecting urban men who have sex with men (MSM).

Sexually transmitted diseases (STDs) are a group of infectious or communicable diseases in which the primary mode of transmission is through sexual contact [1] and are among the major causes of illnesses in the world especially in the developing countries [2] and [3]. Syphilis, Bacterial vaginosis, Hepatis B virus, gonorrhoea are few to mention among the (STDs). Syphilis, was identified as a distinct infection from gonorrhoea by Phillippe Ricord in 1838. It is a human infectious disease, transmitted almost always through sexual contact, caused by the spirochete *Treponema palladium* sub-species *palladium*, and first recognized in Western Europe following Christopher Columbu's return

from the Americas in 1500.

Syphilis is a multistage disease that progresses, when untreated, from primary to secondary, latent and finally to tertiary infection. The primary stage symptoms of syphilis involves the presents of a single chancre (a firm, painless, non-itchy ulceration). The primary mode of transmission is by direct sexual contact with lesions of individuals with primary or secondary syphilis. Infection rates patterns of known cases ranged from 20-85% in contact tracing studies. Secondary syphilis with a diffuse rash which involves the palms of the hands and soles of the feet. Latent syphilis with a Plittle to no symptoms and the tertiary syphilis with gummas, neurological or cardiac symptoms. As its name implies, latent syphilis has no clinical manifestations. Early latent syphilis is infection of less than two years duration. An infection of more than two years duration without clinical evidence of treponemal infection is referred to as last latent syphilis. WHO has based this division on the infectiousness of syphilis and its response to therapy. Syphilis is thought to have infected 12 million additional people worldwide in 1999, with greater than 90% of cases in the developing world. After decreasing dramatically since the widespread availability of penicillin in the 1940s, rates of infection have increased since the turn of the millennium in many countries, often in combination with human immunodeficiency virus (HIV). This has been attributed partly to increased promiscuity, prostitution, decreasing use of condoms, and unsafe sexual practices among men who have sex with men.

[4] [5] [6]. Vertical transmission from infected mother to fetus is the other major mode of transmission. Most infants with congenital infection are infected in utero, but contact with a genital lesion during delivery can also result in infection transmission. [13].

Mathematical modelling is used to determine how more asymptomatic infections among repeat cases would impact projected syphilis transmission in the population. Mathematical model of (STDs) transmission were first developed in 1970s [7]; [8] in response to concern over the dramatic increases in the number of reported gonorrhoea cases in the USA during the 1960s and 1970s. After there model, researchers developed mathematical models to simulate the spread of a wide range of (STDs), such as syphilis HIV/AIDS, gonorrhoea, Hepatitis B virus [9].

Syphilis is a multistage disease that progress when untreated, from primary to secondary and finally to tertiary infection. It is a disease of a considerable public health importance because, if not treated, it can lead to various cardiovascular and neurological diseases.

However, it is estimated that 15% of blindness was due to syphilis in 1900, when the disease was still in curable. But as at today, research has taken a new a dimension. Syphilis can be treated by a single dose of antibiotics. Incidence of syphilis in the for example in the USA during the world warII was over 500,000 infections per year. However, it reduces between 1945 -2000 to 31,575 per year.

Resurgence of syphilis in several countries has been observed in several instances since the 1970s. The recent outbreak can be explained by many factors such as the gay liberation movement of the 1970s, changes in the campaign programme among others.

[10] Studied a syphilis model using a mathematical model that includes all stages of the disease. They assumed that infected individuals acquire temporary immunity only after recovery from the latent and tertiary infections. In this paper, we develop a model for the transmission of syphilis in a heterogeneous settings with complications.

## 2. Model Formulation

The model sub-divides the total human population at time  $t$  denoted by  $N(t)$  into six compartments of susceptible male  $S_m(t)$ , susceptible female  $S_f(t)$ , infected male  $I_m(t)$ , infected female  $I_f(t)$ , complications  $C(t)$  and Treated  $T(t)$ , where  $N(t)$  is given as

$$N(t) = S_m(t) + S_f(t) + I_m(t) + I_f(t) + C(t) + T(t)$$

The susceptibles are individuals that have not contracted the infection but may be infected through sexual contacts. The population recruits into the susceptible classes at the rate  $\pi_m$  for susceptible male and  $\pi_f$  for the susceptible female. Infected individuals are those with the infection and can transmit the infection by sexual act to the susceptibles,  $\alpha_1$  represent contact rate at which susceptible male move to infected male, similarly  $\alpha_2$  is the contact rate of movement of susceptible female into infected female class. The complications are individuals in the

population with the infection at the latent stage that can leads to other diseases or death,  $\beta_1, \beta_2$  are the rate of progression of infected male and female into the complications class respectively. Treated are people in the population that have recovered due to treatment,  $r_1, r_2$  represent the recovery / treated rate of infected male and infected female while  $v$  is the treated rate of complications class. We assume that the death rate is not negligible and so the nature death rate is represented by  $\mu$  and due to untreated syphilis which can lead to death, we represent the syphilis induced death rate by  $\delta$ .

The model equation is given as:

$$\frac{dS_m}{dt} = \pi_m - \alpha_1 I_f S_m - \mu S_m \tag{1}$$

$$\frac{dS_f}{dt} = \pi_f - \alpha_2 I_m S_f - \mu S_f \tag{2}$$

$$\frac{dI_m}{dt} = \alpha_1 I_f S_m - (r_1 + \beta_1 + \mu) I_m \tag{3}$$

$$\frac{dI_f}{dt} = \alpha_2 I_m S_f - (r_2 + \beta_2 + \mu) I_f \tag{4}$$

$$\frac{dC}{dt} = \beta_1 I_m + \beta_2 I_f - (v + \mu + \delta) C \tag{5}$$

$$\frac{dT}{dt} = r_1 I_m + r_2 I_f + v C + \mu T \tag{6}$$

## 3. Basic Properties of the Model

For the syphilis transmission model to be epidemiologically meaningful, it is important to show that the state variables of the model remain non – negative for all non – negative initial conditions.

Consider the feasible region

$$\phi = \{(S_m, S_f, I_m, I_f, C, T) \in \mathcal{R}_+^6: S_m + S_f + I_m + I_f + C + T \leq \frac{\pi_m + \pi_f}{\mu}\}$$

Where  $\frac{dN}{dt} = \pi_m + \pi_f - \mu N$

Lemma 1: The closed  $\phi$  is positively invariant and attracting:

Proof:

$\frac{dN}{dt} = \pi_m + \pi_f - \mu N$ . The total human population is bounded by  $\frac{\pi_m + \pi_f}{\mu}$

$$\frac{dN}{dt} + \mu N = \pi_m + \pi_f$$

$$N e^{\mu t} = \pi_m + \pi_f \int (e^{\mu t} + k) dt$$

$$N(t) = \frac{\pi_m + \pi_f}{\mu} + k e^{-\mu t}$$

Hence, at  $t = 0$ ,

$$N(0) = \frac{\pi_m + \pi_f}{\mu} + \left(N_0 - \frac{\pi_m + \pi_f}{\mu}\right) e^{-\mu t}, \text{ where}$$

$$N(0) = \frac{\pi_m + \pi_f}{\mu} \text{ if } N(t) = \frac{\pi_m + \pi_f}{\mu}$$

Hence, the region is positively invariant and attracts all solutions in  $\mathcal{R}^5$ .

#### 4. Existence and Uniqueness of Solution for the Model

Theorem 1 ( see [11]). Let  $D^1$  denotes the region

$$|t - t_0| \leq a, \|x - x_0\| \leq b, x = (x_1, x_2, \dots, x_n), x_0 = (x_{10}, x_{20}, \dots, x_{n0}) \quad (7)$$

Suppose that  $f(t, x)$  satisfies the Lipchitz condition

$$\|f(t, x_1) - f(t, x_2)\| \leq k \|x_1 - x_2\|,$$

Whenever, the pairs  $(t, x_1)$  and  $(t, x_2)$  belongs to  $D^1$ , where  $k$  is positive constant. Then, there exists a constant  $> 0$ , such that there exists a unique continuous vector solution  $\bar{x}(t)$  of the system in the interval  $|t - t_0| \leq \delta$ .

Lemma 2: If  $f(t, x)$  has continuous partial derivative  $\frac{\partial f_i}{\partial x_j}$  on a bounded closed convex domain  $R$ , then it satisfies a Lipchitz condition in  $R$ .

We are interested in the region

$$1 \leq \varepsilon \leq R \quad (8)$$

We look for a bounded solution of the form

$$0 < R < \infty \quad (9)$$

We shall prove the following existence theorem:

Theorem 2: Let  $D^1$  denote the region defined in (7) such that (8) and (9) hold. Then there exists a solution of model system below which is bounded in the region  $D^1$ .

$$f_1 = \pi_m - \alpha_1 I_f S_m - \mu S_m$$

$$f_2 = \pi_f - \alpha_2 I_m S_f - \mu S_f$$

$$f_3 = \alpha_1 I_f S_m - (r_1 + \beta_1 + \mu) I_m$$

$$f_4 = \alpha_2 I_m S_f - (r_2 + \beta_2 + \mu) I_f$$

$$f_5 = \beta_1 I_m + \beta_2 I_f - (v + \mu + \delta) C$$

$$f_6 = r_1 I_m + r_2 I_f + vC + \mu T$$

It suffices to show that  $\frac{\partial f_i}{\partial x_j}, i, j = 1, 2, 3, 4, 5, 6$  are continuous.

Consider the partial derivatives

$$\frac{\partial f_1}{\partial S_m} = -\alpha_1 I_f - \mu, \left| \frac{\partial f_1}{\partial S_m} \right| = |-\alpha_1 I_f - \mu| \leq \infty,$$

$$\frac{\partial f_1}{\partial S_f} = 0, \left| \frac{\partial f_1}{\partial S_f} \right| = |0| \leq \infty,$$

$$\frac{\partial f_1}{\partial I_m} = 0, \left| \frac{\partial f_1}{\partial I_m} \right| = |0| \leq \infty,$$

$$\frac{\partial f_1}{\partial I_f} = -\alpha_1 S_m, \left| \frac{\partial f_1}{\partial I_f} \right| = |-\alpha_1 S_m| \leq \infty,$$

$$\frac{\partial f_1}{\partial C} = 0, \left| \frac{\partial f_1}{\partial C} \right| = |0| \leq \infty,$$

$$\frac{\partial f_1}{\partial T} = 0, \left| \frac{\partial f_1}{\partial T} \right| = |0| \leq \infty,$$

Also,

$$\frac{\partial f_2}{\partial S_m} = 0, \left| \frac{\partial f_2}{\partial S_m} \right| = |0| \leq \infty,$$

$$\frac{\partial f_2}{\partial S_f} = -\alpha_2 I_m - \mu, \left| \frac{\partial f_2}{\partial S_f} \right| = |-\alpha_2 I_m - \mu| \leq \infty,$$

$$\frac{\partial f_2}{\partial I_m} = -\alpha_2 S_f, \left| \frac{\partial f_2}{\partial I_m} \right| = |-\alpha_2 S_f| \leq \infty,$$

$$\frac{\partial f_2}{\partial I_f} = 0, \left| \frac{\partial f_2}{\partial I_f} \right| = |0| \leq \infty,$$

$$\frac{\partial f_2}{\partial C} = 0, \left| \frac{\partial f_2}{\partial C} \right| = |0| \leq \infty,$$

$$\frac{\partial f_2}{\partial T} = 0, \left| \frac{\partial f_2}{\partial T} \right| = |0| \leq \infty,$$

Also,

$$\frac{\partial f_3}{\partial S_m} = -\alpha_1 I_f, \left| \frac{\partial f_3}{\partial S_m} \right| = |-\alpha_1 I_f| \leq \infty,$$

$$\frac{\partial f_3}{\partial S_f} = 0, \left| \frac{\partial f_3}{\partial S_f} \right| = |0| \leq \infty,$$

$$\frac{\partial f_3}{\partial I_m} = -(r_1 + \beta_1 + \mu), \left| \frac{\partial f_3}{\partial I_m} \right| = |-(r_1 + \beta_1 + \mu)| \leq \infty,$$

$$\frac{\partial f_3}{\partial I_f} = \alpha_1 S_m, \left| \frac{\partial f_3}{\partial I_f} \right| = |\alpha_1 S_m| \leq \infty,$$

$$\frac{\partial f_3}{\partial C} = 0, \left| \frac{\partial f_3}{\partial C} \right| = |0| \leq \infty,$$

$$\frac{\partial f_3}{\partial T} = 0, \left| \frac{\partial f_3}{\partial T} \right| = |0| \leq \infty,$$

Also,

$$\frac{\partial f_4}{\partial S_m} = 0, \left| \frac{\partial f_4}{\partial S_m} \right| = |0| \leq \infty,$$

$$\frac{\partial f_4}{\partial S_f} = \alpha_2 I_m, \left| \frac{\partial f_4}{\partial S_f} \right| = |\alpha_2 I_m| \leq \infty,$$

$$\frac{\partial f_4}{\partial I_m} = \alpha_2 S_f, \left| \frac{\partial f_4}{\partial I_m} \right| = |\alpha_2 S_f| \leq \infty,$$

$$\frac{\partial f_4}{\partial I_f} = -(r_2 + \beta_2 + \mu), \left| \frac{\partial f_4}{\partial I_f} \right| = |-(r_2 + \beta_2 + \mu)| \leq \infty,$$

$$\frac{\partial f_4}{\partial C} = 0, \left| \frac{\partial f_4}{\partial C} \right| = |0| \leq \infty,$$

$$\frac{\partial f_4}{\partial T} = 0, \left| \frac{\partial f_4}{\partial T} \right| = |0| \leq \infty,$$

And,

$$\frac{\partial f_5}{\partial S_m} = 0, \left| \frac{\partial f_5}{\partial S_m} \right| = |0| \leq \infty,$$

$$\frac{\partial f_5}{\partial S_f} = 0, \left| \frac{\partial f_5}{\partial S_f} \right| = |0| \leq \infty,$$

$$\frac{\partial f_5}{\partial I_m} = \beta_1, \left| \frac{\partial f_5}{\partial I_m} \right| = |\beta_1| \leq \infty,$$

$$\frac{\partial f_5}{\partial I_f} = \beta_2, \left| \frac{\partial f_5}{\partial I_f} \right| = |\beta_2| \leq \infty,$$

$$\frac{\partial f_5}{\partial C} = -(v + \mu + \delta), \left| \frac{\partial f_5}{\partial C} \right| = |-(v + \mu + \delta)| \leq \infty,$$

$$\frac{\partial f_5}{\partial T} = 0, \left| \frac{\partial f_5}{\partial T} \right| = |0| \leq \infty,$$

Lastly,

$$\frac{\partial f_6}{\partial S_m} = 0, \left| \frac{\partial f_6}{\partial S_m} \right| = |0| \leq \infty,$$

$$\frac{\partial f_6}{\partial S_f} = 0, \left| \frac{\partial f_6}{\partial S_f} \right| = |0| \leq \infty,$$

$$\frac{\partial f_6}{\partial I_m} = r_1, \left| \frac{\partial f_6}{\partial I_m} \right| = |r_1| \leq \infty,$$

$$\frac{\partial f_6}{\partial I_f} = r_2, \left| \frac{\partial f_6}{\partial I_f} \right| = |r_2| \leq \infty,$$

$$\frac{\partial f_6}{\partial C} = v, \left| \frac{\partial f_6}{\partial C} \right| = |v| \leq \infty,$$

$$\frac{\partial f_6}{\partial T} = -\mu, \left| \frac{\partial f_6}{\partial T} \right| = |-\mu| \leq \infty,$$

As clearly shown above, all the partial derivatives of the whole system of equation exists, finite and bounded, hence by theorem (2), the model system has a unique solution in  $D^1$ .

## 5. Conclusion

The main objective of this research is to develop a

mathematical model to better the understanding and effect of different approaches to control syphilis. However, in doing this, we must examine the existence and uniqueness of the model which is significant in any mathematical epidemiology. In this paper, we have considered a deterministic model for the transmission of syphilis in a heterogeneous setting with complications. We showed that there exists a positive invariant. Also, the existence and uniqueness of the model is examined using Lipchitz condition.

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