

Common Fixed Point Theorem for Four Self-Maps Satisfying (CLR_{ST}) Property for Generalized (ψ, ϕ) –Weakly Contraction in B-metric Space

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Abstract: This paper is devoted to obtain fixed point results. Fixed point theory is very wide field which have many application in different areas. The concept of altering distance to find fixed point results have been explored by many authors. In the present paper the author state altering distance function and ultra-altering distance function and the coincidence point for two self-mappings that satisfy the (CLR_{ST}) - property with the help of altering distance function and ultra-altering distance function in the context of b-metric spaces and achieve a unique common fixed point for two weakly compatible pairs. Many discoveries can also be derived from these main results in the framework of metric spaces.

Keywords: B-metric Spaces, Coincidence Point, Weakly Compatible, Ultra-Altering Distance Function, Common Fixed Point

1. Introduction and Preliminaries

Fixed point theory is a branch of nonlinear analysis that can be applied successfully to a wide range of contexts in social and natural Sciences. IT is one of the most acclaimed scientific considerations with application in many disciplines, particularly in engineering, physics, economics and medical science etc. This theory is also beneficial for solving varies kind of integral and differential equations. Although some results had seen introduced, it is usually considered that this field of study was born in 1922, when Banach presented a celebrated theorem in order to guarantee that a nonlinear operator had generalization, in many different frame works, have been done.

On the otherhand side, Bakhtin [3] and Czerwik [4] develop the notion of b – metric space and established some fixed point theorems in b – metric spaces. Subsequently, several results appeared in this direction [11, 12, 13, 14, 15]. The concept of b – metric space was introduced by Czerwik [4] and formally defined a b – metric space with a view of generalizing the Banach contraction mapping theorem. The well-known Banach contraction principle assures the existence and uniqueness of

fixed points of certain self-maps in metric spaces. It is well known that fixed point theory has wide application in applied Science. Banach contraction principle which states that if (X, d) is complete metric space and $f: X \rightarrow X$ is a contraction map then f has a unique fixed point, it is a fundamental result in this theory. Due to its importance and simplicity several authors have obtained many interesting extensions and generalization of Banach contraction principle, some generalizations of contraction condition was conducted. Because of its wide applications, several researchers have extended, improved and generalized the result in many directions.

Definition 1.1 [4] let X be a non empty set and $s \geq 1$ be given real number. A function $d: X \times X \rightarrow [0, +\infty)$ is a b – metric if for any $p, q, r \in X$, the following conditions are satisfied

$$(b1) \ d(p, q) = 0 \text{ if and only if } p = q$$

$$(b2) \ d(p, q) = d(q, p);$$

$$(b3) \ d(p, q) \leq s[d(p, r) + d(r, q)].$$

A pair (X, d) is called a b – metric space with parameter s . It should be noted that the class of b – metric space is

effectively larger than that of metric spaces. Indeed, a b -metric is metric if and only if $s = 1$.

Example 1.2

1) Let $X = \mathbb{R}$. Define a mappings $d: X \times X \rightarrow [0, \infty)$ by

$$d(p, q) = (p - q)^2 \text{ for all } p, q \in X.$$

Then (X, d) is a b -metric space with coefficient $s = 2$.

2) let $X = \{1, 2, 3\}$. Define a mapping $d: X \times X \rightarrow [0, \infty)$ by $d(1, 1) = d(2, 2) = d(3, 3) = 0$, $d(1, 2) = d(2, 1) = 2$, $d(2, 3) = d(3, 2) = 1$ and $d(1, 3) = d(3, 1) = 6$. Then (X, d) is a b -metric space with coefficient $s = 2$.

In 2015, Ozterk et al [5] give the notion of property $b - (E, A)$ property in b -metric space as follows:

Definition 1.3. [5] A pair (S, T) of self-maps on a b -metric space (X, d) is said to be satisfy $b - (E, A)$ property if there exists a sequence $\{y_n\}$ in X such that, $\lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ty_n = p$, for some p in X .

Definition 1.4. [8] Let $\{y_n\}$ be a sequence in a b -metric space (X, d) .

a) $\{y_n\}$ is called b -convergent if and only if there is $p \in X$ such that $d(y_n, p) \rightarrow 0$ as $n, m \rightarrow \infty$.

b) $\{y_n\}$ is called b -Cauchy sequence if and only if $d(y_n, y_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proposition 1.5 [3] In b -metric space (X, d) the following assertions hold:

1) A b -convergent sequence has a unique limit.

2) Each b -convergent sequence is b -Cauchy.

3) In general a b -metric space is not continuous.

Definition 1.6 [1] A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

1) ψ is non-decreasing and continuous,

2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.7 [7] An ultra-altering distance function is continuous, non-decreasing mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) > 0$ or $t = 0$ and $\varphi(0) \geq 0$.

Definition 1.8 [6] Let A and B be given self-mappings on a non-void set X . The pair (A, B) is said to be weakly compatible if A and B commute at their coincidence points

(i.e. $ABp = BAp$ whenever $Ap = Bp$).

In 2011, Sintunavarat et al. [2] introduce the notion of CLR_{ST} -property in the context of metric space as follows.

Definition 1.9 [2] Two self-mapping A and B of a metric space (X, d) are said to satisfy (CLR_A) property if there exists a sequence $\{y_n\}$ in X such that,

$$\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} By_n = Ap$$

for some p in X .

In 2009, Zhang and Song [9] used generalized φ -weak contraction which is defined for two mappings and gave conditions for existence of a common fixed point.

Theorem 1.10 [9] (see [7]). Let (X, d) be complete metric space, and $T, S: X \rightarrow X$ two mappings such that for all $x, y \in X$,

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y))$$

Where $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$.

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \frac{[d(y, Tx) + d(x, Sy)]}{2}\}$$

then there exists the unique point $u \in x$ such that $u = Tu = Su$.

In 2020, Sahi Arora et al [10] define C-class function and prove the below theorem,

Definition 1.11 [10] A mapping $F: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is called C-class function it is continuous and fulfill the conditions:

1) $F(s, t) \leq s$

2) $F(s, t) = s$, implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

Theorem 1.12 (see [10]) Let (X, d) be a b -metric spaces and $P, Q, S, T: X \rightarrow X$ be mapping with $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$, Such that:

$\psi(d(Px, Qy)) \leq F(\psi(M(x, y)), \varphi(M(x, y)))$, for all $x, y \in X$.

Where,

$$M(x, y) = \max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty), \frac{[d(Px, Ty) + d(Sx, Qy)]}{2s}\}$$

then P, Q, S, T has a unique common fixed point in X .

2. Main Result

In the following theorem, we show the existence and uniqueness of common fixed point for four self-maps.

Theorem 2.1. Let (X, d) be a b -metric space with $s \geq 1$ and $A, B, S, T: X \rightarrow X$ be self-maps with $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ such that

$$\psi(d(Ax, By)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \text{ for each } x, y \in X: \quad (1)$$

Where,

$$M_s(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(Sx, By)}{2s}\}.$$

Suppose that the pair (A, S) and (B, T) satisfy the (CLR_S) and (CLR_T) properties respectively, then the pair (A, S) and (B, T) have a coincidence point in X . Moreover, if the pair (A, S) and (B, T) are weakly compatible, then A, B, S, T have a unique

common fixed point.

Proof: if the pair (A, S) satisfy the CLR_S property, then there exist a sequence $\{x_n\}$ in X , such that $\lim_{n \rightarrow \infty} A x_n = \lim_{n \rightarrow \infty} S x_n = p$, where, $p \in S(X)$. As $A(X) \subseteq T(X)$, there exist a sequence $\{y_n\}$ in X such that $A x_n = T y_n$. Hence, $\lim_{n \rightarrow \infty} T y_n = p$.

We will show that $\lim_{n \rightarrow \infty} B y_n = p$

Putting $x = x_n$ and $y = y_n$ in inequality (1), we arrive at

$$\psi(d(Ax_n, By_n)) \leq \psi(M_s(x_n, y_n)) - \varphi(M_s(x_n, y_n)),$$

Where

$$\begin{aligned} M_s(x_n, y_n) &= \max\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \frac{d(Ax_n, Ty_n) + d(Sx_n, By_n)}{2s}\} \\ &= \max\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \frac{d(Ax_n, Ax_n) + d(Sx_n, By_n)}{2s}\} \\ &= \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \frac{d(Sx_n, By_n)}{2s}\right\} \\ &\leq \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), s\left(\frac{d(Sx_n, Ax_n) + d(Ax_n, By_n)}{2s}\right)\right\} \\ &= \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \left(\frac{d(Sx_n, Ax_n) + d(Ax_n, By_n)}{2}\right)\right\} \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} M_s(x_n, y_n) &= \max\{d(p, p), d(p, p), d(By_n, p), \frac{d(p, p) + d(p, By_n)}{2s}\} \\ &= \max\{d(By_n, p), \frac{d(p, By_n)}{2}\} = d(By_n, p), \end{aligned}$$

This implies that

$$M_s(x_n, y_n) \leq d(By_n, p)$$

Making $n \rightarrow \infty$ in (2), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(d(Ax_n, By_n)) &\leq \lim_{n \rightarrow \infty} \psi(M_s(x_n, y_n)) - \lim_{n \rightarrow \infty} \varphi(M_s(x, y)) \\ \psi\left(\lim_{n \rightarrow \infty} d(Ax_n, By_n)\right) &\leq \psi\left(\lim_{n \rightarrow \infty} d(By_n, p)\right) - \varphi\left(\lim_{n \rightarrow \infty} d(By_n, p)\right) \end{aligned}$$

This implies that, $\lim_{n \rightarrow \infty} d(By_n, p) = 0$,

Hence, $\lim_{n \rightarrow \infty} B y_n = p$.

Now, since the pair (A, S) and (B, T) satisfy the CLR_{ST} property so there exist a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} A x_n = \lim_{n \rightarrow \infty} S x_n = \lim_{n \rightarrow \infty} B y_n = \lim_{n \rightarrow \infty} T y_n = p$, where, $p \in S(X) \cap T(X)$, so there exist a point $u \in X$ such that $Su = p$, we have to show that $Au = Su$.

Putting $x = u$ and $y = y_n$ in equation (1) above, we get,

$$\psi(d(Au, By_n)) \leq \psi(M_s(u, y_n)) - \varphi(M_s(u, y_n))$$

Where

$$\begin{aligned} M_s(u, y_n) &= \max\{d(Su, Ty_n), d(Au, Sx_n), d(By_n, Ty_n), \frac{d(Au, Ty_n) + d(Su, By_n)}{2s}\} \\ &= \max\{d(u, Ty_n), d(Au, p), d(By_n, Ty_n), \frac{d(Au, Ty_n) + d(Su, By_n)}{2s}\} \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} M_s(u, y_n) &= \max\{d(p, p), d(Au, p), d(p, p), \frac{d(Au, p) + d(p, p)}{2s}\} \\ &= \max\{d(Au, p), \frac{d(Au, p)}{2s}\} = d(Au, p) \end{aligned}$$

This implies that

$$\psi(d(Au, p)) \leq \psi(d(Au, p)) - \varphi(d(Au, p))$$

Hence, $\psi(d(Au, p)) = 0$.

Therefore, $Au = p = Su$

Thus, u is a point of coincidence of the pair (A, S) . As $p \in T(X)$, there exist a point $v \in X$ such that $Tv = p$.

We assert that, $Bv = Tv$

Putting $x = u$ and $y = v$ in equation (1) above, we get

$$\psi(d(Au, Bv)) \leq \psi(M_s(u, v)) - \varphi(M_s(u, v)), \text{ for each } x, y \in X:$$

Where

$$\begin{aligned} M_s(u, v) &= \max\{d(Su, Tv), d(Au, Su), d(Bv, Tv), \frac{d(Au, Tv) + d(Su, Bv)}{2s}\} \\ &= \max\{d(p, p), d(p, p), d(Bv, p), \frac{d(p, p) + d(p, Bv)}{2s}\} \\ &= \max\{d(Bv, p), \frac{d(Bu, p)}{2s}\} = d(Bv, p) \end{aligned}$$

This implies that

$$\psi(d(Bv, p)) \leq \psi(d(Bv, p)) - \varphi(d(Bv, p))$$

Hence, $\psi(d(Bv, p)) = 0$. This implies that, $Bv = p = Tv$.

Hence, v is a point of coincidence of point (B, T) . Since the pair (A, T) is weakly compatible, then $Ap = Sp$.

Now, will show that p is common fixed point of A and S .

Putting $x = p$ and $y = v$ in equation (1), we get

$$\psi(d(Ap, Bv)) \leq \psi(M_s(p, v)) - \varphi(M_s(p, v)), \text{ for each } x, y \in X:$$

Where

$$\begin{aligned} M_s(p, v) &= \max\{d(Sp, Tv), d(Ap, Sp), d(Bv, Tv), \frac{d(Ap, Tv) + d(Sp, Bv)}{2s}\} \\ &= \max\{d(Ap, P), d(Ap, Ap), d(P, P), \frac{d(p, p) + d(Ap, p)}{2s}\} \\ &= \max\{d(Ap, p), 2 \frac{d(Bu, p)}{2s}\} \\ &= d(Ap, p) \end{aligned}$$

This implies that

$$\psi(d(Ap, p)) \leq \psi(d(Ap, p)) - \varphi(d(Ap, p))$$

Thus, $\psi(d(Ap, p)) = 0$,

This gives, $Ap = p = Sp$. Hence p is common fixed point of the pair (A, S) .

Again let us show that p is common fixed point of B and T .

Putting $x = u$ and $y = p$ in equation (1), we get

$$\psi(d(Au, Bp)) \leq \psi(M_s(u, p)) - \varphi(M_s(u, p)), \text{ for each } x, y \in X:$$

Where

$$\begin{aligned} M_s(u, p) &= \max\{d(Su, Tp), d(Au, Sp), d(Bu, Tp), \frac{d(Au, Tp) + d(Su, Bp)}{2s}\} \\ &= \max\{d(p, Tp), d(p, p), d(p, p), \frac{d(Au, Tp) + d(p, p)}{2s}\} \\ &= \max\{d(p, Tp), \frac{d(Au, Tp)}{2s}\} \\ &= d(p, Tp) \end{aligned}$$

This implies that

$$\psi(d(p, Tp)) \leq \psi(d(p, Tp)) - \varphi(d(p, Tp))$$

Thus, $\psi(d(p, Tp)) = 0$,

This gives, $Tp = p = Bp$.

Hence, p is common fixed point of A, B, S, T .

If there exists another fixed point $t \in X$, then from

$$\psi(d(p, t)) = \psi(d(Ap, Bt)) \leq \psi(M_s(p, t)) - \varphi(M_s(p, t)),$$

Where,

$$\begin{aligned} M_s(p, t) &= \max\{d(Sp, Tt), d(Ap, St), d(Bp, Tt), \frac{d(Ap, Tt) + d(Sp, Bt)}{2s}\}. \\ &= \max\{d(p, t), d(p, p), d(p, p), \frac{d(p, t) + d(p, t)}{2s}\}. \\ &= \max\{d(p, t), \frac{d(p, p)}{s}\} \\ &= d(p, t) \end{aligned}$$

This implies that

$$\psi(d(p, t)) \leq \psi(d(p, t)) - \varphi(d(p, t))$$

Thus

$$\psi(d(p, t)) = 0.$$

Hence, we conclude that $p = t$.

The proof is completed.

Corollary 2.2 Let (X, d) be a b-metric space and $P, T : X \rightarrow X$ be mappings, such that,

$$\psi(d(Px, Py)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \text{ for each } x, y \in X:$$

Where,

$$M_s(x, y) = \max\{d(Tx, Ty), d(Px, Tx), d(Py, Ty), \frac{d(Px, Ty) + d(Tx, Py)}{2s}\}.$$

Suppose that the pair (P, T) satisfy the (CLR_{ST}) properties. Then the pair (P, T) and (B, T) have a common coincidence point in X . Moreover, if the pair (P, T) is weakly compatible, then P, T have a unique common fixed point.

3. Conclusion

In this paper the desired results on the coincidence point for weakly compatible self-mapping satisfying (CLR_{ST}) -property obtained and achieve a unique common fixed point for two weakly compatible with the help of altering distance function and ultra-altering distance function in the frame work of b-metric spaces was achieved.

Availability of Data and Materials

On request, the data used to support the findings of this study can be obtained from the corresponding author.

Conflict of Interest

The author declares no conflict of interest.

References

- [1] Khan M. S, Swaleh M and Sessa S. 1984, Fixed point theorems by altering distance between the Points Butten of the Australian Mathematical Society 30 1-9.
- [2] Sintunavarat W and Kumam P 2011, Common fixed theorems for a pair of weakly compatible Mappings in Fuzzy metric spaces, Journal of Applied mathematics 637958 1-14.
- [3] Boriceanu M. 2009, Strict fixed point theorems for multivalued operator in b-metric spaces, Int. J. Mod. Math 4. 285-301.
- [4] S. Czerwik, 1993, Contraction mapping in b-metric space, Acta Mathematician Informatics University Ostraviensis, 1, 5-11.
- [5] Ozturk. V and Turkoglu. D 2015, Common fixed point theorems for mappings satisfying (E, A) property in b -metric spaces J. Nonlinear Sci. Appl. 8 1127-1133.
- [6] G. Jungck 1986, Compatible mapping and Common fixed points. Int. Math. Sci 9 771-779.
- [7] Ansori A H. Chandok S and Ionesco C. 2014. Fixed point theorem on b -metric spaces for Weakly contraction with Auxiliary function, Journal of inequality and Application 429 1-9.
- [8] Boriceanu, M. and Bota, M. and Petrusel, A. (2010). Multivalued fractals in b -metric spaces. Cent. Eur. J. Math, 8 (2), 367-377.
- [9] Q. Zhang. Y. Song, 2009, Fixed point theorem for generalized φ -weak contraction, Applied Mathematics Letters, 22 75-78.
- [10] Sahi Arora, 2020, common fixed point theorems for four self-maps satisfying $(CLR)_{ST}$ -property in b-metric spaces, J. Phys. Conf. Ser. 1531 012083.

- [11] S. Phiangsungnoen and P. Kumamam, (2018), On stability of fixed point inclusion for multivalued type contraction mappings in dislocate b-metric space with application ApplSci, 114, <https://doi.org/10.1002/mma.4871>.
- [12] W, Kumam, P, Sukprasert, P. Kumam, A, Shoaib, A. Shahzad and Q. Mahmood, (2018), Some fuzzy fixed point results for fuzzy mappings in complete b-metric spaces, Cogent Mathematic sanStatist
<https://doi.org/10.1080/25742558.2018.1458933>.
- [13] T. Abdeljawad, E. Karapinar, K. Tas, Existence and uniqueness of a common fixed point on Partial metric spaces, Appl., math. Lett, 24 (2011), 1900-1904. 1.
- [14] S. Phiangsunangnsungnoen and P. Kumamam, (2015), Fuzzy fixed point theorem for Mult-ivalued fuzzy contraction in b-metric spaces, J. Nonlinear Sci. Appl., 8, 5563.
- [15] H. Piri and P. Kumam, (2016). Fixed point theoremsfor generalized F-Suzuki- Contraction in Complete b-metric spaces, Fixed Point Theory and Applications, 90, doi: 10.1186/s13663-016-0577-5.