

# Some New Properties of $W_d$ -fuzzy Implication Algebras

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**Abstract:** Implication is an important logical connective in practically every propositional logic. In 1987, the so-called Fuzzy implication algebras were introduced by Wu Wangming, then various interesting properties of FI-algebras and some subalgebra of Fuzzy implication algebra, such as regular FI-algebras, commutative FI-algebras,  $W_d$ -FI-algebras, and other kinds of FI-algebras were reported. The main aim of this article is to study  $W_d$ -fuzzy implication algebras which are subalgebra of fuzzy implication algebras. We showed that  $W_d$ -fuzzy implication algebras are regular fuzzy implication algebras, but the inverse is not true. The relations between  $W_d$ -fuzzy implication algebras and other fuzzy algebras are discussed. Properties and axiomatic systems for  $W_d$ -fuzzy implication algebras are investigated. Furthermore, a few new results on  $W_d$ -fuzzy implication algebras has been added.

**Keywords:** Fuzzy Implication Algebras,  $W_d$ -Fuzzy Implication, Regular Fuzzy Implication Algebras, Heyting Type Fuzzy Implication Algebras

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## 1. Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [2]. Recently, some authors studied fuzzy implications from different perspectives[16]. Naturally, it is meaningful investigating the common properties of some important fuzzy implications used in fuzzy logic. Consequentially, Professor Wu [1] introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990.

In the past two decades, some authors focused on FI-algebras. Various interesting properties of FI-algebras [3-5], regular FI-algebras [1,6,7], commutative FI-algebras [8],  $W_d$ -FI-algebras [9], and other kinds of FI-algebras [10] were reported, and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed [1,11,12]. Relationships between FI-algebra and BCK-algebra [13,14], MV-algebras [15], Rough set algebras [16,17], and were partly investigated, and FI-algebras were axiomatized [18]. In the recent work, the relationship between these FI-algebras and several famous fuzzy algebras were systematically discussed[3,19-24].

The organization of the paper is as follows: preliminary

notions and results are introduced in section 2; section 3 relationships between  $W_d$ -FI algebras and several classes of important fuzzy algebras are discussed; Main properties of  $W_d$ -FI algebras is included in section 4. Lastly, the paper introduces several conclusions and pointers for further research.

## 2. Preliminaries

In this section, we summarize some definitions and results about  $W_d$ -FI algebras, which will be used in the following sections of the paper.

First, we recall some definitions and properties about  $W_d$ -FI algebras.

**Definition 2.1.** (see [1]) Let  $X$  be a set with a binary operation  $\rightarrow$ , and  $0 \in X$ . An fuzzy implication algebra, shortly, FI-algebra is an algebra  $(X, \rightarrow, 0)$  of type  $(2,0)$  satisfying

- (I1)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ ;
- (I2)  $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$ ;
- (I3)  $x \rightarrow x = 1$ ;
- (I4) if  $x \rightarrow y = y \rightarrow x = 1$ , then  $x = y$ ;
- (I5)  $0 \rightarrow z = 1$ , for all  $x, y, z \in X$ , where  $1 = 0 \rightarrow 0$ .

**Definition 2.2.** (see [1]). A Hilbert fuzzy implication algebra, shortly, HFI-algebra is an algebra  $(X, \rightarrow, 0)$  of type  $(2,0)$  which satisfies the following conditions for every  $x, y, z \in X$ :

- (H1)  $x \rightarrow (y \rightarrow x) = 1$ ;
  - (H2)  $(x \rightarrow (y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) = 1$ ;
  - (H3) if  $1 \rightarrow x = 1$ , then  $x = 1$ ;
  - (H4) if  $x \rightarrow y = y \rightarrow x = 1$ , then  $x = y$ ;
  - (H5)  $0 \rightarrow x = 1$ ,
- where  $1 = 0 \rightarrow 0$ .

On an FI-algebra  $X$ , one can define a binary relation  $\leq$  and operators  $C, T, S$  as follows.

$$x \leq y \iff x \rightarrow y = 1, x, y \in X; \quad (1)$$

$$C(x) = x \rightarrow 0, x \in X; \quad (2)$$

$$T(x, y) = C(x \rightarrow C(y)),$$

$$S(x, y) = C(x) \rightarrow y, x, y \in X. \quad (3)$$

Usually, we also say that  $X$  is an FI-algebra for convenience.

Obviously, the relation " $\leq$ " is a partial ordering on  $X$ , i.e., the relation is reflexive, antisymmetric and transitive (see [1]). In fuzzy logic, the property (1) is called the ordering property.

The operator " $C$ " defined in the above definition is a negation on  $X$ , i.e., the operator is order-inverting and satisfies  $C(0) = 1$ , and  $C(1) = 0$ . " $\leq$ " and  $C$  are called the partial ordering and the negation induced by the FI-algebra  $X$ , respectively.

If an FI-algebra  $X$  form a lattice with respect to the partial order " $\leq$ ", then we called FI-lattice.

**Definition 2.3.** (see [1,3]). Let  $X$  be an FI-algebra.

(i)  $X$  is regular FI-algebra, or an RFI-algebra, if  $CC(x) = x$ , for all  $x \in X$ .

(ii)  $X$  is commutative, or a CFI-algebra, if the binary operation  $\odot$  defined by (3) is commutative, or the following condition (I6) holds for all  $x, y \in X$ :

$$(I6) (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

Then  $W_d$ -Fuzzy implication algebra is an algebra of type  $(2,0)$ . The notion was first formulated in 1996 by Deng and some properties were obtained (see [9]). This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu (see[1]).

**Definition 2.4.** (see [9]). A  $(2,0)$ -type algebra  $(X, \rightarrow, 0)$  is a  $W_d$ -Fuzzy implication algebra, shortly,  $W_d$ -FI algebra, if the following conditions hold for all  $x, y, z \in X$ :

- (W1)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ ;
- (W2)  $(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$ ;
- (W3)  $x \rightarrow x = 1$ ;
- (W4) if  $x \rightarrow y = y \rightarrow x = 1$ , then  $x = y$ ;
- (W5)  $0 \rightarrow x = 1$ , where  $1 = 0 \rightarrow 0$ .

**Example 1** Consider  $X = [0, 1]$ , for every  $x, y \in X$ , defined  $x \rightarrow y = 1$ , then  $(X, \rightarrow, 0)$  is a  $W_d$ -FI algebras. **Example 2** Let  $X = \{0, a, 1\}$  be a finite set of distinct elements. We define

$\rightarrow 0$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then  $(X, \rightarrow, 0)$  is a FI- algebra, but not  $W_d$ -FI algebras. In fact,  $(1 \rightarrow a) \rightarrow 0 = a \rightarrow 0 = 0$ , but  $(0 \rightarrow a) \rightarrow 1 = 1 \rightarrow 1 = 1$ , so (W2) does not hold.

### 3. Relationships Between $W_d$ -FI Algebras and Two Classes of Important Fuzzy Algebras

**Lemma 3.1.** Let  $(X, \rightarrow, 0)$  is a  $W_d$ -FI algebra, then for any  $x, y, z \in X$ , the following holds:

(W6)  $x \rightarrow 1 = 1, 1 \rightarrow x = x$ , for all  $x \in X$ ;

(W7) if  $x \rightarrow y = 1, y \rightarrow z = 1$ , then  $x \rightarrow z = 1$ , for all  $x, y, z \in X$ ;

(W8)  $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$ , for all  $x, y, z \in X$ ;

(W9)  $(x \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow z, (x \rightarrow z) \rightarrow (y \rightarrow z) = y \rightarrow x$ ;

(W10)  $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = x$ .

*Proof.* (W6) Indeed  $x \rightarrow 1 = x \rightarrow (0 \rightarrow x) = 0 \rightarrow (x \rightarrow 0) = 0 \rightarrow 1 = 1$ . Thus, we have verified that  $x \rightarrow 1 = 1$ .

Besides,  $(1 \rightarrow x) \rightarrow x = (x \rightarrow x) \rightarrow 1 = 1 \rightarrow 1 = 1$ ,  $x \rightarrow (1 \rightarrow x) = 1 \rightarrow (x \rightarrow x) = 1 \rightarrow 1 = 1$ . that is  $1 \rightarrow x = x$ .

(W7) If  $x \rightarrow y = 1, y \rightarrow z = 1$  holds, then  $x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = ((x \rightarrow z) \rightarrow y) \rightarrow x = ((y \rightarrow z) \rightarrow x) \rightarrow x = (1 \rightarrow x) \rightarrow x = x \rightarrow x = 1$ , Thus,  $x \rightarrow z = 1$ .

(W8) Indeed  $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (y \rightarrow z) \rightarrow (((x \rightarrow z) \rightarrow y) \rightarrow x) = (y \rightarrow z) \rightarrow (((y \rightarrow z) \rightarrow x) \rightarrow x) = (y \rightarrow z) \rightarrow ((x \rightarrow x) \rightarrow (y \rightarrow z)) = (y \rightarrow z) \rightarrow (1 \rightarrow (y \rightarrow z)) = (y \rightarrow z) \rightarrow (y \rightarrow z) = 1$ . Hence, for all  $x, y, z \in X$ ,  $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$ .

The proof of (W9), (W10) is similar to previous ones.

**Lemma 3.2.** Any  $W_d$ -FI algebra must be an FI - algebra. but the inverse is not true.

*Proof.* From the definition 2.3 and (3) of Lemma 3.1, it is easy to see that any  $W_d$ -FI algebra must be an FI - algebra. By example 2,  $W_d$ -FI algebra be an proper subalgebra of FI - algebra, but FI algebra must be not  $W_d$ -FI algebra.

**Theorem 3.1.** Any  $W_d$ -FI algebra must be an RFI- algebra, but the inverse is not true(see[9]).

We have see that  $W_d$ -FI algebra classes are subclasses of RFI-algebras.

**Theorem 3.2.**  $W_d$ -FI algebra must be not CFI- algebra.

*Proof.* It is easy to proof that if  $x \neq y$ , then the condition (I6) does not hold, i.e., suppose

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

then clearly

$$(y \rightarrow y) \rightarrow x = (x \rightarrow x) \rightarrow y.$$

Thus, we have  $x = y$ , which contradict to assertion.

**Theorem 3.3.** (see[9]) Relations between  $W_d$ -FI algebra and HFI - algebra as following :

(1) If  $(X, \rightarrow, 0)$  is a  $W_d$ -FI algebra such that, for all  $x, y, z \in X$ ,

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$$

holds, then  $(X, \rightarrow, 0)$  is a HFI-algebra.

(2) If  $(X, \rightarrow, 0)$  is a HFI-algebra such that, for all  $x, y, z \in X$ ,

$$(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$$

holds, then  $(X, \rightarrow, 0)$  is a  $W_d$ -FI algebra.

## 4. Main Properties of $W_d$ -FI Algebras

On a  $W_d$ -FI algebra  $X$ , one define a binary relation  $\leq$  as follows,  $x \leq y$  if and only if  $x \rightarrow y = 1, x, y \in X$ .

Obviously, the relation " $\leq$ " is a partial ordering on  $X$ .

**Theorem 4.1.** Let  $X$  be a  $W_d$ -FI algebra, then exist a partial ordering in  $X$ .

*Proof.* Indeed,  $\forall x, y \in X, x \rightarrow y = (1 \rightarrow x) \rightarrow y = (y \rightarrow x) \rightarrow 1 = 1$ . Thus, we have verified that  $x \leq y$ . Therefore, for any  $W_d$ -FI algebra must be exist a partial ordering in  $X$ .

**Theorem 4.2.** Let  $X$  be a  $W_d$ -FI algebra, and  $x, y, z \in X$ . Then

(W11) If  $x \leq y$ , then  $z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$ ;

(W12)  $x \leq CC(x)$ ;

(W13)  $CCC(x) = C(x)$ ;

(W14)  $C(x) \rightarrow y = C(y) \rightarrow x$ ;

(W15) (Commutativity)  $T(x, y) = T(y, x), S(x, y) = S(y, x)$ ;

(W16) (Associativity)  $T(T(x, y), z) = T(x, T(y, z)), S(S(x, y), z) = S(x, S(y, z))$ ;

(W17) (Monotonicity) If  $x \leq y$ , then  $T(x, z) \leq T(y, z), S(x, Z) \leq S(y, z)$ ;

(W18) (Identity)  $T(x, 1) = x, S(x, 0) = x$ ;

(W19) (Duality)  $S(x, y) = C(T(C(x), C(y))), T(x, y) = C(S(C(x), C(y)))$ ;

(W20)  $S(x, C(x)) = 1, S(x, C(x)) = 0$ ;

(W21)  $x \rightarrow (y \rightarrow z) = T(x, y) \rightarrow z$ ;

(W22)  $T((z \rightarrow x), (z \rightarrow y)) = z \rightarrow T(x, y)$ ;

(W23)  $C(x \rightarrow y) = x$ .

*Proof.* (W16)  $T(T(x, y), z) = T(C(x \rightarrow C(y)), z) = C(C(x \rightarrow C(y) \rightarrow C(z))) = C(((x \rightarrow C(y)) \rightarrow 0) \rightarrow C(z)) = C((((x \rightarrow C(y)) \rightarrow 0) \rightarrow (z \rightarrow 0))) = C(z \rightarrow (x \rightarrow C(y))) = C(x \rightarrow (z \rightarrow C(y))) = C(x \rightarrow (z \rightarrow (y \rightarrow 0))) = C(x \rightarrow (y \rightarrow (z \rightarrow 0))) = T(x, T(y, z))$ .

Similarly, we have  $S(S(x, y), z) = S(x, S(y, z))$ .

(W17) Due to  $x \leq y \Leftrightarrow x \rightarrow y = 1$ , then for all  $x, y, z \in X$ , it is  $C(x \rightarrow C(z)) \rightarrow C(y \rightarrow C(z)) = ((x \rightarrow C(z)) \rightarrow 0) \rightarrow ((y \rightarrow C(z)) \rightarrow 0) = (y \rightarrow C(z)) \rightarrow$

$(x \rightarrow C(z)) = x \rightarrow y = 1$ . Hence,  $T(x, z) \leq T(y, z)$ .  $S(x, Z) \leq S(y, z)$ . Similarly, we have  $S(x, z) \leq S(y, z)$ .

(W18)  $T(x, 1) = C(x \rightarrow C(1)) = (x \rightarrow C(1)) \rightarrow 0 = (0 \rightarrow C(1)) \rightarrow x = 1 \rightarrow x = x$ .  $S(x, 0) = C(x) \rightarrow 0 = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$ .

(W19)  $C(T(C(x), C(y))) = C(C(C(x) \rightarrow CC(y))) = C(C(C(x) \rightarrow y)) = C(C(x) \rightarrow y) \rightarrow 0 = (0 \rightarrow 0) \rightarrow (C(x) \rightarrow y) = 1 \rightarrow (C(x) \rightarrow y) = (C(x) \rightarrow y) = S(x, y)$ ;  
 $C(S(C(x), C(y))) = C(S(C(x), C(y))) = C(CC(x) \rightarrow C(y)) = C(x \rightarrow C(y)) = T(x, y)$ .

(W20)  $S(x, C(x)) = C(x) \rightarrow C(x) = 1, T(x, C(x)) = C(x \rightarrow CC(x)) = C(x \rightarrow x) = C(1) = 1 \rightarrow 0 = 0$ .

(W21)  $T(x, y) \rightarrow z = C(x \rightarrow C(z)) \rightarrow z = ((x \rightarrow C(y)) \rightarrow 0) \rightarrow z = ((x \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow z = (z \rightarrow 0) \rightarrow (x \rightarrow (y \rightarrow 0)) = x \rightarrow ((z \rightarrow 0) \rightarrow (y \rightarrow 0)) = x \rightarrow (y \rightarrow z)$ .

(W22)  $T((z \rightarrow x), (z \rightarrow y)) = C((z \rightarrow x), C(z \rightarrow y)) = ((z \rightarrow x) \rightarrow ((z \rightarrow y) \rightarrow 0)) \rightarrow 0 = z \rightarrow x$ , and  $z \rightarrow T(x, y) = z \rightarrow C(x \rightarrow C(y)) = z \rightarrow ((x \rightarrow (y \rightarrow 0)) \rightarrow 0) = z \rightarrow x$ .

(W23)  $C(x \rightarrow y) = (x \rightarrow y) \rightarrow 0 = (0 \rightarrow y) \rightarrow x = 1 \rightarrow x = x$ .

**Theorem 4.3.** Let  $X$  be a  $W_d$ -FI algebra, and  $x, y, z \in X$ . Then  $1 \rightarrow x = x, (x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$  imply  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ .

*Proof.*  $x \rightarrow (y \rightarrow z) = (1 \rightarrow x) \rightarrow (y \rightarrow z) = ((y \rightarrow z) \rightarrow x) \rightarrow 1 = ((x \rightarrow z) \rightarrow y) \rightarrow 1 = (1 \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow (x \rightarrow z)$ .

Hence,  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ .

**Theorem 4.4.** A  $(2,0)$ -type algebra  $(X, \rightarrow, 0)$  is a  $W_d$ -fuzzy implication algebra if and only if it satisfies that

(W1')  $(x \rightarrow y) \rightarrow z = (z \rightarrow y) \rightarrow x$ ;

(W2')  $1 \rightarrow x = x$ ;

(W3')  $x \rightarrow x = 1$ ;

(W4') If  $x \rightarrow y = y \rightarrow x = 1$ , then  $x = y$ ;

(W5')  $0 \rightarrow x = 1$ , where  $1 = 0 \rightarrow 0$ .

*Proof.* Immediate from theorem 4.2 and definition 2.3.

Condition (W3) and (W6) states that 1 are a logical unit and the greatest element of  $W_d$ -FI algebra. Note that a logical unit is always unique. We say that an  $W_d$ -FI algebra  $X$  has a negation if  $X$  admits a smallest element 0 such that the map:  $C : x \mapsto C(x)$  is bijective, where  $C(x) = x \rightarrow 0$ .

For an  $W_d$ -FI algebra with negation, we define two binary operation on  $X$  as follows: for any  $x, y \in X$ ,

$$x \perp y = C(x) \rightarrow y,$$

$$x \top y = C(x \rightarrow C(y)).$$

**Theorem 4.5.** Let  $X$  be a  $W_d$ -FI algebra. Then for any  $x, y, z \in X$  we have:

1)  $x \perp y = y \perp x, x \top y = y \top x$ ;

2)  $(x \perp y) \perp z = x \perp (y \perp z), (x \top y) \top z = x \top (y \top z)$ ;

3)  $x \perp 1 = 1, x \top 1 = x, x \perp 0 = x$ ;

4)  $x \top C(x) = 0, x \perp C(x) = 1$ ;

5)  $x \perp y = C(C(x) \top C(y)), x \top y = C(C(x) \perp C(y))$ ;

6)  $C(x) \rightarrow C(y) = y \rightarrow x, C(x) \rightarrow y = C(y) \rightarrow x$ ;

*Proof.* 1) Applying (W2) and the definition of operator  $\perp$ , we have  $x \perp y = C(x) \rightarrow y = (x \rightarrow 0) \rightarrow y = (y \rightarrow 0) \rightarrow x = y \perp x$ . By (W1) and the definition of operator  $\top$ , we can obtain that  $x \top y = C(x \rightarrow C(y)) = (x \rightarrow (y \rightarrow 0)) \rightarrow 0 = (y \rightarrow (x \rightarrow 0)) \rightarrow 0 = C(y \rightarrow C(x)) = y \top x$ .

2) Using (W2) and the definition of operator  $\perp$ , we get  $(x \perp y) \perp z = C(x \perp y) \rightarrow z = ((x \rightarrow 0) \rightarrow y) \rightarrow z = (z \rightarrow 0) \rightarrow (x \rightarrow 0) \rightarrow y = (x \rightarrow 0) \rightarrow ((x \rightarrow 0) \rightarrow y) = C(x) \rightarrow (C(z) \rightarrow y) = x \perp (z \perp y) = x \perp (y \perp z)$ .

Similarly,  $(x \top y) \top z = C((x \top y) \rightarrow C(z)) = C(C(x \rightarrow C(y)) \rightarrow (z \rightarrow 0)) = c(((x \rightarrow (y \rightarrow 0)) \rightarrow 0) \rightarrow (z \rightarrow 0)) = C(((z \rightarrow 0) \rightarrow 0) \rightarrow (x \rightarrow (y \rightarrow 0))) = C(z \rightarrow (x \rightarrow (y \rightarrow 0)))$ ,

$x \top (y \top z) = x \top (C(y \rightarrow C(z))) = C(x \rightarrow CC(y \rightarrow C(z))) = C(x \rightarrow (y \rightarrow (z \rightarrow 0))) = C(x \rightarrow (z \rightarrow (y \rightarrow 0))) = C(z \rightarrow (x \rightarrow (y \rightarrow 0)))$ ,

Hence,  $(x \top y) \top z = x \top (y \top z)$ .

3)  $x \perp 1 = C(x) \rightarrow 1 = (x \rightarrow 0) \rightarrow 1 = (1 \rightarrow 0) \rightarrow x = 0 \rightarrow x = 1$ ,

$x \top 1 = C(x \rightarrow C(1)) = c(x \rightarrow (1 \rightarrow 0)) = C(x \rightarrow 0) = CC(x) = x$ ,

$x \perp 0 = C(x) \rightarrow 0 = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$ .

4)  $x \top C(x) = C(x \rightarrow CC(x)) = C(x \rightarrow x) = C(1) = 1 \rightarrow 0 = 0$ ,

$x \perp C(x) = C(x) \rightarrow C(x) = 1$ ;

5)  $C(C(x) \top C(y)) = C(C(x) \rightarrow CC(y)) = C(C(x) \rightarrow y) C(x) \rightarrow y = x \perp y$ , i.e.  $x \top y = C(C(x) \top C(y))$ ,

$C(C(x) \perp C(y)) = C(CC(x) \rightarrow C(y)) = C(x \rightarrow C(y)) = x \top y$ , i.e.  $x \perp y = C(C(x) \perp C(y))$ ;

6) By (W2), we have  $C(x) \rightarrow C(y) = (x \rightarrow 0) \rightarrow (y \rightarrow 0) = y \rightarrow ((x \rightarrow 0) \rightarrow 0) = y \rightarrow x$  and

$C(x) \rightarrow y = (x \rightarrow 0) \rightarrow y = (y \rightarrow 0) \rightarrow x = C(y) \rightarrow x$ .

## 5. Conclusion

The main aim of this article is to study  $W_d$ -fuzzy implication algebras which are subalgebra of fuzzy implication algebras. We showed that  $W_d$ -fuzzy implication algebras are regular fuzzy implication algebras, but the inverse is not true. The relations between  $W_d$ -fuzzy implication algebras and other fuzzy algebras are discussed. Properties and axiomatic systems for  $W_d$ -fuzzy implication algebras are investigated. Furthermore, a few new results on  $W_d$ -fuzzy implication algebras has been added, two new operations were introduced in  $W_d$ -fuzzy implication algebras and some further properties were given.

The work of this paper clearly suggests that  $W_d$ -fuzzy implication algebras provide a fertile area for future research. In future we will study the following topics:

(1) A HFI algebra form a  $W_d$ -fuzzy implication algebra under what conditions?

(2) A RFI algebra form a  $W_d$ -fuzzy implication algebra under what conditions?

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