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Solitary Wave Solutions for the Boussinesq and Fisher Equations by the Modified Simple Equation Method

Md. Ashrafuzzaman Khan[1], M. Ali Akbar[1], Fethi Bin Muhammad Belgacem[2]

[1]Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh

[2]Department of Mathematics, Faculty of Basic Education, PAAET, Al-Ardhyia, Kuwait

Email address:

akhanmath@yahoo.com (M. A. Khan), ali_math74@yahoo.com (M. A. Akbar), fbmbelgacem@gmail.com (F. B. M. Belgacem)

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Abstract: Although the modified simple equation method effectively provides exact traveling wave solutions to nonlinear evolution equations in the field of engineering and mathematical physics, it has some drawbacks. Particularly, if the balance number is greater than 1, the method cannot be expected to yield any solution. In this article, we present a process to implement the modified simple equation method to solve nonlinear evolution equations for balance number greater than 1, namely with balance number equal to 2. To validate our theory through applications, two equations have been chosen to undergo the proposed process, the Boussinesq and the Fisher equations, to which traveling wave are found and analyzed. For special parameters values, solitary wave solutions are originated from the exact solutions. We analyze the solitary wave properties by the graphs of the solutions. This shows the validity, usefulness, and necessity of the process.

Keywords: Boussinesq Equation, Fisher Equation, Modified Simple Equation Method, Nonlinear Evolution Equations, Solitary Wave Solutions

1. Introduction

The mathematical modeling of complex phenomena that change over time depends closely on the study of a variety of systems of ordinary and partial differential equations. Similar models are developed in diverse fields of study, ranging from the natural and physical sciences, population ecology to economics, infectious disease epidemiology, neural networks, biology, mechanics etc. In spite of the eclectic nature of the fields wherein these models are formulated, different groups of them contribute adequate common attributes that make it possible to examine them within a unified theoretical structure. Such studies make for a large area of functional analysis, usually called the theory of evolution equations (EEs) which may be linear or nonlinear. The latter are usually more challenging than their linear counterparts, and richer in terms of adequately modeling and describing complex phenomena. Therefore, the investigation of solutions to nonlinear evolution equations (NLEEs) plays a very important role to uncover the obscurity of many phenomena and processes throughout the natural sciences. However, one of the essential problems is to obtain their exact solutions. Therefore, in order to find out exact solutions to NLEEs different groups of mathematicians, physicist, and engineers have been working tirelessly. Accordingly, in the recent years, they establish several methods to search exact solutions, for instance, the inverse scattering method [1], the Hirota's bilinear transformation method [2], the Backlund transformation method [3], [4], the Darboux transformation method [5], the Painleve expansion method [6], the Adomian decomposition method [7], [8], the He's homotopy perturbation method [9], [10], the Jacobi's elliptic function method [11], [12], the Miura transformation method [13], the sine-cosine method [14], [15], the homogeneous balance method [16], the tanh-function method [17], [18], the extended tanh-function method [19], [20], the first integration method [21], the F-expansion method [22], the auxiliary equation method [23], the Lie group symmetry method [24], the variational iteration method [25], the ansatz method [26], [27], the Exp-function method [28], [29], the (G'/G)-expansion method [30], [31], [32], [33], [34], [35], the $\exp(-\phi(\eta))$-expansion method [41], [42], and the various versions and improvements of the (G'/G)-expansion method [45], [46], [47], [48], [49], and [50].

The modified simple equation method, [36], [37], [38],

[39], [40], being recently developed, is rising in use. Its computation is straightforward, systematic, and needs not the symbolic computation software to manipulate the algebraic equations. However, the method has some shortcomings. The main problem is that when the balance number is greater than one, the method usually does not give any solution. To the best of our knowledge, till now only two articles are available in the literature concerning higher balance number (for balance number two). In [43], Salam used the MSE method to the modified Liouville equation (wherein the balance number is two) and written down a solution to this equation. However, unfortunately the obtained solution does not satisfy the equation. Also, in Ref. [44], Zayed and Arnous solved the KP-BBM equation by means of the MSE method and found some solutions of this equation. Unfortunately, there is no guideline in this article, how one can solve other NLEEs for the higher balance number. In the present article, we have considered two equations; the balance number for each of these equations is two. If the balance number is greater than one, usually there arise difficulties in solving the NLEEs by means of the MSE method. One cannot use the MSE method in straight away. In this case, we need to take in some strategy. Inserting the assumed solution to the corresponding ordinary differential equation and then equating the coefficients of $s(\xi)^{-j}$, $(j=0,1,2,\cdots,N)$ yields an over-determined set of algebraic and differential equations. During determination of the unknown function, there born a third order linear ordinary differential equation in s and ξ.A polynomial appearing in the solution of s, ξ will make it ineligible for a solitary wave solution, because in this case, we have $|u|\to 0$ as $\xi\to\pm\infty$ [7]. Therefore, the coefficients of the polynomial must be zero. This constraint is essential to solve NLEEs for higher balance number.

The article is organized as follows: In section 2, we summarize the description of the method. In section 3, we employ the method to NLEEs with balance number is 2, and in section 4, we give conclusions.

2. The Modified Simple Equation (MSE) Method

To elaborate on the MSE method, let us consider the nonlinear evolution equation of the form,

$$H(u,u_t,u_x,u_y,u_z,u_{xx},u_{tt},...)=0, \tag{1}$$

where $u=u(x,t)$ is an unidentified function, H is a polynomial in $u(x,t)$ and its partial derivatives, which include the highest order derivatives and nonlinear terms of the highest order, and the subscripts denote partial derivatives. In order to solve Equation (1) by means of the MES method [36], [37], [38], [39], and [40], we have to execute the following steps:

Step 1: The traveling wave variable,

$$u(x,y,z,t)=u(\xi)\text{,with, } \xi=k(x+y+z\pm\omega t), \tag{2}$$

Permits for the change of Eq. (1), into the following ordinary differential equation (ODE):

$$G(u,u',u'',\cdots)=0, \tag{3}$$

Where, G is a polynomial in $u(\xi)$ and its derivatives, wherein $u'(\xi)=\frac{du}{d\xi}$.

Step 2: We suppose that Eq. (3) has the solution in the form,

$$u(\xi)=\sum_{i=0}^{N}a_i\left[\frac{s'(\xi)}{s(\xi)}\right]^i, \tag{4}$$

where a_i, $(i=0,1,2,\cdots,N)$ are unknown constants to be determined, such that $a_N\neq 0$, and $s(\xi)$ is an unknown function to be evaluated. In sine-cosine method, tanh-function method, (G'/G)-expansion method, Jacobi elliptic function method, Exp-function method etc., the solutions are proposed in terms of some functions established in advance, but in the MSE method, $s(\xi)$ is not pre-defined or not a solution of any pre-defined differential equation. Therefore, it is not possible to conjecture from earlier what kind of solutions one may get through this method. This is the individuality and distinction of this method. Therefore, some fresh solutions might be found by this method.

Step 3: The positive integer N appearing in Eq. (4) can be determined by taking into account the homogeneous balance between the highest order nonlinear terms and the derivatives of highest order occurring in Eq. (3). If the degree of $u(\xi)$ is $\deg[u(\xi)]=N$, therefore, the degree of the other expressions will be as follows:

$$\deg[\frac{d^m u(\xi)}{d\xi^m}]=N+m,$$

and,

$$\deg[u^m(\frac{d^l u(\xi)}{d\xi^l})^p]=mN+p(N+l).$$

Step 4: We substitute Eq. (4) into Eq. (3) and then we account the function $s(\xi)$. As a result of this substitution, we get a polynomial of $\left(s'(\xi)/s(\xi)\right)$ and its derivatives. In the resultant polynomial, we equate all the coefficients of $\left(s(\xi)\right)^{-i}$, $(i=0,1,2,...,N)$ to zero. This procedure yields a system of algebraic and differential equations which can be solved for getting the values of a_i $(i=0,1,2,\cdots,N)$, $s(\xi)$ and the value of the other needful parameters.

3. Applications of the MSE Method

In this section, we will execute the MSE method to extract solitary wave solutions to the Boussinesq equation and the Fisher equations which are very important in the fields of surface wave propagation in coastal regions, heat and mass transfer, biology, ecology, gene propagation, physiology, crystallization, plasma physics, and reaction-diffusion systems.

3.1. The Boussinesq Equation

In this sub-section, we will implement the MSE method to find the exact solutions and then the solitary wave solutions to the Boussinesq equation in the form:

$$u_{tt}-u_{xx}+\alpha(u^2)_{xx}+\beta\, u_{xxxx}=0\,. \tag{5}$$

where α and β are real constants.

To construct solitary wave solutions of the Boussinesq equation by applying the MSE method, we use the wave variable

$$u(x,t)=U(\xi),\ \xi=k(x-\omega t)\,. \tag{6}$$

The traveling wave transformation (3.2) reduces Eq. (3.1) to the following ODE:

$$(\omega^2-1)\,U''+\alpha(U^2)''+\beta k^2U^{(iv)}=0\,, \tag{7}$$

where prime denotes the ordinary derivatives with respect to ξ. Now, integrating Eq. (7) twice with respect to ξ and setting the constant of integration to zero, we obtain a new ODE in the form:

$$(\omega^2-1)\,U+\alpha\,U^2+\beta k^2U''=0\,. \tag{8}$$

Balancing the highest order derivative term U'' and the nonlinear term of the highest order U^2, we obtain $N=2$. Therefore, the solution (4) takes the form,

$$U(\xi)=a_0+a_1\left(\frac{s'}{s}\right)+a_2\left(\frac{s'}{s}\right)^2, \tag{9}$$

where $a_0\,, a_1$ and a_2 are constants such that $a_2\neq 0$ and $s(\xi)$ is an unknown function to be determined. Now, it is easy to make out,

$$U'=-\frac{a_1s'^2}{s^2}-\frac{2a_2s'^3}{s^3}+\frac{a_1s''}{s}+\frac{2a_2s's''}{s^2}. \tag{10}$$

$$U''=\frac{2a_1s'^3}{s^3}+\frac{6a_2s'^4}{s^4}-\frac{3a_1s's''}{s^2}-\frac{10a_2s'^2s''}{s^3}+\frac{2a_2s''^2}{s^2}+\frac{a_1s'''}{s}+\frac{2a_2s's'''}{s^2}. \tag{11}$$

Substituting the values of U, U' and U'' from (9)-(11) into Eq. (8) and then equating the coefficients of s^0, s^{-1}, s^{-2}, s^{-3}, s^{-4} to zero, we respectively obtain

$$a_0\,(-1+\omega^2+a_0\alpha)=0, \tag{12}$$

$$a_1\,(-1+\omega^2+2\,a_0\alpha)\,s'+k^2\beta\,s'''=0, \tag{13}$$

$$\begin{aligned}&\alpha\,a_1^2\,(s')^2-3\,k^2\beta\,a_1\,s'\,s''+a_2\,(-1+\omega^2+2\,a_0\alpha)(s')^2\\&+2\,k^2\,\beta\,(s'')^2+k^2\,\beta\,s'\,s'''=0\end{aligned} \tag{14}$$

$$2\,(s')^2(a_1\,(k^2\beta+\alpha\,a_2)\,s'-5\,k^2\beta\,a_2\,s'')=0, \tag{15}$$

$$a_2\,(6\,k^2\beta+\alpha\,a_2)(s')^4=0. \tag{16}$$

From Eq. (12) and Eq. (16), we obtain

$$a_0=0,\ \frac{1-\omega^2}{\alpha}\text{ and } a_2=-\frac{6\,k^2\beta}{\alpha},\text{ since } a_2\neq 0\,.$$

Therefore, we obtain the following two cases arises for the values of a_0.

Case 1: When $a_0=0$, then From Eq. (13)-(15), we get

$$a_1=\pm\frac{6\,k\sqrt{\beta\,(1-\omega^2)}}{\alpha}$$

And $s(\xi)=\dfrac{k^2\beta\,c_1}{\omega^2-1}\,e^{\pm\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\beta}}}+c_2$,

where c_1 and c_2 are integrating constant.

Now, using the values of a_0, a_1, a_2 and $s(\xi)$ into Eq. (9), we obtain the solution

$$U(\xi)=\frac{6k^2\beta\left(-1+\omega^2\right)^2c_1c_2}{\alpha\left(k^2\beta\,c_1e^{\pm\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\beta}}}-\left(-1+\omega^2\right)c_2\right)^2}\,e^{\pm\frac{\xi\sqrt{1-\omega^2}}{k\sqrt{\beta}}}\,. \tag{17}$$

Simplifying the required solution (17), we derive the following close-form solution of the Boussinesq equation:

$$u(x,t)=\frac{6k^2\beta\left(-1+\omega^2\right)^2c_1c_2}{\alpha\left[\begin{aligned}&k^2\beta c_1\left\{\cosh\left(\frac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\pm\sinh\left(\frac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\right\}\\&-\left(-1+\omega^2\right)c_2\left\{\cosh\left(\frac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\mp\sinh\left(\frac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\right\}\end{aligned}\right]^2}\,. \tag{18}$$

Since c_1 and c_2 are arbitrary constants, one may arbitrarily pick their values. If we choose $c_1=-1+\omega^2$ and $c_2=k^2\beta$ then from solution (18), we obtain

$$u_1(x,t)=\frac{3\left(-1+\omega^2\right)^2\cos ech^2\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)}{2\alpha}\,. \tag{19}$$

Again if we choose $c_1=-1+\omega^2$ and $c_2=-k^2\beta$ then from solution (18), we have the following solitary wave solution:

$$u_2(x,t) = -\frac{3\left(-1+\omega^2\right)^2 \operatorname{sech}^2\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)}{2\alpha}. \quad (20)$$

On the other hand, if $c_1 = 1$ and $c_2 = 1$, from solution (18), we derive the solitary wave solutions in the form:

$$u_3(x,t) = \frac{6k^2\beta\left(-1+\omega^2\right)^2}{\alpha\left\{\begin{array}{l}\left(1+k^2\beta+\omega^2\right)\cosh\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\\ \pm\left(-1+k^2\beta+\omega^2\right)\sinh\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\end{array}\right\}^2}. \quad (21)$$

Also when $c_1 = 1$ and $c_2 = -1$, then from solution (18) can be written as the following solitary wave solutions in the form:

$$u_4(x,t) = \frac{-6k^2\beta\left(-1+\omega^2\right)^2}{\alpha\left\{\begin{array}{l}\left(-1+k^2\beta+\omega^2\right)\cosh\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\\ \pm\left(1+k^2\beta-\omega^2\right)\sinh\left(\dfrac{(x-\omega t)\sqrt{1-\omega^2}}{2\sqrt{\beta}}\right)\end{array}\right\}^2}. \quad (22)$$

The solutions (19)-(20) are plotted and shown in Figure 1 and solutions (21)-(22) are plotted and shown in Figure 2 and 3 respectively for $\alpha = \beta = k = 1,\ \omega = 2$.

Case 2: When $a_0 = \dfrac{1-\omega^2}{\alpha}$ then from Eqs.(13)-(15), we obtain,

$$a_1 = \pm\frac{6k\sqrt{\beta(-1+\omega^2)}}{\alpha}, \text{ while}$$

$$s(\xi) = \frac{k^2\beta c_1}{\omega^2-1} e^{\pm\frac{\xi\sqrt{-1+\omega^2}}{k\sqrt{\beta}}} + c_2,$$

Where, c_1 and c_2 are integrating constant.

Now, using the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (9), we obtain the solution in the form:

$$U(\xi) = \frac{\left(-1+\omega^2\right)\left\{k^4\beta^2c_1^2e^{\pm\frac{2\xi\sqrt{-1+\omega^2}}{k\sqrt{\beta}}} - 4k^2\beta c_1c_2\left(-1+\omega^2\right)e^{\pm\frac{\xi\sqrt{-1+\omega^2}}{k\sqrt{\beta}}} + \left(-1+\omega^2\right)^2c_2^2\right\}}{\alpha\left\{k^2\beta c_1 e^{\pm\frac{\xi\sqrt{-1+\omega^2}}{k\sqrt{\beta}}} + \left(-1+\omega^2\right)c_2\right\}^2}. \quad (23)$$

Switching the exponential solution (23) into trigonometric function, we derive the solution of the Eq. (5):

$$u(x,t) = \frac{\left(\omega^2-1\right)\left[\begin{array}{l}k^4\beta^2c_1^2\left\{\cosh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\pm\sinh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}\\ -4k^2\beta c_1c_2\left(-1+\omega^2\right)\\ +\left(-1+\omega^2\right)^2c_2^2\left\{\cosh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\mp\sinh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}\end{array}\right]}{-\alpha\left[\begin{array}{l}k^2\beta c_1\left\{\cosh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right)\pm\sinh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right)\right\}\\ +\left(-1+\omega^2\right)c_2\left\{\cosh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right)\mp\sinh\left(\dfrac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right)\right\}\end{array}\right]^2}. \quad (24)$$

Thus, we get the exact solution (24) to the Boussinesq equation (5). But, since c_1 and c_2 are arbitrary constants, one may randomly pick their values. So, if we take $c_1 = -1+\omega^2$ and $c_2 = k^2\beta$ then the solitary wave solution (24) becomes,

$$u_1(x,t) = -\frac{(\omega^2-1)}{\alpha}\left\{-2+\cosh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}\sec h^2\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right). \quad (25)$$

Again, if we choose $c_1 = -1+\omega^2$ and $c_2 = -k^2\beta$ then from (24), we obtain the following solitary wave solutions in the form:

$$u_2(x,t) = -\frac{(\omega^2-1)}{\alpha}\left\{2+\cosh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}\cos ech^2\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{2\sqrt{\beta}}\right). \quad (26)$$

On the other hand, if we take $c_1 = -1+\omega^2$ and $c_2 = \pm\sqrt{-k^4\beta^2}$ then from (24), we get the solution in the form:

$$u_3(x,t) = \frac{(\omega^2-1)\left\{2+i\sinh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}}{\alpha\left\{1-i\sinh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}}. \quad (27)$$

Also, if $c_1 = -1+\omega^2$ and $c_2 = \mp\sqrt{-k^4\beta^2}$ then we derive the solitary wave solution (24) can be written in the form:

$$u_4(x,t) = \frac{(\omega^2-1)\left\{2-i\sinh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}}{\alpha\left\{1+i\sinh\left(\frac{(x-\omega t)\sqrt{\omega^2-1}}{\sqrt{\beta}}\right)\right\}}. \quad (28)$$

The solutions (25)-(26) are plotted and shown in Figure 4 and solutions (27)-(28) are plotted and shown in Figure 5, for $\alpha = \beta = 1$, $\omega = 2$.

The major advantage of the MSE method is that the calculations are not sophisticated and easy to control. It is not required any computer algebra system to facilitate the calculations as it take to the Exp-function method, the (G'/G)-expansion, the tanh-function method, the homotopy analysis method etc. But the solutions obtained by the MSE method are equivalent to those solutions obtained by the above mentioned method. Since c_1 and c_2 are arbitrary constants for other choices of c_1 and c_2, we might obtain much new and more general exact solutions of Eq. (5) by the MSE method without any aid of symbolic computation software.

Remark 1: Solutions (19)-(22) and (25)-(28) have been verified by putting them back into the original equation and found correct.

3.2. Physical Interpretations of the Boussinesq Equation Solutions

In this sub-section, we will depict the graph and signify the obtained solutions to the Boussinesq equation. The solutions (19) and (20) represent the singular periodic solutions. Periodic solutions are traveling wave solutions that are periodic, such that Fig. 1 shows the shape of the solutions (19) and (20) for $\alpha = \beta = 1$, $\omega = 2$ within $-2 \le x, t \le 2$. Solitons are solitary waves with resilient scattering property. Solutions (21) and (22) are complex solutions, therefore, the modulus and arguments of these solutions have been plotted. The graph of modulus of the solutions (21) and (22) have been shown in Fig. 2, and their arguments have been shown in Figs. 3 and 4 respectively for $\alpha = \beta = k = 1$, $\omega = 2$ within $-2 \le x, t \le 2$. On the other hand, Fig. 5 shows that the solution (25) is the bell shape soliton and solution (26) is singular bell shape soliton. Also, the solutions (27) and (28) are complex solutions, therefore, the modulus and arguments of these solutions have been plotted. The graph of modulus and arguments of that solutions have been shown in Fig. 6 and 7 respectively for $\alpha = \beta = 1$, $\omega = 2$, such that, $-2 \le x, t \le 2$.

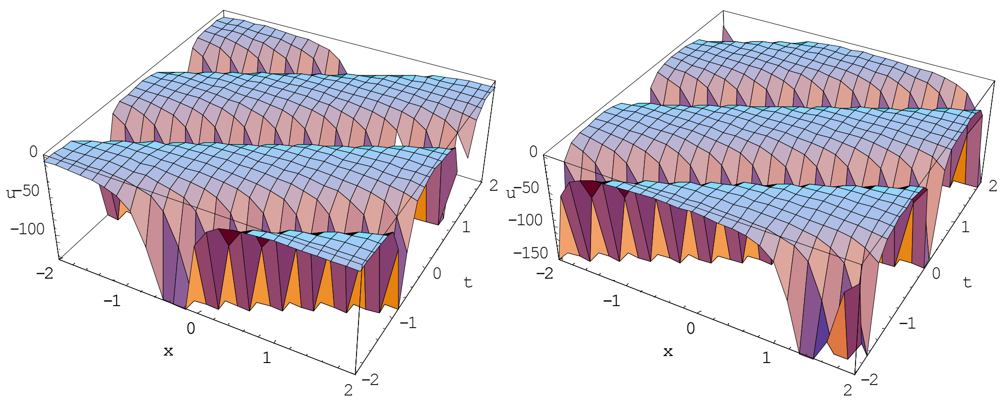

Fig. 1. *Periodic solutions u_1 and u_2 in (19) and (20) to the Boussinesq equation (5).*

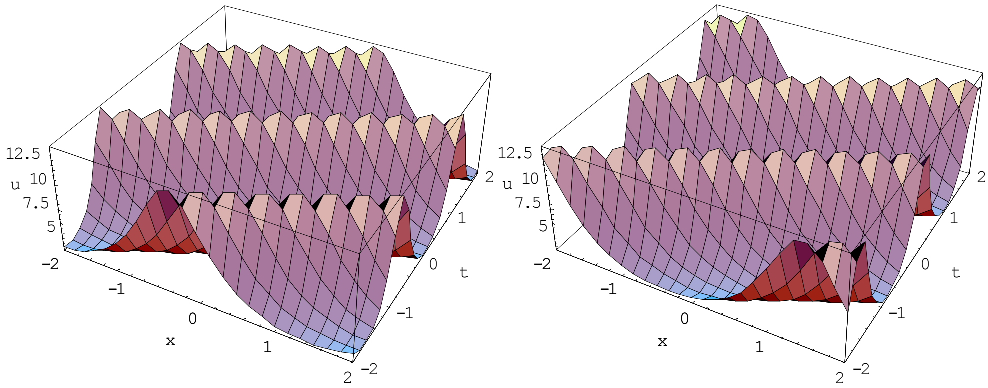

Fig. 2. *Plot of the modulus of the solutions* u_1 *and* u_2 *in (21) and (22) to the Boussinesq equation (5).*

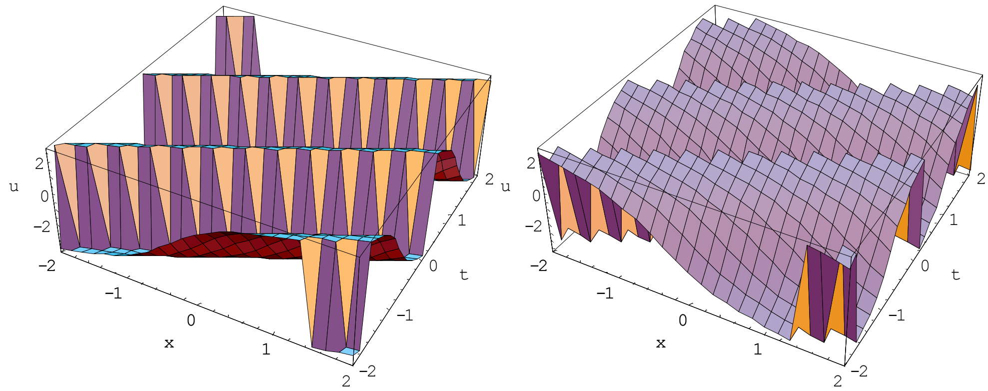

Fig. 3. *Plot of the arguments of the solution* u_3 *in (21) to the Boussinesq equation (5).*

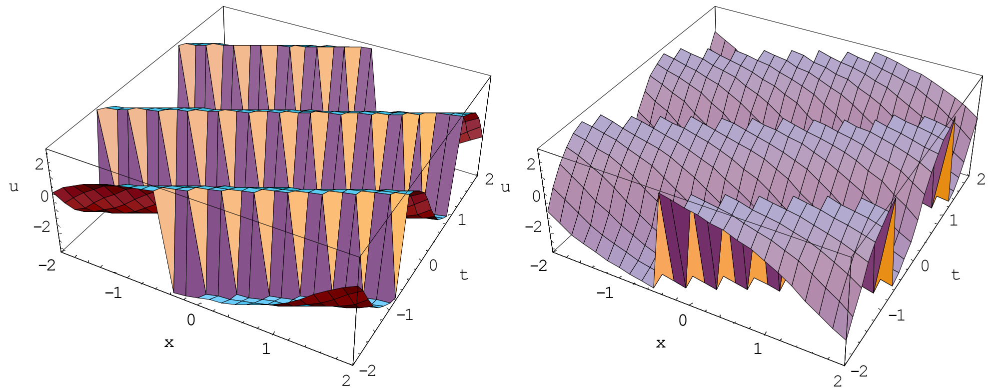

Fig. 4. *Plot of the arguments of the solution* u_4 *in (22) to the Boussinesq equation (5).*

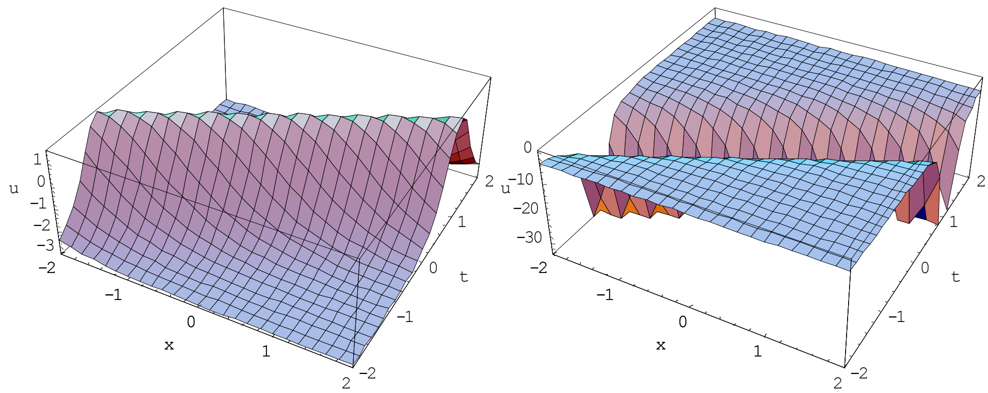

Fig. 5. *Bell shape solution* u_1 *in (25) and singular bell shape solution* u_2 *in (26) to the Boussinesq equation.*

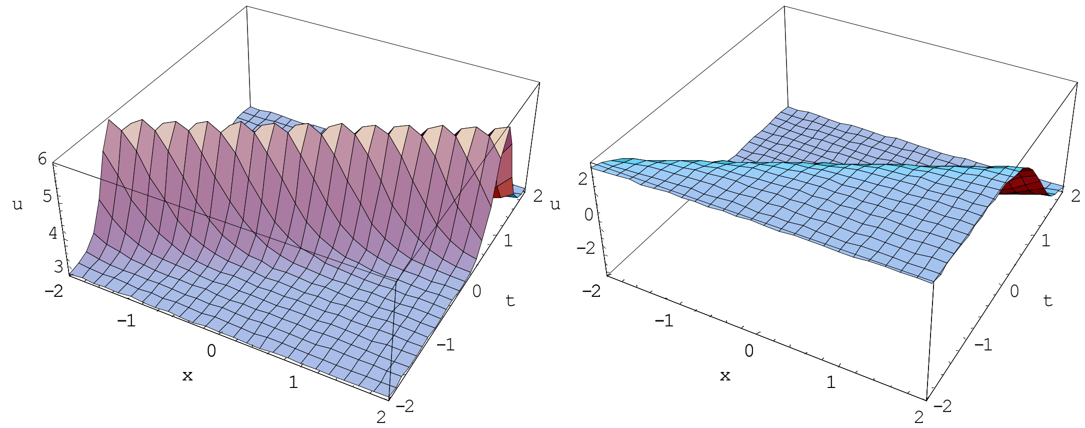

Fig. 6. *Plot of the modulus and arguments of the solutions u_3 in (27) to the Boussinesq equation (5).*

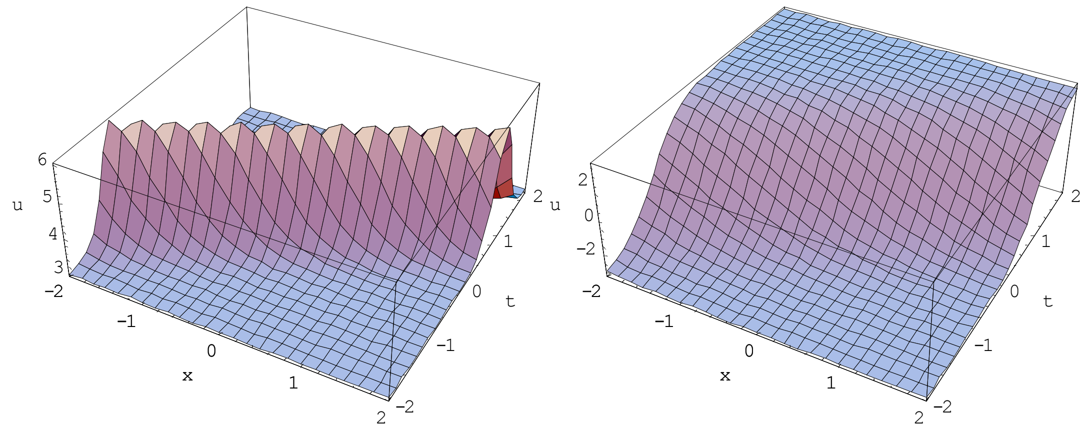

Fig. 7. *Plot of the modulus and arguments of the solutions u_4 in (28) to the Boussinesq equation (5).*

3.3. The Fisher Equation

In this sub-section, we will make use of the MSE method to obtain new and more general exact solutions and then the solitary wave solutions to the Fisher equation in the form:

$$u_t - u_{xx} - u(1-u) = 0. \tag{29}$$

The traveling wave transformation in Eq. (6) helps reduce Eq. (29) to the following ODE,

$$\omega k\, U' + k^2 U'' + U(1-U) = 0, \tag{30}$$

where prime denotes the derivatives with respect to ξ. Balancing the highest order derivative term U'' and highest order nonlinear term U^2, we have $N=2$. Therefore, the form of the solution of Eq. (30) is similar to the form of the solution in Eq. (9).

Substituting the values of U, U' and U'' from (9)-(11) into Eq. (30) and then equating the coefficients of $s^0, s^{-1}, s^{-2}, s^{-3}, s^{-4}$ to zero, we respectively obtain the following equations,

$$-a_0(-1+a_0) = 0. \tag{31}$$

$$a_1\left\{(1-2a_0)s' + k\left(\omega s'' + k s'''\right)\right\} = 0. \tag{32}$$

$$\begin{aligned}&-a_1^2(s')^2 - k a_1 s'\left(\omega s' + k s''\right)\\ &\quad + a_2\left\{(1-2a_0)(s')^2 + 2k^2(s'')^2 + 2k s'\left(\omega s'' + k s'''\right)\right\} = 0\end{aligned}. \tag{33}$$

$$2(s')^2\left\{a_1(k^2-a_2)s' - k a_2\left(\omega s' + 5k s''\right)\right\} = 0. \tag{34}$$

$$a_2(6k^2 - a_2)(s')^4 = 0. \tag{35}$$

From Eq. (31), (34) and (35), we obtain

$a_0 = 0, 1$; $a_2 = 6k^2$, since $a_2 \neq 0$, and,

$$s(\xi) = \frac{30k^2 c_1}{-6k\omega - 5a_1} e^{\frac{\xi(-6k\omega - 5a_1)}{30k^2}} + c_2,$$

where c_1 and c_2 are constants of integration. Hence the following two cases arise for the values of a_0.

Case 1: When $a_0 = 0$, then from Eq. (32)-(33), we obtain

$$\left\{a_1 = 0,\ \omega = \pm\frac{5}{\sqrt{6}}\right\}, \left\{a_1 = \pm 6i\,k,\ \omega = 0\right\},$$

$$\left\{a_1 = \pm 2i\sqrt{6}k,\ \omega = \mp\frac{5i}{\sqrt{6}}\right\}.$$

Therefore, three cases arises depending on the values for a_1.

Case 1(a): When $a_1 = 0$ and $\omega = \pm\frac{5}{\sqrt{6}}$, then using the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (3.5), we get

$$U(\xi) = \frac{6k^2 c_1^2 e^{\mp\frac{\xi}{k}\sqrt{\frac{2}{3}}}}{\left(\sqrt{6}\,k\,c_1\,e^{\mp\frac{\xi}{\sqrt{6}k}} \mp c_2\right)^2}. \quad (36)$$

Simplifying the solution (36) of the Fisher equation (29), we obtain

$$u(x,t) = \frac{6k^2c_1^2\left\{\cosh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right) \pm \sinh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right)\right\}}{\left[\begin{array}{l}\sqrt{6}\,k\,c_1\left\{\cosh\left(\frac{5t}{12}-\frac{x}{2\sqrt{6}}\right)\pm\sinh\left(\frac{5t}{12}-\frac{x}{2\sqrt{6}}\right)\right\} \\ +c_2\left\{\mp\cosh\left(\frac{5t}{12}-\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}-\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]^2}. \quad (37)$$

Hence we get the more general exact solution (37) of the Eq. (29). Since, c_1 and c_2 are arbitrary constants, we might randomly opt their values. Therefore, if we take $c_2 = \pm k\sqrt{6}$ and $c_1 = 1$, we derive the solitary wave solution of the Fisher equation (29) from Eq. (37) as follows:

$$u_1(x,t) = \frac{1}{4}\operatorname{cos}ech^2\left(\frac{x}{2\sqrt{6}}-\frac{5t}{12}\right)\left\{\cosh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right)\pm\sinh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right)\right\}. \quad (38)$$

Again, if choose $c_2 = \mp k\sqrt{6}$ and $c_1 = 1$ then from the solution Eq. (37),we obtain the following solitary wave solution:

$$u_2(x,t) = \frac{1}{4}\sec h^2\left(\frac{x}{2\sqrt{6}}-\frac{5t}{12}\right)\left\{\cosh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right)\pm\sinh\left(\frac{5t}{6}-\frac{x}{\sqrt{6}}\right)\right\}. \quad (39)$$

Solution (38)is plotted and shown in Figure 8 and solution (39)is plotted and shown in Figure 9.

Case 1(b): When $a_1 = \pm 6ik$ and $\omega = 0$ then substituting the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (9), we get

$$U(\xi) = \mp\frac{6ik\,c_1c_2\,e^{\mp\frac{i\xi}{k}}}{\left(k\,c_1\,e^{\mp\frac{i\xi}{k}} \mp i\,c_2\right)^2}. \quad (40)$$

Now, simplifying the exponential solution (40), we obtain

$$u(x,t) = \mp\frac{6ik\,c_1c_2}{\left[k\,c_1\left\{\cos\left(\frac{x}{2}\right)\mp i\sin\left(\frac{x}{2}\right)\right\}+c_2\left\{\mp i\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right\}\right]^2}. \quad (41)$$

Since, in this case $\omega = 0$, *i.e.* the wave speed is zero, the wave is no more traveling wave. Therefore, we are not interested to discuss this case. Hence it is rejected.

Case 1(c): When $a_1 = \pm 2i\sqrt{6k}$ and $\omega = \mp\frac{5i}{\sqrt{6}}$ then substituted the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (9), we get

$$U(\xi) = \frac{2k\,c_1\,e^{\pm\frac{i\xi}{\sqrt{6}k}}\left(3k\,c_1\,e^{\pm\frac{i\xi}{\sqrt{6}k}} \pm i\sqrt{6}\,c_2\right)}{\left(\sqrt{6}\,k\,c_1\,e^{\pm\frac{i\xi}{\sqrt{6}k}} \pm i\,c_2\right)^2}. \quad (42)$$

Now, simplifying the solution (42), we obtain the following trigonometric function solution:

$$u(x,t) = \frac{\left(\begin{array}{l}2kc_1\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ \times\left[\begin{array}{l}3k\,c_1\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ \pm i\sqrt{6}\,c_2\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]\end{array}\right)}{\left[\begin{array}{l}\sqrt{6}\,k\,c_1\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ \pm i\,c_2\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]^2}. \quad (43)$$

Thus, we get the more general exact solution (43) of the Fisher equation. Since c_1 and c_2 are constants of integration, we may intuitively choose their values. Therefore, if we choose $c_2 = \sqrt{6}k$, and, $c_1 = \pm i$, then the solution(43)is simplified to yield,

$$u_1(x,t) = \frac{1}{4}\left\{3+2\tanh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\tanh^2\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}. \quad (44)$$

Again if $c_2 = \sqrt{6}k$ and $c_1 = \mp i$ then from the exact solution (43) we derived the following solitary wave solution,

$$u_2(x,t) = \frac{1}{4}\left\{3+2\coth\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\coth^2\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}. \quad (45)$$

Another way is that we choose $c_1 = \pm i\sqrt{6}$ and $c_2 = 3k$ then from solution (43), we get,

$$u_3(x,t) = \frac{4\left\{1+\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{4+5\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+3\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (46)$$

Also, if we take $c_1 = \mp i\sqrt{6}$ and $c_2 = 3k$ then the solution (43) becomes

$$u_4(x,t)=\frac{4\left\{-1+\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{-4+5\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+3\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (47)$$

These solutions (44)-(47) are plotted as shown in Figure 10 to 15, respectively.

Case 2: When $a_0=1$ then from Eq. (32)-(33), we obtain

$$\left\{a_1=0,\ \omega=\pm\frac{5i}{\sqrt{6}}\right\},\ \left\{a_1=\pm 2\sqrt{6}\,k,\ \omega=\mp\frac{5}{\sqrt{6}}\right\},$$
$$\{a_1=\pm 6k,\ \omega=0\}.$$

Again for values of a_1 we can discuss the following three cases.

Case 2(a): When, $a_1=0$, and, $\omega=\pm\frac{5i}{\sqrt{6}}$, then using the values of using the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (9), we obtain exponential solution:

$$U(\xi)=1+\frac{6k^2c_1^2\,e^{\mp i\frac{\xi}{k}\sqrt{\frac{2}{3}}}}{\left(\pm i\sqrt{6}\,k\,c_1\,e^{\mp i\frac{\xi}{\sqrt{6}k}}+c_2\right)^2}. \quad (48)$$

After simplification, from solution (48) we obtain the following trigonometric solution of the Fisher equation (29):

$$u(x,t)=\frac{\left(\begin{array}{l} c_2\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ \times\left[\begin{array}{l}\pm 2i\sqrt{6}\,kc_1\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ +c_2\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]\end{array}\right)}{\left[\begin{array}{l}\pm i\sqrt{6}\,k\,c_1\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\} \\ +c_2\left\{\cosh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]^2}. \quad (49)$$

The solution (49)is more general exact solution of the Fisher equation(29). If we pick $c_2=\pm i\sqrt{6}\,k$ and $c_1=1$, from solution(49), we obtain the following solitary wave solution:

$$u_1(x,t)=\frac{1}{4}\left\{3+2\tanh\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\tanh^2\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}. \quad (50)$$

Again, if we put, $c_2=\mp i\sqrt{6}\,k$, and, $c_1=1$ then the solution(49) of the Fisher equation becomes,

$$u_2(x,t)=\frac{1}{4}\left\{3+2\coth\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)-\coth^2\left(\frac{5t}{12}\pm i\frac{x}{2\sqrt{6}}\right)\right\}. \quad (51)$$

On the other hand, if $c_2=\pm i2\sqrt{6}\,k$ and $c_1=1$ then we derive the following exact solution of Eq. (49):

$$u_3(x,t)=\frac{4\left\{1+\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{4+5\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+3\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (52)$$

Also, if we choose $c_2=\mp i2\sqrt{6}\,k$ and $c_1=1$ then from Eq. (49), we obtain the following solitary wave solution:

$$u_4(x,t)=\frac{4\left\{-1+\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{-4+5\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+3\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (53)$$

On the other hand, if we set $c_1=1$ and $c_2=1$, from Eq. (3.45), we get

$$u_5(x,t)=\frac{2\sqrt{6}\,k\mp\left\{i\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{2\sqrt{6}\,k\pm i\left(6k^2-1\right)\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\mp i\left(1+6k^2\right)\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (54)$$

Finally if we take $c_1=1$ and $c_2=-1$ then the exact solution (49) can be written as:

$$u_6(x,t)=\frac{2\sqrt{6}\,k\pm i\left\{\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\right\}}{2\sqrt{6}\,k\mp i\left(6k^2-1\right)\cosh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)\pm i\left(1+6k^2\right)\sinh\left(\frac{5t}{6}\pm i\frac{x}{\sqrt{6}}\right)}. \quad (55)$$

The solutions (50)-(55) are drawn in Figure 16-24, respectively.

Case 2(b): When $a_1=\pm 2\sqrt{6}\,k$ and $\omega=\mp\frac{5}{\sqrt{6}}$ then substituting the values of a_0, a_1, a_2 and $s(\xi)$ in Eq. (3.5), we get

$$U(\xi)=\frac{c_2^2\,e^{\mp\frac{\xi}{k}\sqrt{\frac{2}{3}}}}{\left(\sqrt{6}\,k\,c_1\mp c_2\,e^{\mp\frac{\xi}{\sqrt{6}k}}\right)^2}. \quad (56)$$

Now, simplifying the solution (56) we obtain the following close-form of the Fisher equation:

$$u(x,t)=\frac{c_2^2\left\{\cosh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)\pm\sinh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)\right\}}{\left[\begin{array}{l}\sqrt{6}\,k\,c_1\left\{\mp\cosh\left(\frac{5t}{12}\pm\frac{x}{2\sqrt{6}}\right)+\sinh\left(\frac{5t}{12}\pm\frac{x}{2\sqrt{6}}\right)\right\} \\ +c_2\left\{\cosh\left(\frac{5t}{12}\pm\frac{x}{2\sqrt{6}}\right)\pm\sinh\left(\frac{5t}{12}\pm\frac{x}{2\sqrt{6}}\right)\right\}\end{array}\right]^2}. \quad (57)$$

Since c_1 and c_2 are arbitrary constants, so we may take $c_2=\pm k\sqrt{6}$ and $c_1=1$, then the general solution(57) can be written as the following form:

$$u_1(x,t)=\frac{1}{4}\operatorname{cosech}^2\left(\frac{5t}{12}\pm\frac{x}{2\sqrt{6}}\right)\left\{\cosh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)\right\}. \quad (58)$$

Again, if we set $c_2=\mp k\sqrt{6}$ and $c_1=1$ then from Eq. (57), we obtain the following solitary wave solution,

$$u_2(x,t)=\frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2\sqrt{6}}\pm\frac{5t}{12}\right)\left\{\cosh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)+\sinh\left(\frac{5t}{6}\pm\frac{x}{\sqrt{6}}\right)\right\}. \quad (59)$$

Solutions (58) - (59) are plotted as shown in Figure 21-22, respectively.

Case 2(c): When $a_1=\pm 6k$ and $\omega=0$, then substituting the values of a_0, a_1, a_2 and $s(\xi)$ into Eq. (3.5), we derive the exponential form general solution:

$$U(\xi)=\frac{k^2c_1^2\pm 4kc_1c_2\ e^{\mp\frac{\xi}{k}}+c_2^2\ e^{\mp\frac{2\xi}{k}}}{\left(k\,c_1\mp i\,c_2\ e^{\mp\frac{\xi}{k}}\right)^2}. \quad (60)$$

Simplifying the exponential solution transformed to the trigonometric function in the following close-form solution of the Eq. (29):

$$u(x,t)=\frac{k^2c_1^2\{\cosh(x)\mp\sinh(x)\}\pm 4kc_1c_2+c_2^2\{\cosh(x)\pm\sinh(x)\}}{\left[k\,c_1\left\{\cosh\left(\frac{x}{2}\right)\mp\sinh\left(\frac{x}{2}\right)\right\}\mp c_2\left\{\cosh\left(\frac{x}{2}\right)\pm\sinh\left(\frac{x}{2}\right)\right\}\right]^2}. \quad (61)$$

This is the more general exact solution (61)to the Fisher equation(29). But, this is not the traveling wave solution, since the wave velocity is null. So, we are not interested to discuss the general solution (61) to the Fisher equation (29).

Imperative is it now to point out that solutions derived by the MSE method are equipotential to those solutions obtained by the previously mentioned method. Since, c_1 and c_2 are arbitrary constants, we may obtain new and/or more general exact solutions to Eq. (29) by the MSE method without any aid of symbolic computation software.

Remark 2: Solutions (38)-(39), (41),(44)-(47), (50)-(55), (58)-(59) and (61) have been confirmed by setting them into the original equation.

3.4. Physical Interpretations of the Fisher Equation Solutions

In this sub-section, we discuss the physical interpretation of the solutions to the Fisher equation. The solution (38) represents the bell shape soliton and the solution (39) represents the kink. The bell shape soliton is a localized surface "wave envelope" that causes a temporary increase in wave amplitude and the kink waves are traveling waves which arise from one asymptotic state to another. The kink solutions are approach to a constant at infinity. Figs. 8 and 9 show the shape of the solutions (38) and (39) within $-10\le x,\, t\le 10$ .Solutions (44)-(47) and (50)-(55) are complex solutions, therefore, the modulus and arguments of these solutions have been plotted. The graph of modulus of the solutions (44)-(47) and (50)-(55) have been shown in Figs. 10, 13, 16, 19 and 22 respectively. On the other hand, the graph of arguments of the solutions (44)-(47) and (50)-(55) have been shown in Figs. 11, 12, 14, 15, 17, 18, 20,21, 23 and 24 respectively. These solutions are plotted for $k=1$ .within $-10\le x,\, t\le 10$. Solutions (58) represent the bell shape solitons and solution (59) represents the kink. The shapes of these solutions are plotted in Figs. 25 and 26 within $-10\le x,\, t\le 10$.

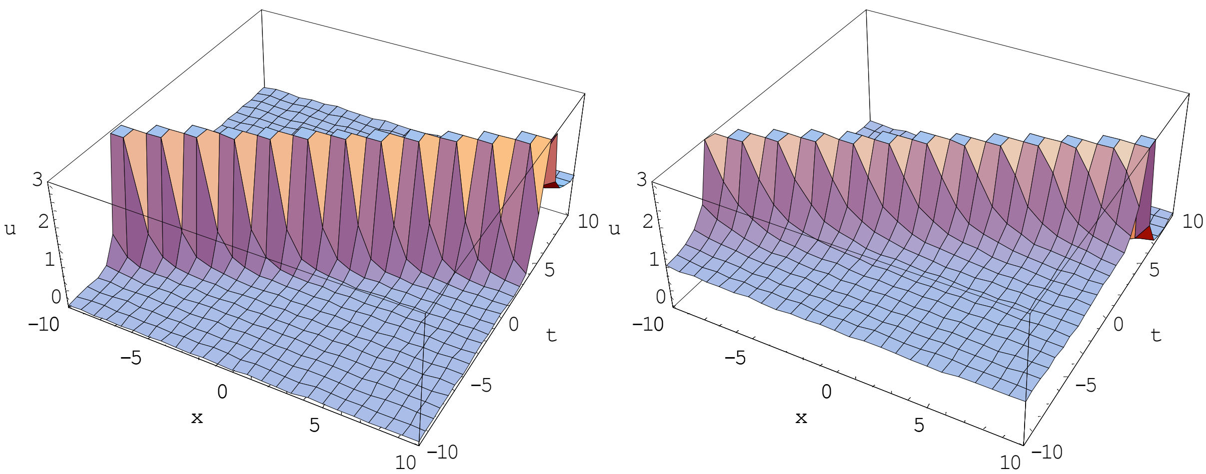

Fig. 8. *Bell shape solitary wave solution given in (38) to the Fisher equation (29).*

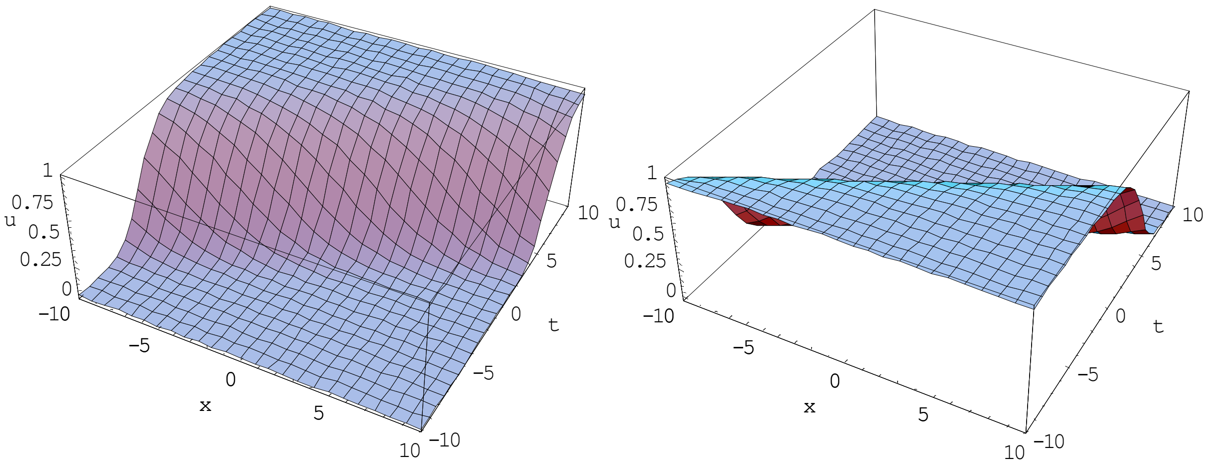

Fig. 9. *Kink soliton obtained from (39) to the Fisher equation (29).*

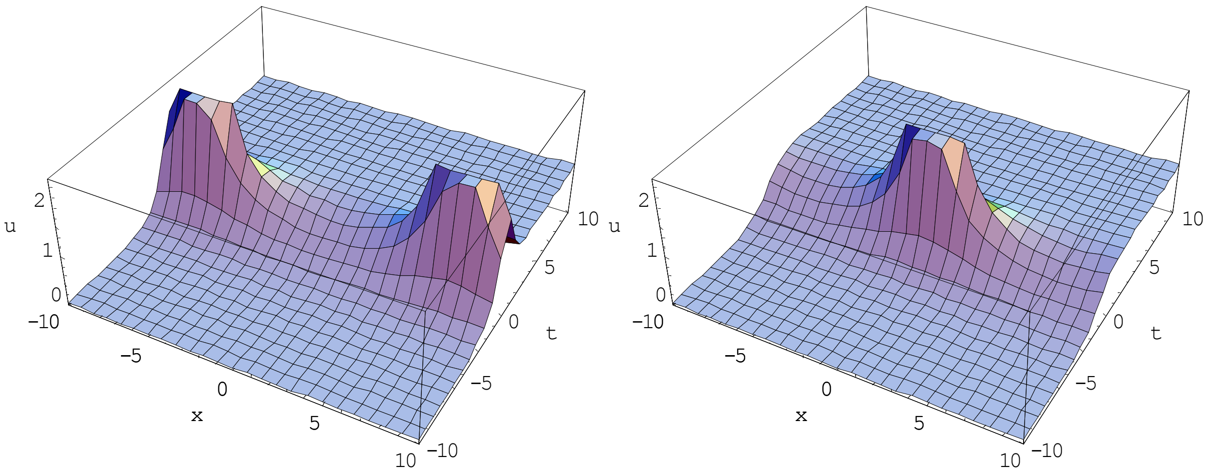

Fig. 10. *Plot of the modulus of the solitary wave solutions* u_1 *and* u_2 *given in 44) and (45) to the Fisher equation (29).*

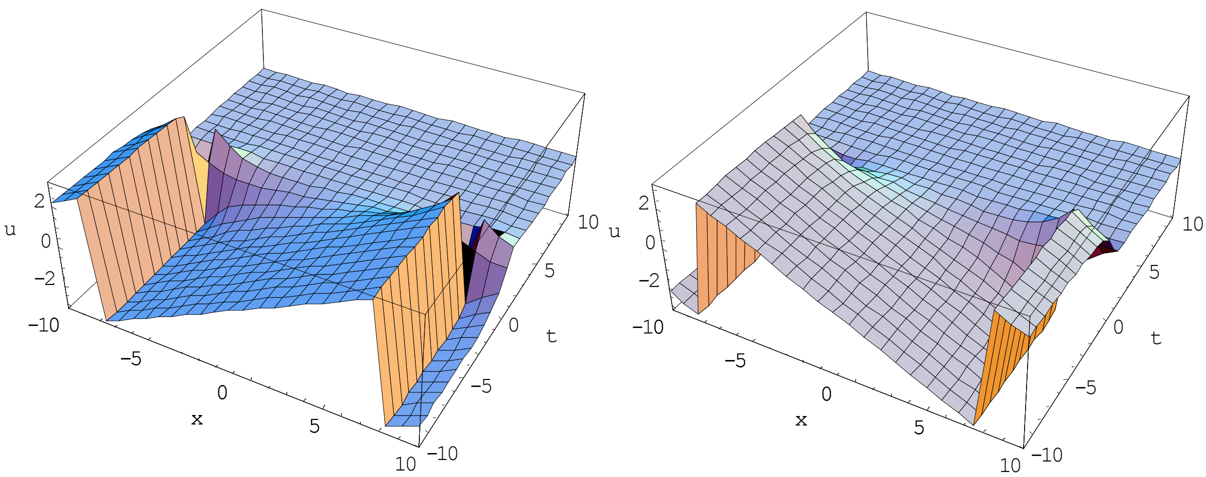

Fig. 11. *Plot of the arguments of the solitary wave solution* u_1 *in (44) to the Fisher equation (29).*

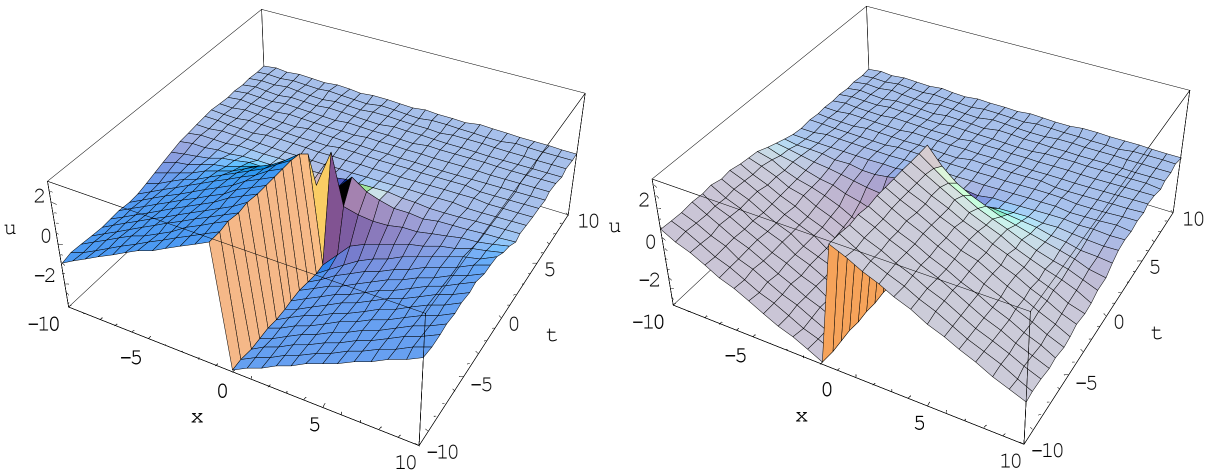

Fig. 12. *Plot of the arguments of the soliton (45) to the Fisher equation (29).*

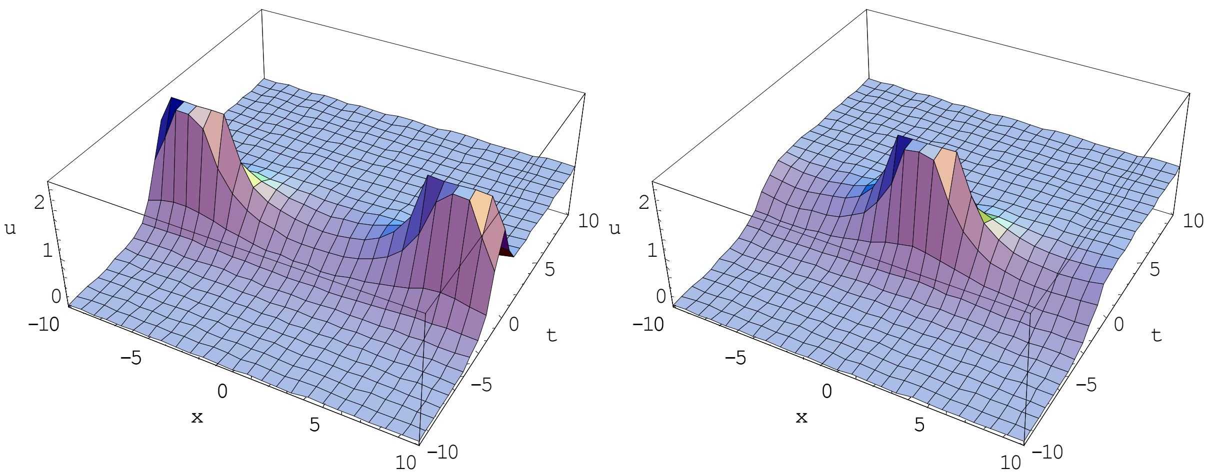

Fig. 13. *Graph of the modulus of the solitary wave solitons* u_3 *and* u_4 *to the Fisher equation.*

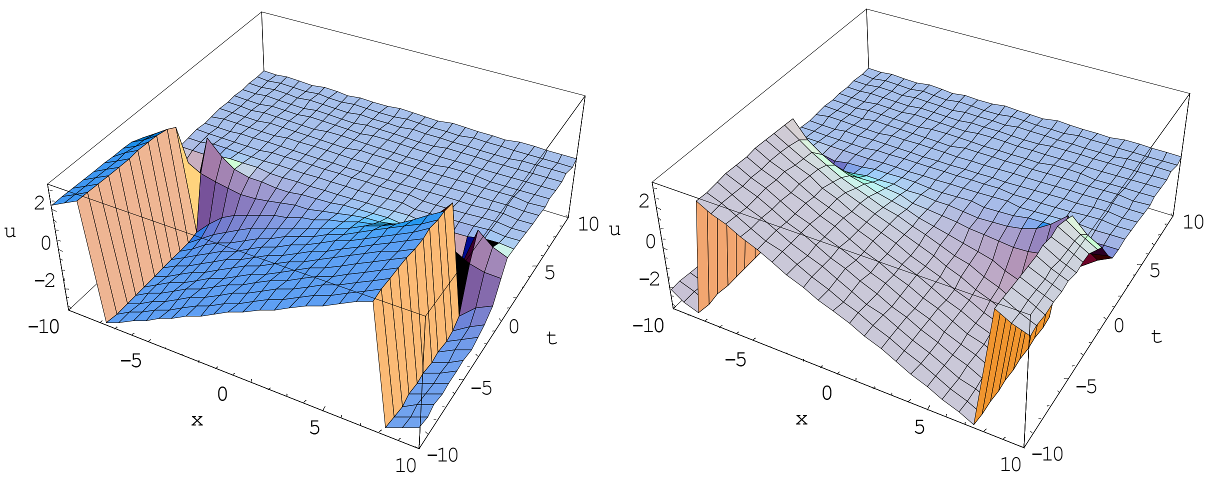

Fig. 14. *Plot of the arguments of the solitary wave solution* u_3 *in (46) to the Fisher equation (29).*

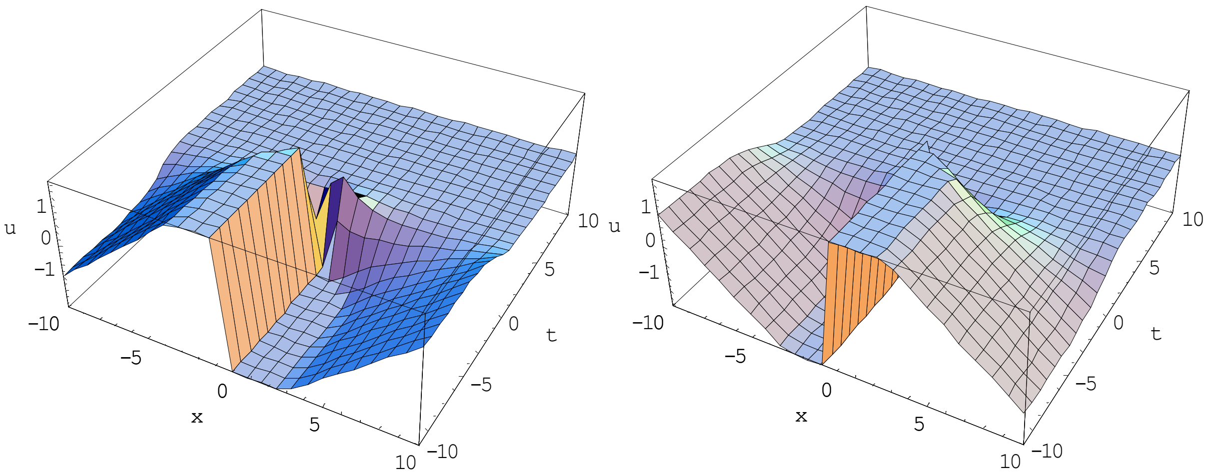

Fig. 15. *Figure of the arguments of the soliton u_4 in (47) to the Fisher equation.*

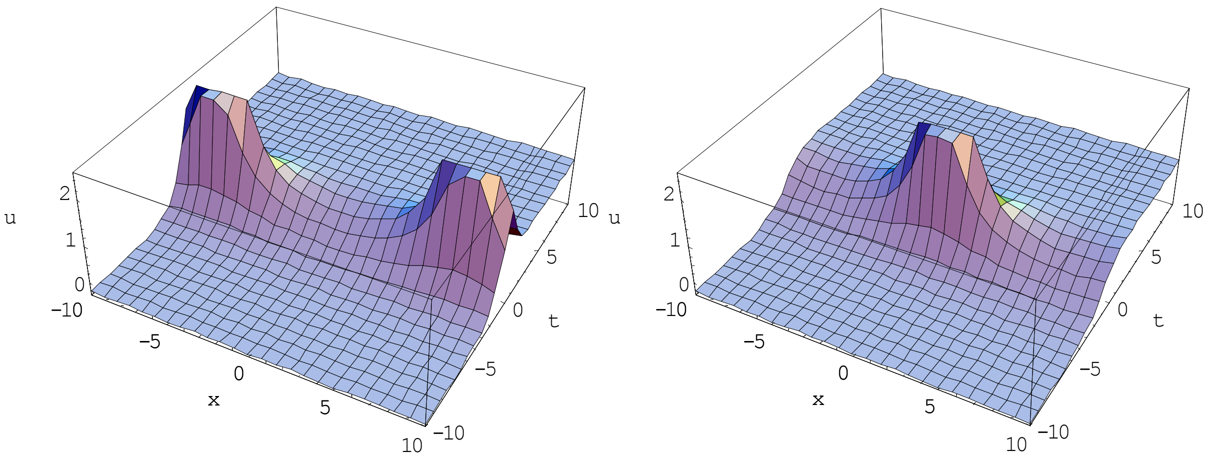

Fig. 16. *Figure of the modulus of the solitons u_1 in (50) and u_2 in (51) to the Fisher equation (29).*

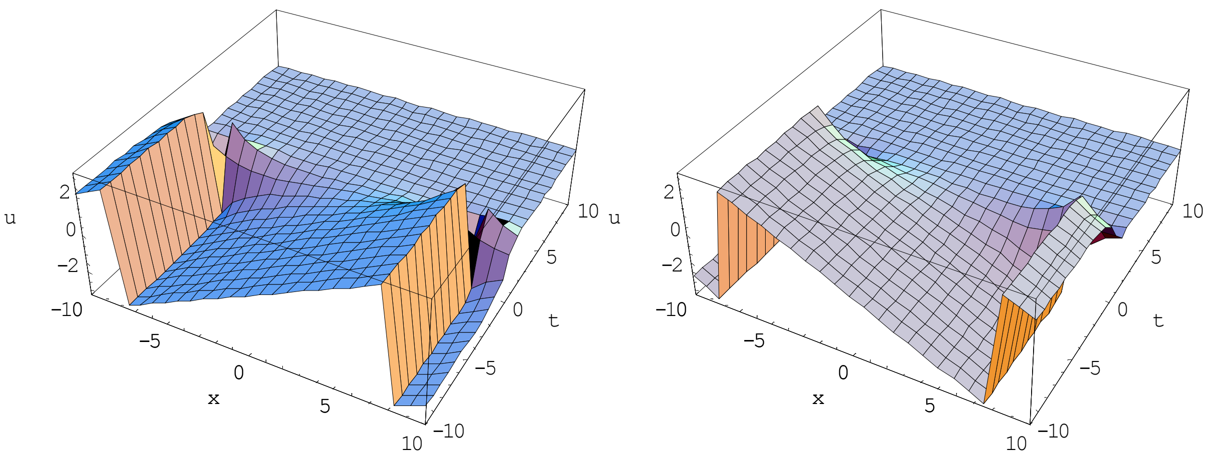

Fig. 17. *Sketch of the arguments of the solitary wave solution u_1 in (50) to the Fisher equation.*

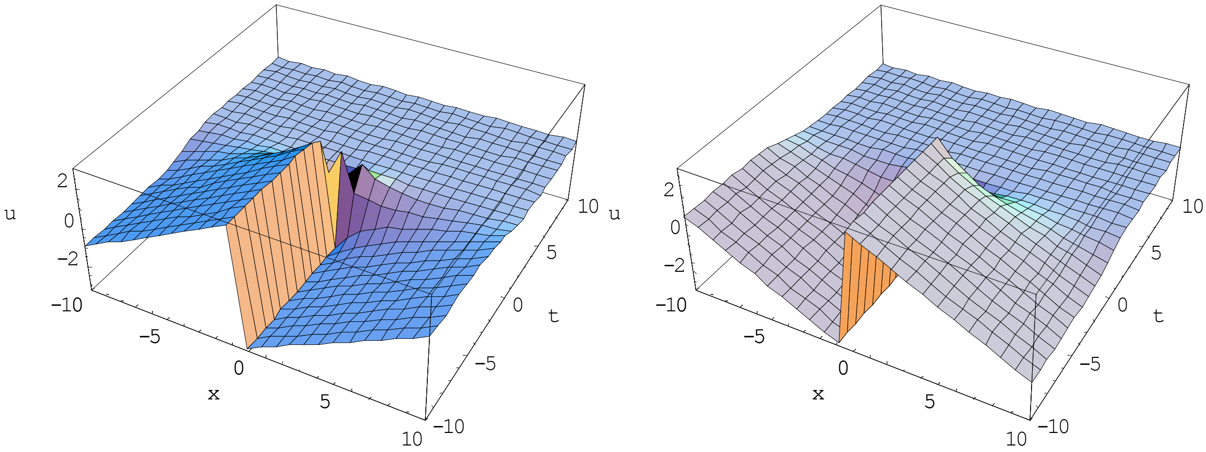

Fig. 18. *Sketch of the arguments of the soliton* u_2 *in (51) to the Fisher equation.*

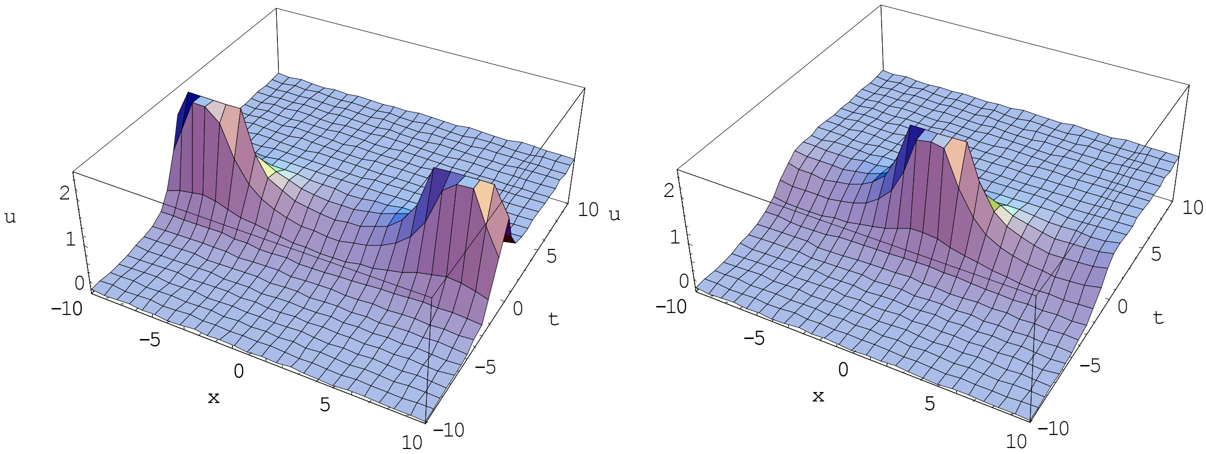

Fig. 19. *Plot of the modulus of the solitons (52) and (53) to the Fisher equation.*

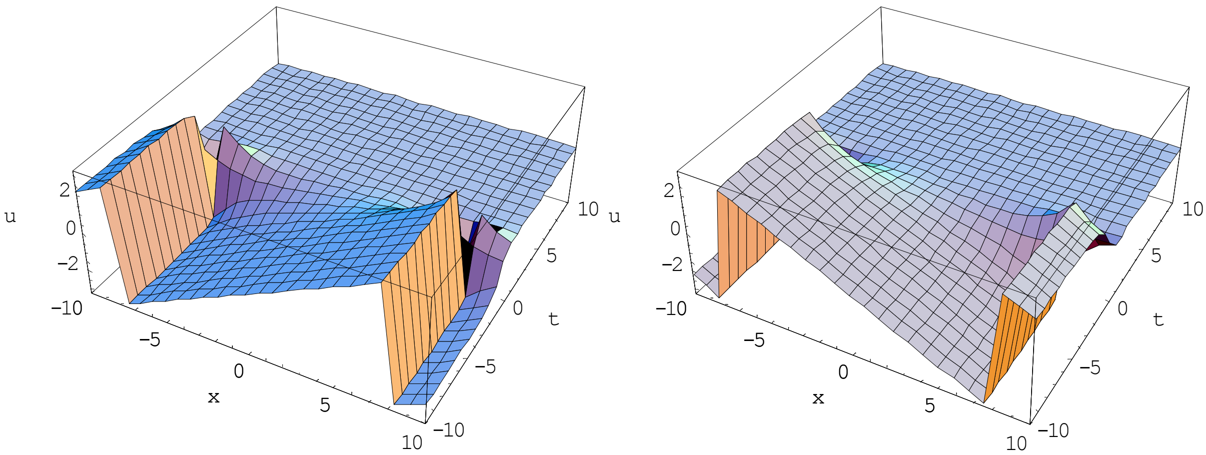

Fig. 20. *To the figure of the arguments of the solution* u_3 *in (52) to the Fisher equation.*

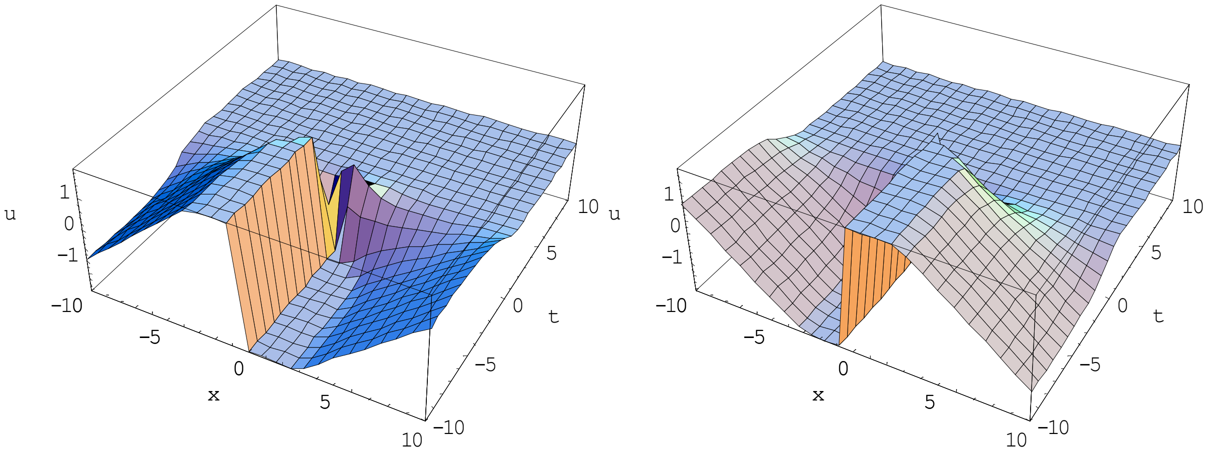

Fig. 21. *Sketch of the arguments of the soliton u_4 in (53) to the Fisher equation.*

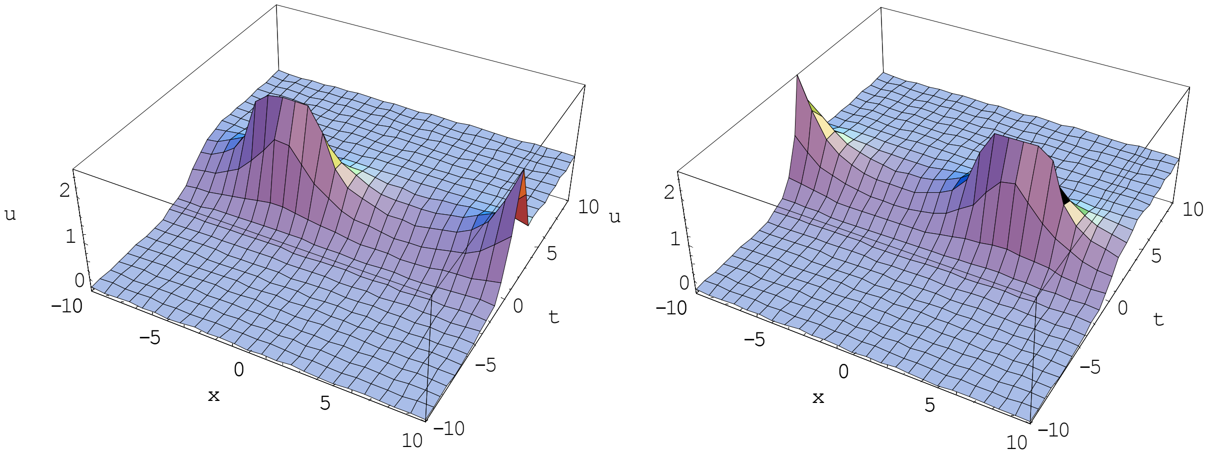

Fig. 22. *Plot of the modulus of the solution u_5 in (54) and u_6 in (55) to the Fisher equation (29).*

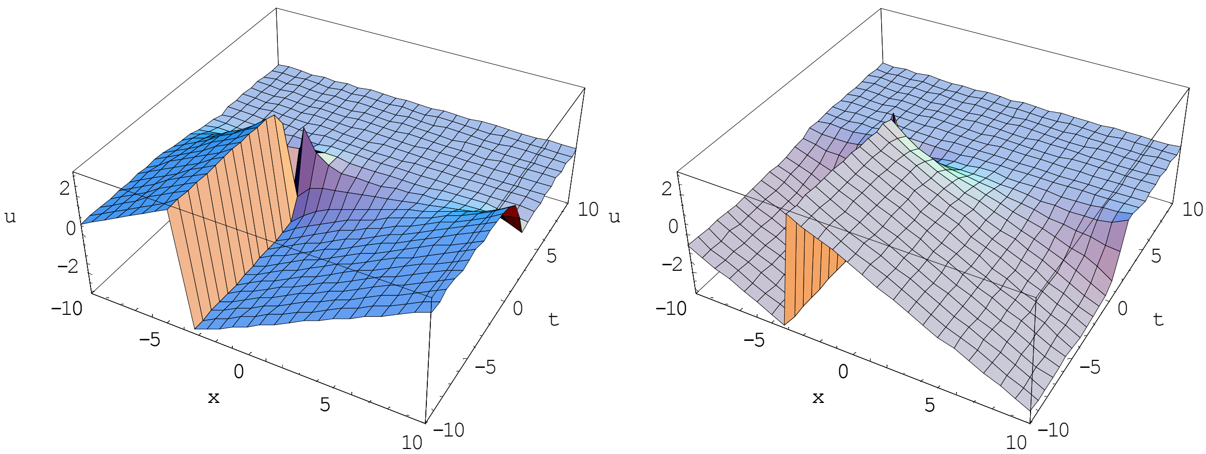

Fig. 23. *Plot the arguments of the soliton u_5 in (54) to the Fisher equation.*

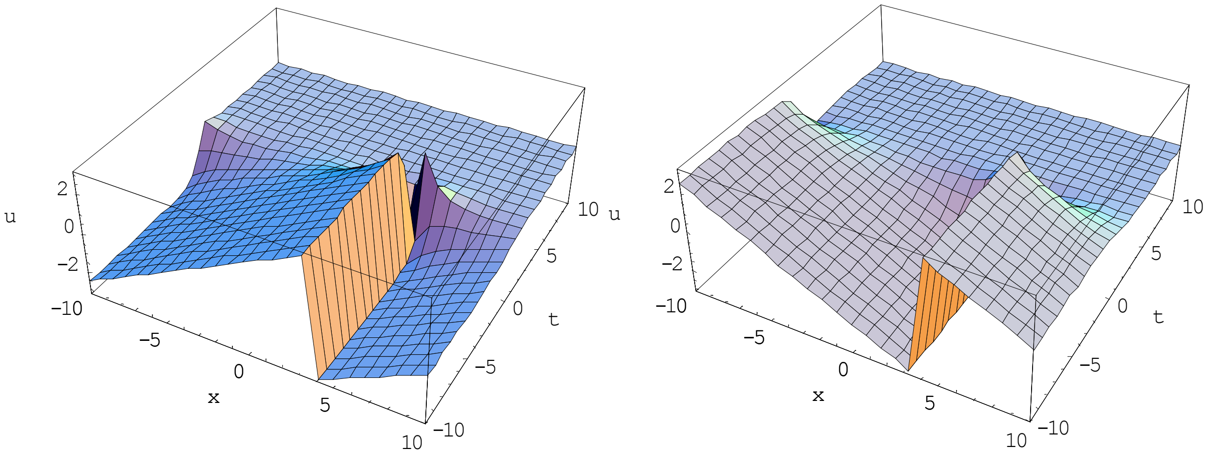

Fig. 24. *Plot of the arguments of the solitary wave solution u_6 in (55) to the Fisher equation (29).*

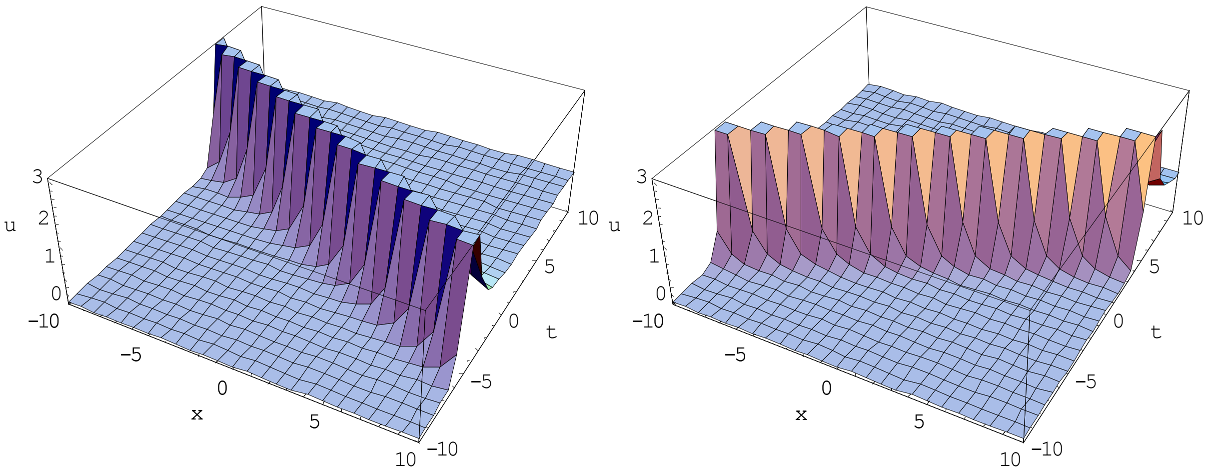

Fig. 25. *Sketch of the Bell shape solitary wave soliton (58) to the Fisher equation (29).*

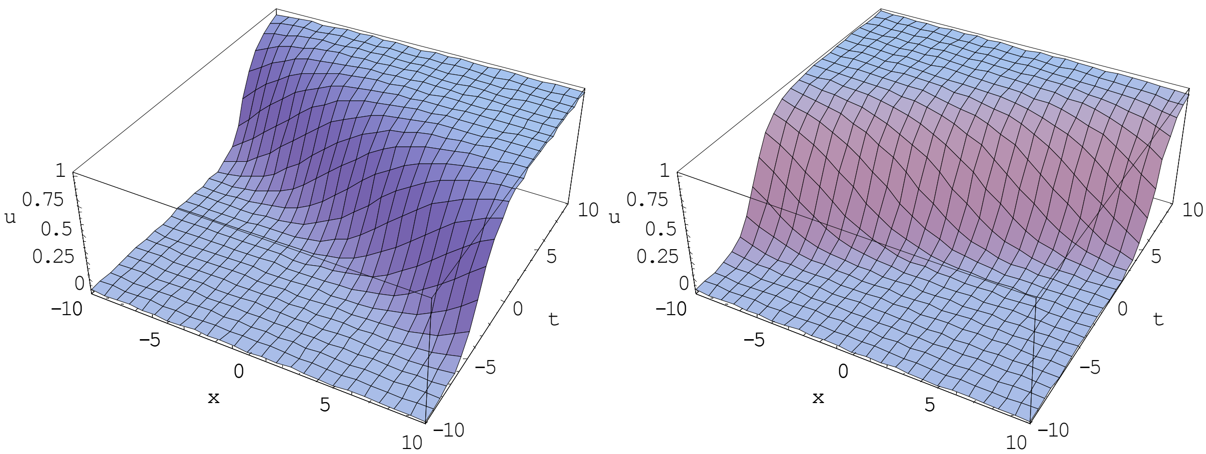

Fig. 26. *Kink solitons obtained from the solitary wave solution u_2 in (59) to the Fisher equation (29).*

4. Conclusions

In this article, we considered the Boussinesq equation and the Fisher equations for both of them the balance number is two. If the balance number is greater than one, in general the MSE method does not provide any solution. For this case, we have established the procedure in order to implement the MSE method to solve NLEEs for balance number two. If the solution of $s(\xi)$ consists of polynomial of the wave variable ξ, it will not be the solitary wave solution, since it does not meet the condition $|u| \to 0$ as $\xi \to \pm\infty$ for solitary wave solution. In this case, each coefficient of the polynomial must be zero. This constraint is crucial to solve NLEEs for higher balance number. By using this achieved process, we solved the above mentioned NLEEs and found some new traveling wave solutions. When the parameters receive special values, solitary wave solutions are derived from the exact solutions. Although the method has been applied in two equations, it can clearly be applied to many other nonlinear evolution equations whose balance number is equal to 2.

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References

[1] M. J. Ablowitz and P.A. Clarkson, Soliton, nonlinear evolution equations and inverse scattering, Cambridge University Press, New York, 1991.

[2] R. Hirota, The direct method in soliton theory, Cambridge Univ. Press, Cambridge, 2004.

[3] C. Rogers and W.F. Shadwick, Backlund transformations and their applications, Vol. 161 of Mathematics in Science and Engineering, Academic Press, New York, USA, 1982.

[4] L. Jianming, D. Jie and Y. Wenjun, Backlund transformation and new exact solutions of the Sharma-Tasso-Olver equation, Abstract Appl. Analysis, (2011) ID 935710, 8 pages.

[5] V. B. Matveev, M.A. Salle, Darboux transformation and solitons, Springer, Berlin, 1991.

[6] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys., 24 (1982) 522-526.

[7] A. M. Wazwaz, Partial Differential equations: Method and Applications, Taylor and Francis, 2002.

[8] M. A. Helal and M.S. Mehana, A comparison between two different methods for solving Boussinesq-Burgers equation, Chaos, Solitons Fract., 28 (2006) 320-326.

[9] D. D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Phys. Lett. A, 355 (2006) 137-141.

[10] D. D. Ganji, G.A. Afrouzi and R.A. Talarposhti, Application of variational iteration method and homotopy perturbation method for nonlinear heat diffusion and heat transfer equations, Phys. Lett. A, 368 (2007) 450-457.

[11] G. Xu, An elliptic equation method and its applications in nonlinear evolution equations, Chaos, Solitons Fract., 29 (2006) 942-947.

[12] E. Yusufoglu and A. Bekir, Exact solution of coupled nonlinear evolution equations, Chaos, Solitons Fractals, 37, (2008) 842-848.

[13] T. L. Bock and M.D. Kruskal, A two-parameter Miura transformation of the Benjamin-One equation, Phys. Lett. A, 74 (1979) 173-176.

[14] A. M. Wazwaz, A sine-cosine method for handle nonlinear wave equations, Appl. Math. Comput. Modeling, 40 (2004) 499-508.

[15] E. Yusufoglu, and A. Bekir, Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method, Int. J. Comput. Math., 83 (12) (2006) 915-924.

[16] M. Wang, Solitary wave solutions for variant Boussinesq equations, Phy. Lett. A, 199 (1995) 169-172.

[17] W. Malfliet and W. Hereman, The tanh method II: Perturbation technique for conservative systems, Phys. Scr., 54 (1996) 563-569.

[18] H. A. Nassar, M.A. Abdel-Razek and A.K. Seddeek, Expanding the tanh-function method for solving nonlinear equations, Appl. Math., 2 (2011) 1096-1104.

[19] A.J.M. Jawad, M.D. Petkovic, P. Laketa and A. Biswas, Dynamics of shallow water waves with Boussinesq equation, Scientia Iranica, Trans. B: Mech. Engr., 20(1) (2013) 179-184.

[20] M. A. Abdou, The extended tanh method and its applications for solving nonlinear physical models, Appl. Math. Comput., 190 (1) (2007) 988-996.

[21] N. Taghizadeh and M. Mirzazadeh, The first integral method to some complex nonlinear partial differential equations, J. Comput. Appl. Math., 235 (2011) 4871-4877.

[22] M. L. Wang and X.Z. Li, Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations, Phys. Lett. A, 343 (2005) 48-54.

[23] Sirendaoreji, Auxiliary equation method and new solutions of Klein-Gordon equations, Chaos, Solitions Fract., 31 (2007) 943-950.

[24] A. L. Guo and J. Lin, Exact solutions of (2+1)-dimensional HNLS equation, Commun. Theor. Phys., 54 (2010) 401-406.

[25] S. T. Mohyud-Din, , M.A. Noor and K.I. Noor, Modified Variational Iteration Method for Solving Sine-Gordon Equations, World Appl. Sci. J., 6 (7) (2009) 999-1004.

[26] H. Triki, A. Chowdhury and A. Biswas,Solitary wave and shock wave solutions of the variants of Boussinesq equation, U.P.B. Sci. Bull., Series A, 75(4) (2013) 39-52.

[27] H. Triki, A.H. Kara and A. Biswas, Domain walls to Boussinesq type equations in (2+1)-dimensions, Indian J. Phys., 88(7) (2014) 751-755.

[28] J. H. He and X.H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons Fract., 30 (2006) 700-708.

[29] H. Naher, A.F. Abdullah and M.A. Akbar, New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method, J. Appl. Math., 2012 (2012) Article ID 575387, 14 pages.

[30] M. Wang, X. Li and J. Zhang, The (G'/G) -expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A, 372 (2008) 417-423.

[31] J. Zhang, F. Jiang and X. Zhao, An improved (G'/G) -expansion method for solving nonlinear evolution equations, Inter. J. Comput. Math., 87 (8) (2010) 1716-1725.

[32] J. Feng, W. Li and Q. Wan, Using (G'/G) -expansion method to seek the traveling wave solution of Kolmogorov-Petrovskii-Piskunov equation, Appl. Math. Comput., 217 (2011) 5860-5865.

[33] M. A. Akbar, N.H.M. Ali and E.M.E. Zayed, Abundant exact traveling wave solutions of the generalized Bretherton equation via (G'/G) -expansion method, Commun. Theor. Phys., 57 (2012) 173-178.

[34] R. Abazari, The (G'/G) -expansion method for Tziteica type nonlinear evolution equations, Math. Comput. Modelling, 52 (2010) 1834-1845.

[35] M. A. Akbar, N.H.M. Ali and S.T. Mohyud-Din, Further exact traveling wave solutions to the (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equation, J. Comput. Analysis Appl., 15 (3) (2013) 557-571.

[36] A. J. M. Jawad, M.D. Petkovic and A. Biswas, Modified simple equation method for nonlinear evolution equations, Appl. Math. Comput., 217 (2010) 869-877.

[37] E. M. E. Zayed and S.A.H. Ibrahim, Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method, Chin. Phys. Lett., 29 (6) (2012) 060201.

[38] K. Khan, M.A. Akbar and M.N. Alam, Traveling wave solutions of the nonlinear Drinfel'd-Sokolov-Wilson equation and modified Benjamin-Bona-Mahony equations, J. Egyptian Math. Soc., 21 (2013) 233-240.

[39] K. Khan and M. Ali Akbar, Exact and solitary wave solutions for the Tzitzeica-Dodd-Bullough and the modified Boussinesq-Zakharov-Kuznetsov equations using the modified simple equation method, Ain Shams Engr. J., 4 (2013) 903-909.

[40] K. Khan, M.A. Akbar, Traveling wave solutions of some coupled nonlinear evolution equations, ISRN Math. Phys., 2013 (2013) Art. ID 685736, 8 pages.

[41] K. Khan, M.A. Akbar, Application of $\exp(-\phi(\xi))$ -expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation, World Appl. Sci. J., 24(10) (2013) 1373-1377.

[42] M. G. Hafez, M.N. Alam and M.A. Akbar, Traveling wave solutions for some important coupled nonlinear physical models via the coupled Higgs equation and the Maccari system, J. King Saud Univ.-Sci., 27(2), (2015) 105-112

[43] M. A. Salam, Traveling wave solution of modified Liouville equation by means of modified simple equation method, ISRN Appl. Math., Vol. 2012, ID 565247, 4 pages.

[44] E. M. E. Zayed and A.H. Arnous, Exact traveling wave solutions of nonlinear PDEs in mathematical physics using the modified simple equation method, Appl. Appl. Math.**:** An Int. J., 8(2) (2013) 553-572.

[45] M. N. Alam, F.B.M. Belgacem, Application of the Novel (G'/G) -Expansion Method to the Regularized Long Wave Equation, Waves, Wavelets and Fractals–Advanced Analysis, 1, (2015) 20-37.

[46] M.N. Alam, F.B.M. Belgacem, Akbar M.A., Analytical treatment of the evolutionary (1+1) dimensional combined KdV-mKdV equation via novel (G'/G)-expansion method, Journal of Applied Mathematics and Physics, (2015) (in press).

[47] M. N. Alam, M. G. Hafez, F.B.M. Belgacem, M.A. Akbar, Applications of the novel(G'/G) expansion method to find new exact traveling wave solutions of the nonlinear coupled Higgs field equation, Nonlinear Studies, 22(4), (2015) 613-633.

[48] M. N. Alam and F.B.M. Belgacem, Microtubules nonlinear models dynamics investigations through the (G'/G)-expansion method implementation, Mathematics, (MDPI), (2016) 13 pages.

[49] M. N. Alam and F.B.M. Belgacem, New generalized (G'/G)-expansion method Applications to coupled Konno-Oono and right-handed non-commutative Burgers equations, Advances in Pure Mathematics, 6(3), (2016) 168-179.

[50] M. N. Alam and F.B.M. Belgacem, Exact Traveling Wave Solutions for the (1+1)-Dim Compound KdVB Equation by the Novel (G'/G)-Expansion Method, Int. J. Modern Nonlinear Theor. Appl., 5 (1), (2016) 28-39.