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Witten Complex of Transitive Digraph and Its Convergence

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Abstract: Digraphs are generalization of graphs in which each edge is given one or two directions. For each digraph, there exists a transitive digraph containing it. Moreover, all the formal linear combinations of allowed elementary paths form a basis of the path complex for a transitive digraph. Hence, the study of discrete Morse theory on transitive digraphs is very important for the further study of discrete Morse theory on general digraphs. As we know, the definition of discrete Morse function on a digraph is different from that on a simplical complex or a cell complex: each discrete Morse function on a digraph is a discrete flat Witten-Morse function. In this paper, we deform the usual boundary operator, replacing it with a boundary operator with parameters and consider the induced Laplace operators. In addition, we consider the eigenvectors of the eigenvalues of the Laplace operator that approach to zero when the parameters approach infinity, define the generation space of these eigenvectors, and further give the Witten complex of digraphs. Finally, we prove that for a transitive digraph, Witten complex approaches to its Morse complex. However, for general digraphs, the structure of Morse complex is not as simple as that of transitive digraphs and the critical path is not directly related to the eigenvector with zero eigenvalue of Laplace operator. This is explained in the last part of the paper.

Keywords: Transitive Digraph, Discrete Morse Function, Witten-Morse Function, Path Homology, Witten Complex

1. Introduction

Digraph is an important topological model of complex network. It is determined by a binary set consisting of vertices set and directed edge set. That is, digraph $G = (V, E)$ is consisted of a finite set V and a non-empty subset E of $V \times V \setminus \{\mathrm{diag}\}$. Denote each directed edge $(u, v) \in E$ as $u \to v$. A *transitive digraph* is a digraph whose directed edges satisfy transitivity. In other words, if $u \to v$ and $v \to w$ are two directed edges, $u \to w$ is also a directed edge (u, v, w are distinct).

Let G be a digraph. An *elementary* n-path (or n-path) is a sequence $v_0v_1 \cdots v_n$ of $n + 1$ vertices on the vertex set V of G. An *allowed elementary* n-path on G is an elementary path $v_0v_1 \cdots v_n$ such that $v_i \to v_{i+1}$ is a directed edge of G, $0 \leq i \leq n - 1$. A *directed loop* on G is an allowed elementary path $v_0v_1 \cdots v_nv_0$ with all $v_i, i = 0, \cdots, n$ are distinct, $n \geq 1$. Obviously, digraphs and transitive digraphs without directed loops can be regarded as hypergraphs and simplicial complexes, respectively.

There have been rich topological research on simplicial complex. Discrete Morse theory is an important mathematical theory and tool. It can greatly reduce the number of cells and simplices, simplify the calculation of homology groups, and can be applied to topological data analysis. The discrete Morse theory of simplicial complex or cell complex was first proposed by R. Forman in 1998 (cf. [5, 8]) and since then, it has been improved and developed (cf. [6, 7]). Subsequently, R. Ayala et al. (cf. [1, 2, 3, 4]) studied the discrete Morse theory on graphs based on [5]. Recently, by regarding the path space of a digraph as a graded submodule of the path space of its transitive closure, we consider the discrete Morse theory on digraphs (cf. [17, 19]).

There are many different definitions of homology on digraphs. In this paper, the homology of digraphs mentioned refers to the path homology defined by A. Grigor'yan, etc (cf. [9, 10, 11, 12, 13, 14, 15]). Next, we give a brief review of path homology of digraphs.

Let R be an arbitrary commutative ring with a unit. Let $\Lambda_n(V)$ be the R-module generated by all n-paths on V. The i-th face map is defined as the R-linear map

$$d_i : \Lambda_n(V) \longrightarrow \Lambda_{n-1}(V)$$

which sends $v_0v_1\cdots v_n$ to $v_0\cdots\hat{v_i}\cdots v_n$, where $\hat{v_i}$ means omission of the vertex v_i. Define the boundary operator as: $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$. Then it is easy to verify that ∂_n is an R-linear map from $\Lambda_n(V)$ to $\Lambda_{n-1}(V)$ satisfying $\partial_n\partial_{n+1} = 0$ for each $n \geq 0$. Hence $\{\Lambda_n(V), \partial_n\}_{n\geq 0}$ is a chain complex.

Let $P_n(G)$ be the R-module generated by all allowed elementary n-paths on G. Then $P_n(G)$ is a submodule of $\Lambda_n(V)$. However, the image of an allowed elementary path under the boundary operator ∂ does not have to be allowed. That is, $\partial P_n(G) \subset P_{n-1}(G)$ may not hold. Hence, consider the R-module $\Omega_n(G)$ which is generated by all the ∂-invariant n-paths in $P_n(G)$. Obviously, $\Omega_n(G)$ is a submodule of $P_n(G)$ satisfying $\partial\Omega_n(G) \subset \Omega_{n-1}(G)$. The path homology of G is defined as the homology of chain complex $\{\Omega_n(G), \partial_n\}_{n\geq 0}$ and denoted as $H_*(G; R)$. Therefore,

$$H_m(G; R) = H_m(\{\Omega_n(G), \partial_n\}_{n\geq 0}), \quad m \geq 0.$$

In this paper, based on the existing discrete Morse theory and path homology theory on digraphs, we mainly prove that the Witten complex of a transitive digraph approaches to its Morse complex in Theorem 3.1 and Corollary 3.1, which is not necessarily true for general digraphs.

2. Preliminaries

In this section, we mainly review the definition of discrete Morse functions on digraphs.

Let G be a digraph. Define a partial order $<$ among all allowed elementary paths on G. Let α and β be two allowed elementary paths on G. Then if β can be obtained from α by removing some vertices, we call $\alpha > \beta$ or $\beta < \alpha$.

Definition 2.1. (cf. [19]) A nonnegative map $f : V(G) \longrightarrow [0, +\infty)$ is called a *discrete Morse function* on G, if for any allowed elementary path $\alpha = v_0v_1\cdots v_n$ on G, both of the followings hold:

$$\text{(i).}\ \#\Big\{\gamma^{(n+1)} > \alpha^{(n)} \mid f(\gamma) = f(\alpha)\Big\} \leq 1;$$
$$\text{(ii).}\ \#\Big\{\beta^{(n-1)} < \alpha^{(n)} \mid f(\beta) = f(\alpha)\Big\} \leq 1.$$

where

$$f(\alpha) = f(v_0v_1\cdots v_n) = \sum_{i=0}^{n} f(v_i).$$

Particularly, if both inequalities (i) and (ii) in Definition 2.1 hold strictly for α, α is called *critical*. Precisely,

Definition 2.2. An allowed elementary n-path $\gamma^{(n)}$ is called *critical*, if both of the followings hold:

$$\text{(i)'}\ \#\Big\{\beta^{(n-1)} < \alpha^{(n)} \mid f(\beta) = f(\alpha)\Big\} = 0,$$
$$\text{(ii)'}\ \#\Big\{\gamma^{(n+1)} > \alpha^{(n)} \mid f(\gamma) = f(\alpha)\Big\} = 0.$$

It follows from Definition 2.2 that an allowed elementary n-path is not critical if and only if either of the following conditions holds

(i)" there exists an allowed elementary path $\beta^{(n-1)}$ such that $\beta < \alpha$ and $f(\beta) = f(\alpha)$;

(ii)" there exists an allowed elementary path $\gamma^{(n+1)}$ such that $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$.

Then we give an important property of discrete Morse functions on digraphs.

Lemma 2.1. (cf. [17, Lemma 2.5]) Let G be a digraph and f a discrete Morse function on G as defined in Definition 2.1. Then for any allowed elementary path in G, there exists at most one index such that the corresponding vertex is with zero value.

By Lemma 2.1,

Lemma 2.2. [19] Let f be a discrete Morse function on digraph G. Then for any allowed elementary path $\alpha = v_0v_1\cdots v_n$ on G, (i)" and (ii)" cannot both be true.

Definition 2.3. (cf. [8, Definition 0.6]) A function $f : V(G) \longrightarrow [0, +\infty)$ is called a *discrete Witten-Morse function* on G, if for any allowed elementary path α, both of the followings hold:

(i) $f(\alpha) < \text{average}\{f(\gamma_1), f(\gamma_2)\}$ where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$;

(ii) $f(\alpha) > \text{average}\{f(\beta_1), f(\beta_2)\}$ where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

Note that each Witten-Morse function is, in fact, a Morse function.

Definition 2.4. (cf. [8, Definition 0.7]) A discrete Witten-Morse funtion is *flat*, if for any allowed elementary path α,

(i) $f(\alpha) \leq \min\{f(\gamma_1), f(\gamma_2)\}$ where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$;

(ii) $f(\alpha) \geq \max\{f(\beta_1), f(\beta_2)\}$ where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

Proposition 2.1. (cf. [20]) Each discrete Morse function on a digraph is a discrete flat Witten-Morse function.

3. Witten Complexes of Transitive Digraphs

In this section, we prove that Witten complex of a transitive digraph approaches to its Morse complex.

Let G be a transitive digraph and R a field . Similar to [8], consider the chain complex

$$0 \longrightarrow \Omega_n(G) \xrightarrow{\partial} \Omega_{n-1}(G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_0(G) \longrightarrow 0. \quad (1)$$

Define a chain homomorphism

$$e^{tf} : \Omega_n(G) \longrightarrow \Omega_n(G)$$

by setting

$$e^{tf}(\alpha) = e^{tf(\alpha)}\alpha \tag{2}$$

for any allowed elementary path α on G, and extending linearly to $\Omega(G)$. Replace the boundary operator ∂ with

$$\partial_t = e^{tf}\partial e^{-tf}.$$

Then

$$\begin{aligned}\partial_t\alpha &= e^{tf}\partial e^{-tf}(\alpha)\\ &= e^{tf}\partial e^{-tf(\alpha)}\alpha\\ &= e^{-tf(\alpha)}e^{tf}(\partial\alpha)\\ &= \sum_{\beta<\alpha,\beta\in\Omega(G)}\langle\partial\alpha,\beta\rangle e^{t[f(\beta)-f(\alpha)]}\beta.\end{aligned}$$

Hence, $\partial_t\alpha \in \Omega(G)$ which implies that

$$0 \longrightarrow \Omega_n(G) \xrightarrow{\partial_t} \Omega_{n-1}(G) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} \Omega_0(G) \longrightarrow 0 \tag{3}$$

is still a chain complex. Moreover, by a similar argument to [18, Section 5.1, P.54], we have that

Proposition 3.1. Let G be a transitive digraph. Then for each $t \in R$, the complexes (1) and (3) have the same path homology. That is,

$$H_m(\{\Omega_n(G), \partial_n\}_{n\geq 0}) \cong H_m(\{\Omega_n(G), \partial_t\}_{n\geq 0}).$$

Proof Note that $\Omega(G) = P(G)$ for transitive digraphs. For any $x \in \mathrm{Ker}\partial$, under the map (2), we have that

$$\begin{aligned}\partial_t e^{tf}(x) &= e^{tf}\partial(x)\\ &= e^{tf}(\partial x)\\ &= 0.\end{aligned}$$

That is, $e^{tf}(x) \in \mathrm{Ker}\partial_t$. And if $x = \partial y \in \mathrm{Im}\partial$, then

$$\begin{aligned}\partial_t e^{tf}(y) &= e^{tf}(\partial y)\\ &= e^{tf}(x).\end{aligned}$$

Hence, $e^{tf}(x) \in \mathrm{Im}\partial_t$.

Therefore, the mapping (2) is an invertible map from n-paths which are closed but not exact in the usual sense of ∂ to n-paths which are closed but not exact in the sense of ∂_t.

The proposition is proved.

Let

$$\Delta_n(t) = \partial_t\partial_t^* + \partial_t^*\partial_t$$

be the Laplace operator induced by ∂_t where ∂_t^* is the adjoint of ∂_t with respect to the inner product on the chain spaces $\Lambda_*(V)$ such that all paths are orthonormal. Then by [16, Section 3.1],

$$\mathrm{Ker}(\Delta_n(t)) \cong H_m(\{\Omega_n(G), \partial_t\}_{n\geq 0}).$$

Hence, by Proposition 3.1,

$$\mathrm{Ker}(\Delta_n(t)) \cong H_m(\{\Omega_n(G), \partial_n\}_{n\geq 0}). \tag{4}$$

Denote $W_n(t)$ as the span of the eigenvectors of $\Delta_n(t)$ corresponding to the eigenvalues which tend to 0 as $t \to \infty$. Since $\Delta(t)\partial_t = \partial_t\Delta(t)$, ∂_t preserves the eigenspaces. The Witten complex is defined as

$$0 \longrightarrow W_n(t) \xrightarrow{\partial_t} W_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(t) \longrightarrow 0.$$

Let $\mathrm{Crit}_n(G)$ be the span of the critical n-paths on G. We have the following theorem.

Theorem 3.1. Let G be a transitive digraph and f a discrete Morse function on G. Then

$$\lim_{t\to\infty} W_n(t) = \mathrm{Crit}_n(G).$$

Proof Since G is transitive, $P_n(G) = \Omega_n(G)$ for each $n \geq 0$. By [8, Theorem 2.1] and Proposition 2.1, we have that

$$\begin{aligned}\Delta_n(t)\alpha &= [\sum_{\beta<\alpha}\langle\partial\alpha,\beta\rangle^2 e^{2t(f(\beta)-f(\alpha))}\\ &\quad + \sum_{\gamma>\alpha}\langle\partial\gamma,\alpha\rangle^2 e^{2t(f(\alpha)-f(\gamma))}]\alpha + O(e^{-tc})\end{aligned}$$

for some $c > 0$, where γ, α, β are allowed elementary paths on G. Hence, if and only if α is critical, the eigenvalues of $\Delta_n(t)$

$$\begin{aligned}\langle\Delta_n(t)\alpha,\alpha\rangle &= \sum_{\beta<\alpha}\langle\partial\alpha,\beta\rangle^2 e^{2t(f(\beta)-f(\alpha))}\\ &\quad + \sum_{\gamma>\alpha}\langle\partial\gamma,\alpha\rangle^2 e^{2t(f(\alpha)-f(\gamma))}\end{aligned}$$

tend to 0 as $t \to \infty$.

The theorem is proved.

Corollary 3.1. Let G be a transitive digraph. Then Witten complex $\{W_n(t), \partial_t\}_{n\geq 0}$ approaches to the complex $\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n\geq 0}$.

Proof By [17, Theorem 2.1],

$$H_m(\{\mathrm{Crit_n(G)}, \tilde{\partial}_n\}_{n\geq 0}) \cong H_m(\{\Omega_n(G), \partial_n\}_{n\geq 0}).$$

By (4), for all t,

$$H_m(\{W_n(t), \partial_t\}_{n\geq 0}) \cong H_m(\{\Omega_n(G), \partial_n\}_{n\geq 0}).$$

Therefore, by Theorem 3.1, the assertion is followed.

Remark 3.1. Let G be a transitive digraph. By [17], the chain complex consisting of all $\bar{\Phi}$-invariant chains is the Morse complex of G, where $\bar{\Phi}$ is the discrete gradient flow of G.

Moreover, by [19], the Morse complex of G is equivalent to chain complex $\{\mathrm{Crit}_*(G), \tilde{\partial}_*\}$, where $\tilde{\partial} = (\overline{\Phi}^{\infty})^{-1} \circ \partial \circ \overline{\Phi}^{\infty}$ and $\overline{\Phi}^{\infty}$ is the stabilized map of $\overline{\Phi}$. Hence, by Corollary 3.1, Witten complex of G approaches to its Morse complex.

Remark 3.2. Note that for general digraph G, the image of each ∂-invariant element $x \in \Omega_n(G)$ under ∂_t may be not in $\Omega_{n-1}(G)$. This implies that $\{\Omega(G), \partial_t\}$ is not a chain complex in general.

Finally, we give an example to illustrate Remark 3.2.

Example 3.1. Let G be a square with vertex set $V = \{v_0, v_1, v_2, v_3\}$ and directed edge set $E = \{v_0v_1, v_0v_2, v_1v_3, v_2v_3\}$. Then

$$\begin{aligned}\Omega(G) &= \{v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, \\ &\quad v_1v_3, v_2v_3, v_0v_1v_3 - v_0v_2v_3\}\end{aligned}$$

and

$$\begin{aligned}
& \partial_t(v_0v_1v_3 - v_0v_2v_3) \\
= \; & e^{tf}\partial e^{-tf}(v_0v_1v_3 - v_0v_2v_3) \\
= \; & e^{tf}\partial e^{-tf}(v_0v_1v_3) - e^{tf}\partial e^{-tf}(v_0v_2v_3) \\
= \; & e^{-tf(v_0v_1v_3)}e^{tf}\partial(v_0v_1v_3) - e^{-tf(v_0v_2v_3)}e^{tf}\partial(v_0v_2v_3) \\
= \; & [e^{t[f(v_0v_1)-f(v_0v_1v_3)]}v_0v_1 + e^{t[f(v_1v_3)-f(v_0v_1v_3)]}v_1v_3] \\
& -[e^{t[f(v_0v_2)-f(v_0v_2v_3)]}v_0v_2 + e^{t[f(v_2v_3)-f(v_0v_2v_3)]}v_2v_3] \\
& +[e^{t[f(v_0v_3)-f(v_0v_2v_3)]}v_0v_3 - e^{t[f(v_0v_3)-f(v_0v_1v_3)]}v_0v_3]. \quad (5)
\end{aligned}$$

Since the coefficient of v_0v_3 in (5) may not be zero, it follows that

$$\partial_t(v_0v_1v_3 - v_0v_2v_3) \notin \Omega_1(G).$$

Moreover,

$$\begin{aligned}\partial^* |_{\Omega(G)} (v_0v_1) &= v_0v_1v_3, \\ (\partial |_{\Omega(G)})^*(v_0v_1) &= v_0v_1v_3 - v_0v_2v_3.\end{aligned}$$

Hence,

$$\partial^* |_{\Omega(G)} \neq (\partial |_{\Omega(G)})^*.$$

Therefore, we will further consider the path homology of general digraphs based on the results of [17] instead of techniques from Hodge theory in a subsequent paper.

4. Conclusion

For a transitive digraph, there may exist directed loops on it. Hence, path complexes of transitive digraphs are different from simplicial complexes. However, for a transitive digraph, there exists a basis of path complex which is generated by all allowed elementary paths. In addition, path space of a transitive digraph is a Δ-set. Therefore, Morse complex of transitive digraphs consistent with Morse complex of simplicial complexes. Through the research in this paper, the Witten complex of a transitive digraph has a similar property to the Witten complex of cell complexes. That is, Witten complex of a transitive digraph approaches to its Morse complexes.

For general digraphs which are not transitive, path spaces can be considered as graded set of Δ-set. All allowed elementary paths can not generate a basis of the path complex. Hence, discrete Morse theory on digraphs are different from discrete Morse theory on simplicial complexes.

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