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Robust Time-Varying Kalman State Estimators with Uncertain Noise Variances

Wenjuan Qi*, Zunbing Sheng

School of Mechnical and Electronical Engineering, Heilongjiang University, Harbin, China

Email address:
2015072@hlju.edu.cn (Wenjuan Qi), 2009041@hlju.edu.cn (Zunbing Sheng)

*Corresponding author

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Abstract: This paper addresses the design of robust Kalman estimators (filter, predictor and smoother) for the time-varying system with uncertain noise variances. According to the unbiased linear minimum variance (ULMV) optimal estimation rule, the robust time-varying Kalman estimators are presented. Specially, two robust Kalman smoothing algorithms are presented by the augmented and non-augmented state approaches, respectively. They have the robustness in the sense that their actual estimation error variances are guaranteed to have a minimal upper bound for all admissible uncertainties of noise variances. Their robustness is proved by the Lyapunov equation approach, and their robust accuracy relations are proved. The corresponding steady-state robust Kalman estimators are also presented for the time-invariant system, and the convergence in a realization between the time-varying and steady-state robust Kalman estimators is proved by the dynamic error system analysis (DESA) method and the dynamic variance error system analysis (DVESA) method. A simulation example is given to verify the robustness and robust accuracy relations.

Keywords: Uncertain System, Uncertain Noise Variance, Robust Kalman Filtering, Minimax Estimator, Robust Accuracy, Lyapunov Equation Approach, Convergence

1. Introduction

Kalman filtering is a most widely used methodology in estimation and control fields including signal processing and tracking. In classical Kalman filtering theory, the Kalman filter is designed based on a key assumption that the systems have exact model parameters and noise variances. When there exist uncertainties of the model parameters and/or noise variances, the system model used in constructing the filter differs from the real (true) system model that generates the actual measurement, so that the performance of the Kalman filter will degrade [1], and an inexact model may cause the filter to diverge. In practice, there inevitably exist uncertainties in the system model either due to unmodeled dynamics (modeling error) or due to model parameter drifting, or due to uncertain disturbance. This has motivated many studies of designing robust Kalman filters. An important class of robust Kalman filters is to design a Kalman filter for a family of system models yielded by uncertainties such that its actual filtering error variances are guaranteed to have a minimal or less conservative or optimized upper bound for all admissible uncertainties.

In order to design the robust Kalman filters for the systems with the model parameters uncertainties, two important approaches are the Riccati equation approach [1-6] and the linear matrix inequality (LMI) approach [7, 8]. The former is based on the solution two Raccati equations or one Raccati equation to obtain the optimized (minimal) upper bound of actual filtering error variances, by searching the scaling parameters. The latter is based LMI to obtain the optimized upper bound which is obtained by solving convex optimization problem with the LMI constraints using the Matlab LMI Toolbox. The time-varying (finite-horizon) and the steady-state (infinite-horizon) robust Kalman filters were investigated in [2, 4], where the convergence analysis of the finite-horizon robust Kalman filter is given. The limitation of the above robust Kalman filters is that only model parameters are assumed to be uncertain, while the noise variances are assumed to be exactly known.

Notice that the adaptive Kalman filtering [9-11] for system with unknown but deterministic model parameters and/or noise variances is different from the robust Kalman filtering [1-8], because it only handle one system model with unknown deterministic constant or time varying model parameters and/or noise variances, so that it does not solve the robust Kalman filtering problem for a family of system models yielded by uncertain model parameters and/or noise variances. The adaptive filtering approaches have some limitations, where the correlation approach [9] only is suitable for time-invariant stochastic system, the two-stage coupled adaptive Kalman filtering approach for estimating the state and noise variances [10] may cause the filter to diverge because the filter accuracy is very sensitive with respect to its initial values, and the self-tuning Kalman filtering approach [11] requires the on-line system identification in noisy environment, which is a difficult open problem.

So far, the robust Kalman filter for systems with uncertain noise variances are seldom considered [12, 13]. A minimax robust steady-state filter was presented for the descriptor systems with uncertain noise variances in [12], and a robust Kalman filter was presented for the systems with both the parameters and noise variances uncertainties by the Riccati equation approach in [13]. In this paper, we consider the problem of designing the robust Kalman estimators for time-varying system with uncertain noise variances. Using the minimax robust estimation principle [14,15], based on the worst-case conservative system with the conservative upper bounds of noise variances, according to the unbiased linear minimum variance (ULMV) rule, the robust time-varying Kalman estimators including the filter, predictor and smoother are presented, whose actual estimation error variances are respectively guaranteed to have a minimal upper bound for all admissible uncertainties of noise variances. The five robust weighted fusion Kalman filters [16] have been presented for multisensor uncertain system with uncertain noise variances, according to the minimax robust estimation principle and the ULMV optimal estimation rule, based on the worst-case conservative system with conservative upper bound of noise variances.

Furthermore, the robustness of the proposed robust Kalman estimators is proved by a Lyapunov equation method [14-16], which is completely different from the Riccati equation method and the LMI method. The concept of robust accuracy is discussed and the robust accuracy relations of the robust Kalman estimators are proved. Specially, the corresponding robust steady-state Kalman estimators are presented for time-invariant systems with uncertain noise variances, and the convergence in a realization between the time-varying and steady-state robust Kalman estimators is rigorously proved by using the dynamic error system analysis (DESA) method [17] and the dynamic variance error system analysis (DVESA) method [18], which are different from the convergence analysis method in [19], where the problem of their convergence in a realization was not solved [19].

The remainder of this paper is organized as following: Section 2 gives the problem formulation. The robust time-varying Kalman filter is presented in Section 3. The robust time-varying Kalman predictor is presented in Section 4. The robust time-varying Kalman smoother is proposed in Section 5. The robust steady-state Kalman estimators are presented in Section 6. The robust accuracy comparison is given in Section 7. The simulation example is given in Section 8. The conclusions are proposed in Section 9.

2. Problem Formulation

Consider the linear discrete time-varying system with uncertain noise variances.

$$x(t+1)=\Phi(t)x(t)+\Gamma(t)w(t) \tag{1}$$

$$y(t)=H(t)x(t)+v(t) \tag{2}$$

where t represents the discrete time, $x(t)\in R^n$ is the state, $y(t)\in R^m$ is the measurement, $w(t)\in R^r$ is the input noise, $v(t)\in R^m$ is the measurement noise, $\Phi(t)$, $\Gamma(t)$ and $H(t)$ are known time-varying matrices with appropriate dimensions.

Assumption 1. $w(t)$ and $v(t)$ are uncorrelated white noises with zero means and unknown uncertain actual (true) variances $\bar{Q}(t)$ and $\bar{R}(t)$, $Q(t)$ and $R(t)$ are known conservative upper bounds of $\bar{Q}(t)$ and $\bar{R}(t)$, respectively, i.e.,

$$\mathrm{E}\left[\begin{pmatrix} w(t) \\ v(t) \end{pmatrix}\begin{pmatrix} w(k) & v(k) \end{pmatrix}^{\mathrm{T}}\right]=\begin{bmatrix} \bar{Q}(t) & 0 \\ 0 & \bar{R}(t) \end{bmatrix}\delta_{tk} \tag{3}$$

$$\bar{Q}(t)\le Q(t),\bar{R}(t)\le R(t) \tag{4}$$

where E is the mathematical expectation operator, the superscript T is the transpose. δ_{tk} is the Kronecker δ function, $\delta_{tt}=1,\delta_{tk}=0(t\ne k)$.

Assumption 2. The initial state $x(0)$ is independent of the noises $w(t)$ and $v(t)$ and has mean value μ and unknown uncertain actual variance $\bar{P}(0|0)$ which satisfies

$$\bar{P}(0|0)\le P(0|0) \tag{5}$$

where $P(0|0)$ is a known conservative upper bound of $\bar{P}(0|0)$.

The robust Kalman estimate problem is to design the the time-varying Kalman estimators $\hat{x}(t|t+N)$ for uncertain system (1) and (2), such that its actual filtering error variances $\bar{P}(t|t+N)$ yielded by all admissible uncertainties $\bar{Q}(t)$, $\bar{R}(t)$ and $\bar{P}(0|0)$ satisfying (4) and (5), have a minimal upper bound $P(t|t+N)$, i.e.,

$$\bar{P}(t|t+N) \le P(t|t+N) \quad (6)$$

For $N=0$, $N<0$ or $N>0$, they are called as robust filters, predictors or smoothers, respectively.

3. Robust Time-Varying Kalman Filter

Based on the worst-case conservative system (1) and (2) with the conservative upper bounds $Q(t)$ and $R(t)$ of noise variances, the conservative optimal time-varying Kalman filter is given by [20]

$$\hat{x}(t|t) = \Psi_f(t)\hat{x}(t-1|t-1) + K_f(t)y(t) \quad (7)$$

$$\Psi_f(t) = \left[I_n - K_f(t)H(t)\right]\Phi(t-1) \quad (8)$$

$$K_f(t) = P(t|t-1)H^{\mathrm{T}}(t)\left[H(t)P(t|t-1)H^{\mathrm{T}}(t) + R(t)\right]^{-1} \quad (9)$$

$$P(t+1|t) = \Phi(t)[P(t|t-1) - P(t|t-1)H^{\mathrm{T}}(t)(H(t)P(t|t-1)H^{\mathrm{T}}(t) + R(t))^{-1} \times H(t)P(t|t-1)]\Phi^{\mathrm{T}}(t) + \Gamma(t)Q(t)\Gamma^{\mathrm{T}}(t) \quad (10)$$

$$P(t+1|t) = \Phi(t)P(t|t)\Phi^{\mathrm{T}}(t) + \Gamma(t)Q(t)\Gamma^{\mathrm{T}}(t) \quad (11)$$

$$P(t|t) = \left[I_n - K_f(t)H(t)\right]P(t|t-1) \quad (12)$$

with the initial values $\hat{x}(0|0) = \mu$ and $P(0|0)$, where I_n is the $n \times n$ identity matrix. The notation $\hat{x}(t|t)$ denotes the linear minimum variance estimate of the state $x(t)$ at time t, given the finite-horizon conservative measurements $(y(1), \cdots, y(t))$ from $t=1$ up to t.

The conservative filtering error variance $P(t|t)$ can be rewritten as the Lyapunov equation [20]

$$P(t|t) = \Psi_f(t)P(t-1|t-1)\Psi_f^{\mathrm{T}}(t) + \left[I_n - K_f(t)H(t)\right] \times \Gamma(t-1)Q(t-1)\Gamma^{\mathrm{T}}(t-1)\left[I_n - K_f(t)H(t)\right]^{\mathrm{T}} + K_f(t)R(t)K_f^{\mathrm{T}}(t) \quad (13)$$

with the initial value $P(0|0)$.

The actual prediction and filtering errors are

$$\tilde{x}(t|t-1) = x(t) - \hat{x}(t|t-1) = \Phi(t)\tilde{x}(t-1|t-1) + \Gamma(t-1)w(t-1) \quad (14)$$

$$\tilde{x}(t|t) = x(t) - \hat{x}(t|t) = \left[I_n - K_f(t)H(t)\right]\tilde{x}(t|t-1) - K_f(t)v(t) \quad (15)$$

where $\hat{x}(t|t)$ is the actual Kalman filter, and $\hat{x}(t|t-1) = \Phi(t)\hat{x}(t-1|t-1)$ is the actual Kalman predictor. Substituting (14) into (15) yields

$$\tilde{x}(t|t) = \Psi_f(t)\tilde{x}(t-1|t-1) + \left(I_n - K_f(t)H(t)\right)\Gamma(t-1)w(t-1) - K_f(t)v(t) \quad (16)$$

From (16), according to Assumptions 1-2, and noting thatt $w(t)$ and $v(t)$ are uncorrelated with $\tilde{x}(t|t)$, the actual filtering error variance $\bar{P}(t|t) = \mathrm{E}\left[\tilde{x}(t|t)\tilde{x}^{\mathrm{T}}(t|t)\right]$ is given by the Lyapunov equation

$$\bar{P}(t|t) = \Psi_f(t)\bar{P}(t-1|t-1)\Psi_f^{\mathrm{T}}(t) + \left[I_n - K_f(t)H(t)\right] \times \Gamma(t-1)\bar{Q}(t-1)\Gamma^{\mathrm{T}}(t-1)\left[I_n - K_f(t)H(t)\right]^{\mathrm{T}} + K_f(t)\bar{R}(t)K_f^{\mathrm{T}}(t) \quad (17)$$

with the initial value $\bar{P}(0|0)$. Notice that $\bar{P}(t|t)$ is related to the uncertain variances $\bar{Q}(t-1)$, $\bar{R}(t)$ and $\bar{P}(0|0)$.

Theorem 1. For uncertain system (1) and (2) with Assumptions 1-2, the actual Kalman filter (7) is robust in the sense that for all admissible true noise variances and initial value satisfying (4) and (5), the corresponding actual filtering error variances $\bar{P}(t|t)$ satisfy $\bar{P}(t|t) \le P(t|t)$, and $P(t|t)$ is a minimal upper bound of $\bar{P}(t|t)$.

Proof. The proof of the Theorem is similar to the reference [14], the detail is omitted.

4. Robust Time-Varying Kalman Predictor

4.1. Robust Time-Varying Kalman One-step Predictor

Consider uncertain system (1) and (2) with the Assumption

1-2, the conservative optimal time-varying Kalman one-step predictor with the conservative upper bound variances $Q(t)$ and $R(t)$ satisfying (4) is given by [20]

$$\hat{x}(t|t-1)=\Psi_p(t-1)\hat{x}(t-1|t-2)+K_p(t-1)y(t-1) \quad (18)$$

$$\Psi_p(t-1)=\Phi(t-1)-K_p(t-1)H(t-1) \quad (19)$$

$$K_p(t-1)=\Phi(t-1)P(t-1|t-2)H^{\mathrm{T}}(t-1)\left[H(t-1)P(t-1|t-2)H^{\mathrm{T}}(t-1)+R(t-1)\right]^{-1} \quad (20)$$

with the initial value $\hat{x}(0|-1)=\mu, P(0|-1)=P(0|0)$.

The Kalman one-step predictor (18) with the known actual measurements $y(t)$ is called as the actual Kalman one-step predictor. The conservative prediction error variance $P(t|t-1)$ satisfies the Lyapunov equation

$$P(t|t-1)=\Psi_p(t-1)P(t-1|t-2)\Psi_p^{\mathrm{T}}(t-1)+\Gamma(t-1)Q(t-1)\Gamma^{\mathrm{T}}(t-1)+K_p(t-1)R(t-1)K_p^{\mathrm{T}}(t-1) \quad (21)$$

with the initial value $P(0|-1)=P(0|0)$. The actual estimation error is

$$\tilde{x}(t|t-1)=x(t)-\hat{x}(t|t-1)=\Phi(t)\tilde{x}(t-1|t-1)+\Gamma(t-1)w(t-1) \quad (22)$$

$$\tilde{x}(t-1|t-1)=x(t-1)-\hat{x}(t-1|t-1)=\left[I_n-K_p(t-1)H(t-1)\right]\tilde{x}(t-1|t-2)-K_p(t-1)v(t-1) \quad (23)$$

Substituting (23) into (22) yields

$$\tilde{x}(t|t-1)=\Psi_p(t-1)\tilde{x}(t-1|t-2)+\Gamma(t-1)w(t-1)-K_p(t-1)v(t-1) \quad (24)$$

Hence the actual prediction error variance satisfies the Lyapunov equation

$$\bar{P}(t|t-1)=\Psi_p(t-1)\bar{P}(t-1|t-2)\Psi_p^{\mathrm{T}}(t-1)+\Gamma(t-1)\bar{Q}(t-1)\Gamma^{\mathrm{T}}(t-1)+K_p(t-1)\bar{R}(t-1)K_p^{\mathrm{T}}(t-1) \quad (25)$$

with the initial value $\bar{P}(0|-1)=\bar{P}(0|0)$.

4.2. Robust Time-varying Kalman Multi-step Predictor

Consider uncertain system (1) and (2) satisfying the Assumptions 1-2, the actual time-varying Kalman multi-step predictor with the conservative upper bounds $Q(t)$ and $R(t)$ is given by

$$\hat{x}(t|t+N)=\Phi(t,t+N+1)\hat{x}(t+N+1|t+N), N<0 \quad (26)$$

where $\hat{x}(t|t-1)$ is the actual one-step predictor computed by (18), and we define that

$$\Phi(t,t+N+1)=\Phi(t-1)\Phi(t-2)\cdots\Phi(t+N+1), \Phi(t,t)=I_n \quad (27)$$

The conservative ahead N step prediction error variance $P(t|t+N)$ is given by

$$\begin{aligned}P(t|t+N)&=\Phi(t,t+N+1)P(t+N+1|t+N)\Phi^{\mathrm{T}}(t,t+N+1)\\&+\sum_{i=t+N+2}^{t}\Phi(t,i)\Gamma(i-1)Q(i-1)\Gamma^{\mathrm{T}}(i-1)\Phi^{\mathrm{T}}(t,i)\end{aligned}, N<0 \quad (28)$$

where $P(t|t-1)$ is the conservative one-step prediction error variance, which is computed via (10).

Iterating (1), we have the non-recursive formula

$$x(t)=\Phi(t,t+N+1)x(t+N+1)+\sum_{i=t+N+2}^{t}\Phi(t,i)\Gamma(i-1)w(i-1), N<0 \quad (29)$$

From (26), we have the actual prediction error

$$\begin{aligned}\tilde{x}(t\mid t+N) &= x(t)-\hat{x}(t\mid t+N)\\ &=\Phi(t,t+N+1)\tilde{x}(t+N+1\mid t+N)+\sum_{i=t+N+2}^{t}\Phi(t,i)\Gamma(i-1)w(i-1),\ N<0\end{aligned} \tag{30}$$

So we have the actual ahead N step prediction error variance

$$\bar{P}(t\mid t+N)=\Phi(t,t+N+1)\bar{P}(t+N+1\mid t+N)\Phi^{\mathrm{T}}(t,t+N+1)+\sum_{i=t+N+2}^{t}\Phi(t,i)\Gamma(i-1)\bar{Q}(i-1)\Gamma^{\mathrm{T}}(i-1)\Phi^{\mathrm{T}}(t,i) \tag{31}$$

Theorem 2. For uncertain system (1) and (2) with Assumptions 1-2, the actual ahead N step predictor (26)-(31) is robust in the sense that the corresponding actual multi-step prediction error variances $\bar{P}(t\mid t+N)$ satisfy $\bar{P}(t\mid t+N)\le P(t\mid t+N)$, $N<0$, and $P(t\mid t+N)$ is a minimal upper bound of $\bar{P}(t\mid t+N)$.

Proof. The proof of the Theorem is similar to the reference [15], the detail is omitted.

5. Robust Time-Varying Kalman Smoother

Introduce the augmented state

$$x_a(t)=\left[x^{\mathrm{T}}(t)\quad x^{\mathrm{T}}(t-1)\quad\cdots\quad x^{\mathrm{T}}(t-N)\right]^{\mathrm{T}},\ N>0 \tag{32}$$

and the augmented matrices

$$\Phi_a(t)=\begin{bmatrix}\Phi(t) & 0 & \cdots & 0\\ I_n & 0 & & \\ 0 & \ddots & \ddots & \vdots\\ 0 & \cdots & I_n & 0\end{bmatrix},\ \Gamma_a(t)=\begin{bmatrix}\Gamma(t)\\ 0\\ \vdots\\ 0\end{bmatrix},\ H_a(t)=\left[H(t)\quad 0\quad\cdots\quad 0\right] \tag{33}$$

we have the augmented state system

$$x_a(t+1)=\Phi_a(t)x_a(t)+\Gamma_a(t)w(t) \tag{34}$$

$$y(t)=H_a(t)x_a(t)+v(t) \tag{35}$$

Hence the problem for designing robust time-varying Kalman smoother for uncertain system (1) and (2) under Assumptions 1-2 can be converted in to that of designing the following robust time-varying Kalman filter. From (34) and (35), we have the actual augmented Kalman filter as [20]

$$\hat{x}_a(t\mid t)=\Psi_a(t)\hat{x}_a(t-1\mid t-1)+K_a(t)y(t) \tag{36}$$

$$\Psi_a(t)=\left[I-K_a(t)H_a(t)\right]\Phi_a(t-1),\ K_a(t)=P_a(t\mid t-1)H_a^{\mathrm{T}}(t)\left(H_a(t)P_a(t\mid t-1)H_a^{\mathrm{T}}(t)+R(t)\right)^{-1} \tag{37}$$

$$P_a(t+1\mid t)=\Phi_a(t)P_a(t\mid t)\Phi_a^{\mathrm{T}}(t)+\Gamma_a(t)Q(t)\Gamma_a^{\mathrm{T}}(t),\ P_a(t\mid t)=\left[I-K_a(t)H_a(t)\right]P_a(t\mid t-1) \tag{38}$$

where I denotes the $(N+1)n\times(N+1)n$ identity matrix, and $P_a(t+1\mid t)$ satisfies the augmented Riccati equation

$$P_a(t+1\mid t)=\Phi_a(t)[P_a(t\mid t-1)-P_a(t\mid t-1)H_a^{\mathrm{T}}(t)(H_a(t)P_a(t\mid t-1)H_a^{\mathrm{T}}(t)+R(t))^{-1}\times H_a(t)P_a(t\mid t-1)]\Phi_a^{\mathrm{T}}(t)+\Gamma_a(t)Q(t)\Gamma_a^{\mathrm{T}}(t) \tag{39}$$

The conservative filtering error variance satisfies the Lyapunov equation

$$P_a(t\mid t)=\Psi_a(t)P_a(t-1\mid t-1)\Psi_a^{\mathrm{T}}(t)+\left[I-K_a(t)H_a(t)\right]\times\Gamma_a(t-1)Q(t-1)\Gamma_a^{\mathrm{T}}(t-1)\left[I-K_a(t)H_a(t)\right]^{\mathrm{T}}+K_a(t)R(t)K_a^{\mathrm{T}}(t) \tag{40}$$

The actual filtering error variance is given as

$$\bar{P}_a(t\mid t)=\Psi_a(t)\bar{P}_a(t-1\mid t-1)\Psi_a^{\mathrm{T}}(t)+\left[I-K_a(t)H_a(t)\right]\times\Gamma_a(t-1)\bar{Q}(t-1)\Gamma_a^{\mathrm{T}}(t-1)\left[I-K_a(t)H_a(t)\right]^{\mathrm{T}}+K_a(t)\bar{R}(t)K_a^{\mathrm{T}}(t) \tag{41}$$

Theorem 3. For uncertain system (1) and (2) with Assumptions 1-2, the actual time-varying Kalman smoother $\hat{x}(t|t+N)$ with the fixed lag $N>0$, is given as

$$\hat{x}(t|t+N)=H_x\hat{x}_a(t+N|t+N) \tag{42}$$

$$H_x=[0 \ \cdots \ 0 \ I_n] \tag{43}$$

where $\hat{x}_a(t+N|t+N)$ is the actual Kalman filter with the actual measurement $y(t)$ for the augmented system (34) and (35), and the conservative and actual smoothing error variances are respectively given as

$$P(t|t+N)=H_xP_a(t+N|t+N)H_x^{\mathrm{T}}, \ N>0 \tag{44}$$

$$\bar{P}(t|t+N)=H_x\bar{P}_a(t+N|t+N)H_x^{\mathrm{T}}, \ N>0 \tag{45}$$

where $P_a(t|t)$ and $\bar{P}_a(t|t)$ are computed via (40) and (41). The actual Kalman smoother (42) is robust in the sense that the actual smoothing error variances $\bar{P}(t|t+N)$ satisfy $\bar{P}(t|t+N)\le P(t|t+N)$, $N>0$, and $P(t|t+N)$ is a minimal upper bound of $\bar{P}(t|t+N)$.

Proof. The proof of the Theorem is similar to the reference [16], the detail is omitted.

Remark 1. Applying the augmented state method to design the robust Kalman smoother will increase the computation burden, accompanying with the increasing of the fixed lag N. The main computation burden of the Kalman smoother is to solve the Riccati equation. Generally, the number of multiplications and divisions of the algorithm is defined as the operation count or complexity. The complexity of solving non-augmented Riccati equation (10) is about $O(n^3)$ [21], while the complexity of solving augmented Riccati equation (39) is about $O(((N+1)n)^3)$. The advantage of the augmented state method is that the robustness of the Kalman smoother is easily proved based on the robustness of the augmented Kalman filter, and the actual smoothing error variances are easily computed.

The following Theorem 4 gives an non-augmented state approach for simultaneously obtaining the conservative and actual smoothing error variances $P(t|t+N)$ and $\bar{P}(t|t+N)$.

Theorem 4. For uncertain system (1) and (2) with Assumptions 1-2, the robust Kalman smoother $\hat{x}(t|t+N)$ with the fixed lag N, is given by

$$\hat{x}(t|t+N)=\hat{x}(t|t-1)+\sum_{j=0}^{N}K(t|t+j)\varepsilon(t+j), N>0 \tag{46}$$

or recursively

$$\hat{x}(t|t+k)=\hat{x}(t|t+k-1)+K(t|t+k)\varepsilon(t+k), k=0,1,\cdots,N \tag{47}$$

$$\varepsilon(t+j)=y(t+j)-H(t+j)\hat{x}(t+j|t+j-1) \tag{48}$$

where $\hat{x}(t+j|t+j-1)$ is the robust Kalman one-step predictor. The time-varying smoothing gain is computed as

$$K(t|t+j)=P(t|t-1)\left\{\prod_{k=0}^{j-1}\Psi_p^{\mathrm{T}}(t+k)\right\}H^{\mathrm{T}}(t+j)Q_\varepsilon^{-1}(t+j), j\ge 1, \ K(t|t)=K_f(t) \tag{49}$$

$$Q_\varepsilon(t)=H(t)P(t|t-1)H^{\mathrm{T}}(t)+R(t) \tag{50}$$

The conservative smoothing error variance is given as

$$P(t|t+N)=P(t|t-1)-\sum_{j=0}^{N}K(t|t+j)Q_\varepsilon(t+j)K^{\mathrm{T}}(t|t+j), N>0 \tag{51}$$

or recursively

$$P(t|t+k)=P(t|t+k-1)-K(t|t+k)Q_\varepsilon(t+k)K^{\mathrm{T}}(t|t+k), \ k=0,1,\cdots,N \tag{52}$$

The conservative and actual smoothing error variances respectively satisfy the Lyapunov equations

$$P(t|t+N)=\Psi_N(t)P(t|t-1)\Psi_N^{\mathrm{T}}(t)+\sum_{\rho=0}^{N}\left[K_\rho^w(t)Q(t+\rho)K_\rho^{w\mathrm{T}}(t)+K_\rho^v(t)R(t+\rho)K_\rho^{v\mathrm{T}}(t)\right] \tag{53}$$

$$\bar{P}(t|t+N)=\Psi_N(t)\bar{P}(t|t-1)\Psi_N^{\mathrm{T}}(t)+\sum_{\rho=0}^{N}\left[K_\rho^w(t)\bar{Q}(t+\rho)K_\rho^{w\mathrm{T}}(t)+K_\rho^v(t)\bar{R}(t+\rho)K_\rho^{v\mathrm{T}}(t)\right] \tag{54}$$

with the definitions

$$\Psi_p(t+k,t)=\Psi_p(t+k-1)\cdots\Psi_p(t),\Psi_p(t,t)=I_n,$$

$$\Psi_N(t)=I_n-\sum_{k=0}^{N}K(t|t+k)H(t+k)\Psi_p(t+k,t),$$

$$K_\rho^w(t)=-\sum_{k=\rho+1}^{N}K(t|t+k)H(t+k)\Psi_p(t+k,t+\rho+1)\Gamma(t+\rho),\ \rho=0,\cdots,N-1,K_N^w(t)=0,$$

$$K_\rho^v(t)=\sum_{k=\rho+1}^{N}K(t|t+k)H(t+k)\Psi_p(t+k,t+\rho+1)K_p(t+\rho)-K(t|t+\rho),$$

$$\rho=0,\cdots,N-1,K_N^v(t)=-K(t|t+N) \tag{55}$$

where $\Psi_p(t)$, $K_p(t)$ and $K(t|t+k)$ are defined by (19), (20) and (49).

The robust Kalman smoother (46) is robust in the sense that for all admissible uncertainties satisfying (4) and (5), the corresponding actual smoothing error variances $\bar{P}(t|t+N)$ satisfy

$$\bar{P}(t|t+N)\le P(t|t+N),\ \ N>0 \tag{56}$$

and $P(t|t+N)$ is a minimal upper bound of $\bar{P}(t|t+N)$.

Proof. For the conservative system (1) and (2) with the upper bounds $Q(t)$, $R(t)$ and $P(0|0)$ satisfying (4) and (5), the equations (46)-(52) were proved in [20]. Iterating (24), applying (46) and (48), we have (55) and

$$\tilde{x}(t|t+N)=\Psi_N(t)\tilde{x}(t|t-1)+\sum_{\rho=0}^{N}K_\rho^w(t)w(t+\rho)+\sum_{\rho=0}^{N}K_\rho^v(t)v(t+\rho) \tag{57}$$

where $K_\rho^w(t)$ and $K_\rho^v(t)$ are given in (55), which are obtained by combining the terms with the same classes. Applying (57) directly yields (53) and (54). Subtracting (54) from (53) yields the Lyapunov equation

$$\Delta P(t|t+N)=\Psi_N(t)\Delta P(t|t-1)\Psi_N^{\mathrm{T}}(t)+U(t) \tag{58}$$

$$U(t)=\sum_{\rho=0}^{N}K_\rho^w(t)\Delta Q(t+\rho)K_\rho^{w\mathrm{T}}(t)+K_\rho^v(t)\Delta R(t+\rho)K_\rho^{v\mathrm{T}}(t) \tag{59}$$

with the definitions

$$\Delta P(t|t+N)=P(t|t+N)-\bar{P}(t|t+N),\Delta P(t|t-1)=P(t|t-1)-\bar{P}(t|t-1),$$

$$\Delta Q(t+\rho)=Q(t+\rho)-\bar{Q}(t+\rho),\Delta R(t+\rho)=R(t+\rho)-\bar{R}(t+\rho) \tag{60}$$

Applying (4) yields that $\Delta Q(t+\rho)\ge 0$ and $\Delta R(t+\rho)\ge 0$. From Theorem 2 we have $\Delta P(t|t-1)\ge 0$. Hence $U(t)\ge 0$, so that $\Delta P(t|t+N)\ge 0$, that is, (56) holds. Similar to Theorem 1, we easily prove that $P(t|t+N)$ is a minimal upper bound of $\bar{P}(t|t+N)$. The proof is completed.

Remark 2. Compared with the augmented state approach in Theorem 3, Theorem 4 presents a new robust Kalman smoother based on the non-augmented state approach, and gives a new proof of the robustness of the robust Kalman smoother. In Theorem 4, the two algorithms (51) and (53) for computing the conservative variances $P(t|t+N)$ are given, where the formula (51) can be obtained directly based on (46). However, directly applying (46) cannot obtain the actual variances $\bar{P}(t|t+N)$. Since both the augmented and non-augmented algorithms are derivate based on the projection theory [20], according to the uniqueness of the projection, then they are numerically equivalent. Their numerical equivalence will be verified in Table 3 of the simulation example.

6. Robust Steady-state Kalman Estimators

6.1. Robust Steady-state Kalman Filter

Now we investigate the asymptotic properties of the robust time-varying Kalman estimators, and we shall present the corresponding steady-state robust Kalman estimators. We shall also rigorously prove the convergence in a realization between the robust time-varying and steady-state Kalman estimators, by the DESA method and DVESA method [17, 18].

Theorem 5. For uncertain time-invariant system (1) and (2) with Assumption 1, where $\Phi(t)=\Phi$, $\Gamma(t)=\Gamma$, $H(t)=H$, $Q(t)=Q, R(t)=R$, $\bar{Q}(t)=\bar{Q}$ and $\bar{R}(t)=\bar{R}$ are all the constant matrices, if (Φ, H) is a completely detectable pair and $(\Phi, \Gamma Q^{1/2})$ is a completely stabilizable pair, with $Q=Q^{1/2}(Q^{1/2})^{\mathrm{T}}$, according to the steady-state Kalman filtering theory [23], from (7)-(10), (13) and (17), the robust steady-state Kalman filter is given as

$$\hat{x}^s(t\,|\,t)=\Psi_f\hat{x}^s(t-1\,|\,t-1)+K_f y(t) \tag{61}$$

$$\Psi_f=\left[I_n-K_fH\right]\Phi,\ K_f=\Sigma H^{\mathrm{T}}\left(H\Sigma H^{\mathrm{T}}+R\right)^{-1} \tag{62}$$

where Ψ_f is a stable matrix, the superscript s denotes "steady-state", $y(t)$ is the actual measurement, and Σ satisfies the steady-state Riccati equation

$$\Sigma=\Phi\left[\Sigma-\Sigma H^{\mathrm{T}}\left(H\Sigma H^{\mathrm{T}}+R\right)^{-1}H\Sigma\right]\Phi^{\mathrm{T}}+\Gamma Q\Gamma^{\mathrm{T}} \tag{63}$$

and the conservative variance P satisfies the steady-state Lyapunov equation

$$P=\Psi_fP\Psi_f^{\mathrm{T}}+\left[I_n-K_fH\right]\Gamma Q\Gamma^{\mathrm{T}}\left[I_n-K_fH\right]^{\mathrm{T}}+K_fRK_f^{\mathrm{T}} \tag{64}$$

and the actual variance $\bar{P}$ also satisfies the Lyapunov equation

$$\bar{P}=\Psi_f\bar{P}\Psi_f^{\mathrm{T}}+\left[I_n-K_fH\right]\Gamma\bar{Q}\Gamma^{\mathrm{T}}\left[I_n-K_fH\right]^{\mathrm{T}}+K_f\bar{R}K_f^{\mathrm{T}} \tag{65}$$

and we have

$$\Psi_f(t)\to\Psi_f,\ K_f(t)\to K_f,\ P(t\,|\,t-1)\to\Sigma,\ P(t\,|\,t)\to P,\ \bar{P}(t\,|\,t)\to\bar{P},\ \text{as}\ t\to\infty \tag{66}$$

The robust steady-state Kalman filters (61) are robust in the sense that for all admissible uncertainties of $\bar{Q}$ and $\bar{R}$ satisfying $\bar{Q}\le Q, \bar{R}\le R$, it follows that $\bar{P}\le P$, and P is a minimal upper bound of $\bar{P}$.

The robust time-varying and steady-state Kalman filters $\hat{x}(t\,|\,t)$ and $\hat{x}^s(t\,|\,t)$ given by (7) and (61) have the convergence in a realization, such that

$$\left[\hat{x}(t\,|\,t)-\hat{x}^s(t\,|\,t)\right]\to 0,\ \text{as}\ t\to\infty,\ \text{i.a.r.}, \tag{67}$$

where the notation "i.a.r." denotes the convergence in a realization [17].

Proof. The proof of the Theorem is similar to the reference [14], the detail is omitted.

6.2. Robust Steady-state Kalman Predictor

Similar to the proof of Theorem 5, we easily prove the following Theorem 6-8.

Theorem 6. Under the conditions of Theorem 5, the robust steady-state Kalman one-step predictor is given as

$$\hat{x}^s(t\,|\,t-1)=\Psi_p\hat{x}^s(t-1\,|\,t-2)+K_py(t-1) \tag{68}$$

$$\Psi_p=\Phi-K_pH,\ K_p=\Phi\Sigma H^{\mathrm{T}}\left(H\Sigma H^{\mathrm{T}}+R\right)^{-1} \tag{69}$$

the conservative prediction error variance Σ satisfies the steady-state Riccati equation

$$\Sigma=\Phi\left[\Sigma-\Sigma H^{\mathrm{T}}\left(H\Sigma H^{\mathrm{T}}+R\right)^{-1}H\Sigma\right]\Phi^{\mathrm{T}}+\Gamma Q\Gamma^{\mathrm{T}} \tag{70}$$

which can be rewritten as the Lyapunov equation

$$\Sigma=\Psi_p\Sigma\Psi_p^{\mathrm{T}}+\Gamma Q\Gamma^{\mathrm{T}}+K_pRK_p^{\mathrm{T}} \tag{71}$$

and the actual prediction error variance $\bar{\Sigma}$ is given as

$$\bar{\Sigma}=\Psi_p\bar{\Sigma}\Psi_p^{\mathrm{T}}+\Gamma\bar{Q}\Gamma^{\mathrm{T}}+K_p\bar{R}K_p^{\mathrm{T}} \tag{72}$$

and we have

$$P(t\,|\,t-1)\to\Sigma,\ \bar{P}(t\,|\,t-1)\to\bar{\Sigma},\ \text{as}\ t\to\infty \tag{73}$$

The robust steady-state Kalman one-step predictor (68) is robust $\bar{\Sigma}\le\Sigma$ and Σ is a minimal upper bound of $\bar{\Sigma}$.

The robust steady-state fused Kalman multi-step predictor is given as

$$\hat{x}^s(t|t+N)=\Phi^{-N-1}\hat{x}^s(t+N+1|t+N)\quad ,N\le -2 \quad (74)$$

where $\hat{x}^s(t+1|t)$ is the robust steady-state one-step predictor computed by (68).

The conservative steady-state N step prediction error variances are given as

$$P(N)=\Phi^{-N-1}\Sigma\Phi^{(-N-1)\mathrm{T}}+\sum_{j=2}^{-N}\Phi^{-N-j}\Gamma Q\Gamma^{\mathrm{T}}\Phi^{(-N-j)\mathrm{T}}\ ,N\le -2 \quad (75)$$

The actual steady-state prediction error variances are given as

$$\bar{P}(N)=\Phi^{-N-1}\bar{\Sigma}\Phi^{(-N-1)\mathrm{T}}+\sum_{j=2}^{-N}\Phi^{-N-j}\Gamma\bar{Q}\Gamma^{\mathrm{T}}\Phi^{(-N-j)\mathrm{T}}\ ,N\le -2 \quad (76)$$

and we have

$$P(t|t+N)\to P(N),\ \bar{P}(t|t+N)\to\bar{P}(N)\ \text{,as}\ t\to\infty\ ,N\le -2 \quad (77)$$

The robust local steady-state Kalman multi-step predictor (74) is robust $\bar{P}(N)\le P(N)$, $N\le -2$ and $P(N)$ is a minimal upper bound of $\bar{P}(N)$.

If the measurement data of $y(t)$ are bounded, then the robust time-varying and steady-state Kalman predictors s $\hat{x}(t|t+N)$ and $\hat{x}^s(t|t+N)$, $N<0$ given by (26) and (74) have each other the convergence in a realization, such that

$$\left[\hat{x}(t|t+N)-\hat{x}^s(t|t+N)\right]\to 0\ \text{, as}\ t\to\infty\text{, i.a.r.,} \quad (78)$$

6.3. Robust Steady-state Kalman Smoother

Theorem 7. Under the conditions of Theorem 5, the robust steady-state Kalman smoother $\hat{x}^s(t|t+N)$ is given as

$$\hat{x}^s(t|t+N)=H_x\hat{x}_a^s(t+N|t+N),N>0,\quad H_x=\begin{bmatrix}0 & \cdots & 0 & I_n\end{bmatrix} \quad (79)$$

where $\hat{x}_a^s(t|t)$ is the robust steady-state Kalman filter for the augmented system (34) and (35), which is given as

$$\hat{x}_a^s(t|t)=\Psi_a\hat{x}_a^s(t-1|t-1)+K_a y(t) \quad (80)$$

$$\Psi_a=[I_n-K_aH_a]\Phi_a\ ,K_a=\Sigma_aH_a^{\mathrm{T}}\left(H_a\Sigma_aH_a^{\mathrm{T}}+R\right)^{-1} \quad (81)$$

$$\Sigma_a=\Phi_a\left[\Sigma_a-\Sigma_aH_a^{\mathrm{T}}\left(H_a\Sigma_aH_a^{\mathrm{T}}+R\right)^{-1}H_a\Sigma_a\right]\Phi_a^{\mathrm{T}}+\Gamma_aQ\Gamma_a^{\mathrm{T}} \quad (82)$$

with the definitions

$$\Phi_a=\begin{bmatrix}\Phi & 0 & \cdots & 0\\ I_n & 0 & & \\ 0 & \ddots & \ddots & \vdots\\ 0 & \cdots & I_n & 0\end{bmatrix},\ \Gamma_a=\begin{bmatrix}\Gamma\\ 0\\ \vdots\\ 0\end{bmatrix},\ H_a=\begin{bmatrix}H & 0 & \cdots & 0\end{bmatrix} \quad (83)$$

The conservative and actual steady-state smoothing error variance can be respectively computed as

$$P(N)=H_xP_aH_x^{\mathrm{T}},\bar{P}(N)=H_x\bar{P}_aH_x^{\mathrm{T}},\ N>0 \quad (84)$$

and we have

$$P_a(t|t)\to P_a,\ \bar{P}_a(t|t)\to\bar{P}_a,\ P(t|t+N)\to P(N),\bar{P}(t|t+N)\to\bar{P}(N),\ \text{as}\ t\to\infty \quad (85)$$

where $P(t|t+N)$ and $\bar{P}(t|t+N)$ are computed by (44) and (45), P_a and $\bar{P}_a$ satisfy the Lyapunov equations

$$P_a=\Psi_aP_a\Psi_a^{\mathrm{T}}+[I-K_aH_a]\Gamma_aQ\Gamma_a^{\mathrm{T}}[I-K_aH_a]^{\mathrm{T}}+K_aRK_a^{\mathrm{T}} \quad (86)$$

$$\bar{P}_a=\Psi_a\bar{P}_a\Psi_a^{\mathrm{T}}+[I-K_aH_a]\Gamma_a\bar{Q}\Gamma_a^{\mathrm{T}}[I-K_aH_a]^{\mathrm{T}}+K_a\bar{R}K_a^{\mathrm{T}} \quad (87)$$

The robust steady-state Kalman smoother (79) is robust $\bar{P}(N)\le P(N)$, $N>0$ and $P(N)$ is a minimal upper bound of $\bar{P}(N)$.

From (46)-(56), the robust steady-state Kalman smoother based on the non-augmented state approach is given as

$$\hat{x}^s(t|t+N)=\hat{x}^s(t|t-1)+\sum_{j=0}^{N}K(j)\varepsilon^s(t+j)\ ,N>0 \quad (88)$$

or recursively

$$\hat{x}^s(t\,|\,t+k)=\hat{x}^s(t\,|\,t+k-1)+K(k)\varepsilon^s(t+k),\ k=0,1,\cdots,N \tag{89}$$

$$\varepsilon^s(t+j)=y(t+j)-H\hat{x}^s(t+j\,|\,t+j-1) \tag{90}$$

where $\hat{x}^s(t+j\,|\,t+j-1)$ is the robust steady-state Kalman one-step predictor, and is computed by (68), and

$$K(j)=\Sigma(\Psi_p^{\mathrm{T}})^j H^{\mathrm{T}}Q_\varepsilon^{-1},\ j\ge 1,\ \ K(0)=\Sigma H^{\mathrm{T}}Q_\varepsilon^{-1},\ \ Q_\varepsilon=H\Sigma H^{\mathrm{T}}+R \tag{91}$$

The conservative steady-state smoothing error variance is given as

$$P(N)=\Sigma-\sum_{j=0}^{N}K(j)Q_\varepsilon K^{\mathrm{T}}(j) \tag{92}$$

or recursively

$$P(k)=P(k-1)-K(k)Q_\varepsilon K^{\mathrm{T}}(k),\ \ k=0,1,\cdots,N \tag{93}$$

Theorem 8. Under the conditions of Theorem 5, the robust steady-state Kalman smoother (88) and (90) has the conservative and actual smoothing error variances as

$$P(N)=\Psi_N\Sigma\Psi_N^{\mathrm{T}}+\sum_{\rho=0}^{N}\left[K_\rho^w QK_\rho^{w\mathrm{T}}+K_\rho^v RK_\rho^{v\mathrm{T}}\right] \tag{94}$$

$$\bar{P}(N)=\Psi_N\bar{\Sigma}\Psi_N^{\mathrm{T}}+\sum_{\rho=0}^{N}\left[K_\rho^w \bar{Q}K_\rho^{w\mathrm{T}}+K_\rho^v \bar{R}K_\rho^{v\mathrm{T}}\right] \tag{95}$$

with the definitions

$$\Psi_N=I_n-\sum_{k=0}^{N}K(k)H\Psi_p^k,$$

$$K_\rho^w=-\sum_{k=\rho+1}^{N}K(k)H\Psi_p^{k-\rho-1}\Gamma,\ \rho=0,\cdots,N-1, K_N^w=0,$$

$$K_\rho^v=\sum_{k=\rho+1}^{N}K(k)H\Psi_p^{k-\rho-1}K_p-K(\rho),\ \rho=0,\cdots,N-1, K_N^v=-K(N) \tag{96}$$

where Ψ_p, K_p and $K(k)$ are defined by (69) and (91).

The robust Kalman smoother (88) is robust $\bar{P}(N)\le P(N)$, $N>0$ and $P(N)$ is a minimal upper bound of $\bar{P}(N)$.

If the measurement data of $y(t)$ are bounded, then the robust time-varying and steady-state Kalman smoothers $\hat{x}(t\,|\,t+N)$ and $\hat{x}^s(t\,|\,t+N)$ given by (46)and (88) have each other the convergence in a realization, such that

$$\left[\hat{x}(t\,|\,t+N)-\hat{x}^s(t\,|\,t+N)\right]\to 0,\ \text{as}\ t\to\infty,\ \text{i.a.r.,} \tag{97}$$

7. The Robust Accuracy Comparison

Theorem 9. Under the conditions of Theorem 5, the robust time-varying and steady-state Kalman filters $(N=0)$, predictors $(N<0)$, and smoothers $(N>0)$ have the following accuracy relations with the matrix inequalities as

$$\bar{P}(t\,|\,t+N)\le P(t\,|\,t+N),\ \ N=0, N<0, N>0 \tag{98}$$

$$P(t\,|\,t)\le P(t\,|\,t-1)\le\cdots\le P(t\,|\,t+N+1)\le P(t\,|\,t+N),\ \ N\le -1 \tag{99}$$

$$P(t\,|\,t+N)\le P(t\,|\,t+N-1)\le\cdots\le P(t\,|\,t+1)\le P(t\,|\,t),\ N\ge 1 \tag{100}$$

$$\bar{P}(N)\le P(N),\ \ N=0, N<0, N>0 \tag{101}$$

$$P\le\Sigma\le\cdots\le P(N+1)\le P(N),\ N\le -1 \tag{102}$$

$$P(N) \le P(N-1) \le \cdots \le P(1) \le P\,,\, N \ge 1 \tag{103}$$

with the definitions $P = P(0), \bar{P} = \bar{P}(0), \Sigma = P(-1), \bar{\Sigma} = \bar{P}(-1)$.

Proof. The robustness (98) was proved in the above Theorems. Applying the recursive projection formula [20], we have

$$\hat{x}(t\,|\,t+N+1) = \hat{x}(t\,|\,t+N) + \mathrm{E}\left[x(t)\varepsilon^{\mathrm{T}}(t+N+1)\right]Q_{\varepsilon}^{-1}(t+N+1)\varepsilon(t+N+1)\,,\, N \le -1 \tag{104}$$

where the innovation $\varepsilon(t+N+1)$ and its variance $Q_{\varepsilon}(t+N+1)$ are computed by (48) and (50).

When $N=-1$, we have

$$\hat{x}(t\,|\,t) = \hat{x}(t\,|\,t-1) + K_f(t)\varepsilon(t) \tag{105}$$

From (105) it follows that $\tilde{x}(t\,|\,t) = x(t) - \hat{x}(t\,|\,t)$ is given as

$$\tilde{x}(t\,|\,t) = \tilde{x}(t\,|\,t-1) - K_f(t)\varepsilon(t) \tag{106}$$

Noting that $\tilde{x}(t\,|\,t)$ is orthogonal to $\varepsilon(t)$, and

$$\tilde{x}(t\,|\,t-1) = \tilde{x}(t\,|\,t) + K_f(t)\varepsilon(t) \tag{107}$$

It follows that

$$P(t\,|\,t-1) = P(t\,|\,t) + K_f(t)Q_{\varepsilon}(t)K_f^{\mathrm{T}}(t) \tag{108}$$

Since $Q_{\varepsilon}(t) > 0$, then $K_f(t)Q_{\varepsilon}(t)K_f^{\mathrm{T}}(t) \ge 0$, and from (108) it follows that

$$P(t\,|\,t) \le P(t\,|\,t-1),\ \ N=-1 \tag{109}$$

When $N<-1$, iterating (1) yields

$$x(t) = \Phi^{N-1}x(t+N+1) + \sum_{k=0}^{-N-2}\Phi^k\Gamma w(t-k-1) \tag{110}$$

Substituting (2) into (48) yields

$$\varepsilon(t+N+1) = H\tilde{x}(t+N+1\,|\,t+N) + v(t+N+1) \tag{111}$$

Notice that

$$x(t+N+1) = \tilde{x}(t+N+1\,|\,t+N) + \hat{x}(t+N+1\,|\,t+N) \tag{112}$$

Substituting (110)-(112) into (104) yields the recursive predictor

$$\hat{x}(t\,|\,t+N+1) = \hat{x}(t\,|\,t+N) + \Phi^{-N-1}P(t+N+1\,|\,t+N)H^{\mathrm{T}}Q_{\varepsilon}^{-1}(t+N+1)\varepsilon(t+N+1) \tag{113}$$

which yields

$$\tilde{x}(t\,|\,t+N+1) = \tilde{x}(t\,|\,t+N) - \Phi^{-N-1}P(t+N+1\,|\,t+N)H^{\mathrm{T}}Q_{\varepsilon}^{-1}(t+N+1)\varepsilon(t+N+1) \tag{114}$$

or equivalently

$$\tilde{x}(t\,|\,t+N) = \tilde{x}(t\,|\,t+N+1) + \Phi^{-N-1}P(t+N+1\,|\,t+N)H^{\mathrm{T}}Q_{\varepsilon}^{-1}(t+N+1)\varepsilon(t+N+1) \tag{115}$$

Since $\tilde{x}(t\,|\,t+N+1)$ is orthogonal to $\varepsilon(t\,|\,t+N+1)$, then we have

$$P(t\,|\,t+N) = P(t\,|\,t+N+1) + \Delta \tag{116}$$

$$\varDelta = \varPhi^{-N-1} P(t+N+1\,|\,t+N) H^{\mathrm{T}} Q_{\varepsilon}^{-1}(t+N+1) H P(t+N+1\,|\,t+N) \left(\varPhi^{-N-1}\right)^{\mathrm{T}} \tag{117}$$

where from $Q_{\varepsilon}(t+N+1) > 0$, we have $\varDelta \geq 0$, and from (116) it follows that

$$P(t\,|\,t+N+1) \leq P(t\,|\,t+N),\ \ N < -1 \tag{118}$$

From (109) and (118) we obtain (99). Similarly, (100) can be directly proved from (52). As $t \to \infty$, taking the limit operations for (98)-(100) yields (101)-(103). The proof is completed.

Corollary 1. Under the conditions of Theorem 5, the robust time-varying and steady-state Kalman estimators have the following robust and actual accuracy relations, respectively,

$$\mathrm{tr}\,\bar{P}(t\,|\,t+N) \leq \mathrm{tr}\,P(t\,|\,t+N),\ \ N=0, N<0, N>0 \tag{119}$$

$$\mathrm{tr}\,P(t\,|\,t) \leq \mathrm{tr}\,P(t\,|\,t-1) \leq \cdots \leq \mathrm{tr}\,P(t\,|\,t+N+1) \leq \mathrm{tr}\,P(t\,|\,t+N),\ \ N \leq -1 \tag{120}$$

$$\mathrm{tr}\,P(t\,|\,t+N) \leq \mathrm{tr}\,P(t\,|\,t+N-1) \leq \cdots \leq \mathrm{tr}\,P(t\,|\,t+1) \leq \mathrm{tr}\,P(t\,|\,t),\ \ N \geq 1 \tag{121}$$

$$\mathrm{tr}\,\bar{P}(N) \leq \mathrm{tr}\,P(N),\ \ N=0, N<0, N>0 \tag{122}$$

$$\mathrm{tr}\,P \leq \mathrm{tr}\,\varSigma \leq \cdots \leq \mathrm{tr}\,P(N+1) \leq \mathrm{tr}\,P(N),\ \ N \leq -1 \tag{123}$$

$$\mathrm{tr}\,P(N) \leq \mathrm{tr}\,P(N-1) \leq \cdots \leq \mathrm{tr}\,P(1) \leq \mathrm{tr}\,P,\ \ N \geq 1 \tag{124}$$

Remark 3. Corollary 1 shows that all admissible actual accuracies yielded by the uncertainties of noise variances of each robust Kalman estimator are higher than its robust accuracy, the robust accuracy of the filter is higher than those of the predictors, and is lower than those of the smoothers. For two robust smoothers with different fixed lags N, the robust accuracy of the smoother with larger fixed lag is higher than that of the smoother with smaller fixed lag. For two robust Kalman predictors with different steps $(-N)$ $(N<0)$, the robust accuracy of the predictor with smaller step is higher than that of the predictor with larger step.

8. Simulation Example

Consider a tracking system with uncertain noise variances

$$x(t+1) = \varPhi x(t) + \varGamma w(t) \tag{125}$$

$$y(t) = Hx(t) + v(t) \tag{126}$$

$$\varPhi = \begin{bmatrix} 1 & T_0 \\ 0 & 1 \end{bmatrix}, \varGamma = \begin{bmatrix} 0.5T_0^2 \\ T_0 \end{bmatrix}, H = I_2 \tag{127}$$

where $T_0 = 0.25$ is the sample period, $x(t) = \left[x_1(t), x_2(t)\right]^{\mathrm{T}}$ is the state, $x_1(t)$ and $x_2(t)$ are the position and velocity of target at time tT_0. $w(t)$ and $v(t)$ are independent Gaussion white noises with zero mean and unknown actual variances $\bar{Q}$ and $\bar{R}$ respectively, satisfying $\bar{Q} \leq Q$ and $\bar{R} \leq R$. In the simulation, we consider the following two cases: the case 1 and the case 2. In the case 1, we take $Q=1$, $R=\mathrm{diag}(8, 0.36)$, $\bar{Q}=0.8$, $\bar{R}=\mathrm{diag}(6, 0.25)$, the initial values $x(0)=[0\ \ 0]^{\mathrm{T}}$, $\mu=0$, $P(0\,|\,0)=1.2I_2, \bar{P}(0\,|\,0)=I_2$. In the case 2, we take $Q=1$, $\bar{Q}=0.8+0.2\sin(2\pi t/100)$, $R=I_2$, $\bar{R}=[0.65+0.35\sin(2\pi t/100)]I_2$, the initial values $x(0)=[0\ \ 0]^{\mathrm{T}}$, $\mu=0$, $P(0\,|\,0)=1.2I_2$, $\bar{P}(0\,|\,0)=I_2$, and we take $N=-2,-1,0,1,2$.

The traces of the error variances of the time-varying Kalman estimators are compared in Tables 1-2, and Figures 1-2, which verify the accuracy relations (119)-(121), and the steady-state robust accuracy relations (122)-(124). From Figure 1, we see that the trace of the local and fused robust time-variant Kalman filters can quickly converge to the corresponding steady-state Kalman filters. From Figure 2 we see that the robust and actual accuracy relation (119) of each robust Kalman estimator holds. The traces of the error variances between the augmented state approach and non-augmented state approach are compared in Table 3, which verify that the numerically equivalence of the two approaches.

Table 1. *The accuracy comparison of robust and actual time-varying Kalman estimators at* $t=10$ *for case 1.*

$\mathrm{tr}\,P(t\,|\,t)$	$\mathrm{tr}\,\bar{P}(t\,|\,t)$	$\mathrm{tr}\,P(t\,|\,t-1)$	$\mathrm{tr}\,\bar{P}(t\,|\,t-1)$	$\mathrm{tr}\,P(t\,|\,t-2)$	$\mathrm{tr}\,\bar{P}(t\,|\,t-2)$
0.7410	0.5719	0.8159	0.6271	0.9395	0.7213
$\mathrm{tr}\,P(t\,|\,t+1)$	$\mathrm{tr}\,\bar{P}(t\,|\,t+1)$	$\mathrm{tr}\,P(t\,|\,t+2)$	$\mathrm{tr}\,\bar{P}(t\,|\,t+2)$		
0.6194	0.4743	0.5711	0.4376		

Table 2. *The robust and actual accuracy comparison of robust steady-state Kalman estimators for case 1.*

tr P	tr $\bar{P}$	tr Σ	tr $\bar{\Sigma}$	tr $P(-2)$	tr $\bar{P}(-2)$	tr $P(1)$	tr $\bar{P}(1)$	tr $P(2)$	tr $\bar{P}(2)$
0.5245	0.3815	0.6278	0.4603	0.7541	0.5565	0.4696	0.3417	0.4365	0.3185

Table 3. *The robust and actual accuracy comparison of augmented and non-augmented approaches for case 1.*

	tr $P(1)$	tr $\bar{P}(1)$	tr $P(2)$	tr $\bar{P}(2)$
Augmented state approach	0.469619202285972	0.341683402305256	0.436478175999641	0.318492157779223
Non-augmented state approach	0.469619202285973	0.341683402305506	0.436478175294423	0.317492157779228

In order to give a geometric interpretation of the matrix accuracy relations, The covariance ellipses of robust steady-state Kalman estimators are given in Figure 3. From Figure 3, we see that the ellipses of $\bar{P}(N)$ $(N=-2,-1,0,1,2)$ are all enclosed in these of the conservative upper bound $P(N)$, respectively, which verify the robustness (101). Figure 3 also shows that the ellipse of the smoother with two lag is enclosed in that of the smoother with one lag, they are both enclosed in that of the filter, and the ellipse of filter is enclosed in that of the one-step predictor, and they are all enclosed in that of the two-step predictor, which verifies the accuracy relations (102) and (103).

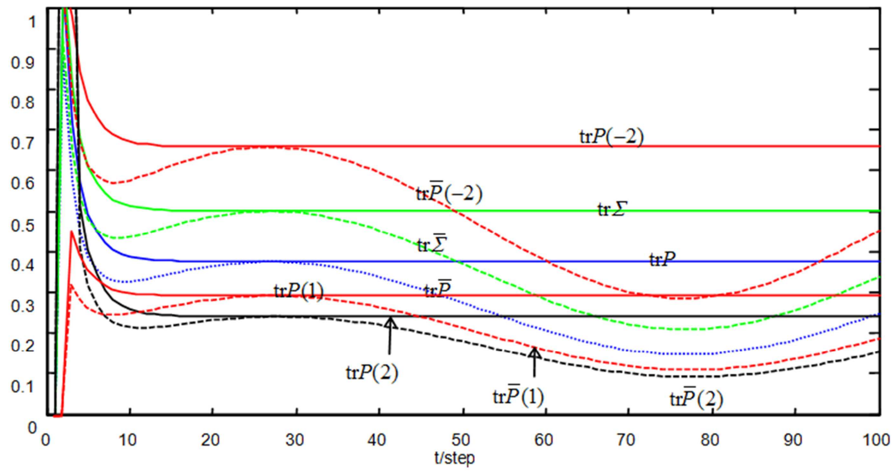

Figure 1. *The robust and actual accuracy relations of the robust time-varying Kalman estimators for case 1.*

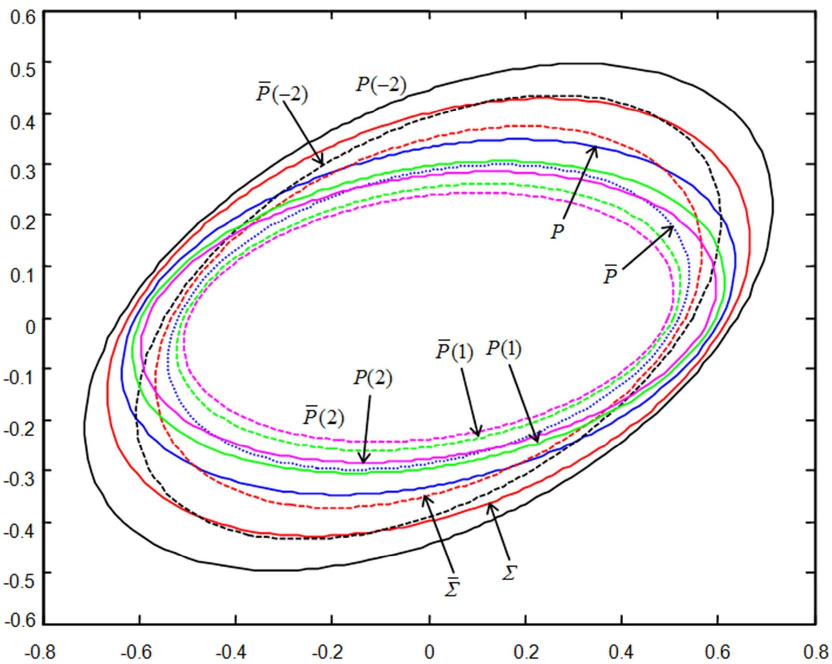

Figure 2. *The robust and actual accuracy relations of the robust time-varying Kalman estimators for case 2.*

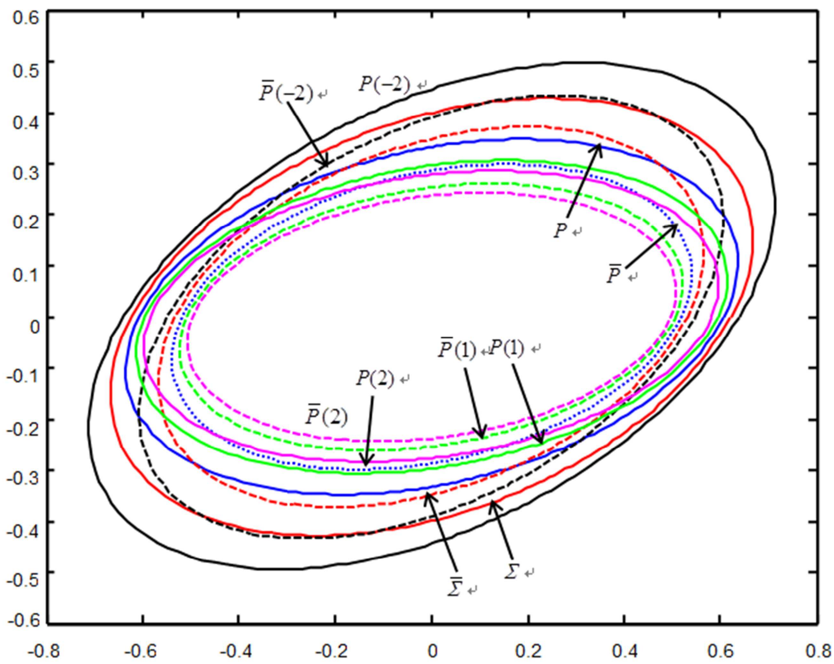

Figure 3. *The covariance ellipses of the actual and conservative steady-state estimation error variances.*

The MSE curves of the robust steady-state Kalman estimators are shown in Figure 4, where the straights lines denote the traces of the theoretical actual error variances, respectively, the curves denote the values of the $\mathrm{MSE}^{(N)}(t)$, $N=-2,-1,0,1,2$. From Figure 4, we see that when ρ and t are sufficiently large, the values of $\mathrm{MSE}^{(N)}(t)$ are close to the corresponding theoretical actual values tr $\overline{P}(N)$. which verifies the accuracy relations (119) and (124).

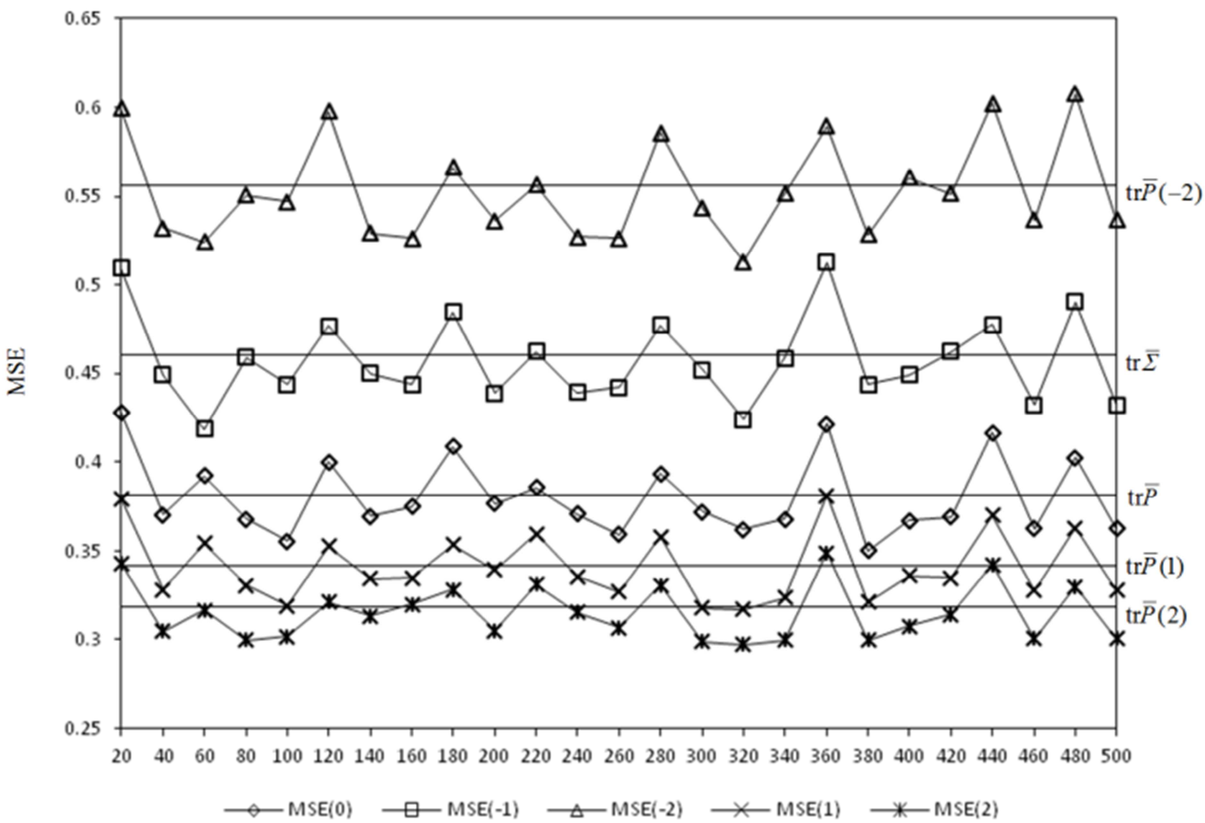

Figure 4. *The MSE curves for robust Kalman estimators for case 1.*

9. Conclusions

For the linear discrete time-varying stochastic system with uncertain noise variances, the minimax robust time-varying Kalman estimators (filter, predictor and smoother) have been presented. The two robust Kalman smoothers have been presented respectively by the augmented state approach and non-augmented state approach. Each estimator has the robustness in the sense that their all actual admissible estimation error variances yielded by the uncertainties of noise variances are guaranteed to have a minimal upper bound. Their robustness was proved by the Lyapunov equation method. Their robust accuracy relations have been proved based on the recursive projection formula. The corresponding robust steady-state Kalman estimators have been also presented for uncertain time-invariant system. The convergence in a realization between the time-varying and steady-state Kalman estimators was proved by the DESA method and the DVESA method.

The above results construct a robust Kalman filtering theory for systems with noise variance uncertainties, which have important theoretical and application meanings. The robust Kalman filters with uncertain model parameters were extended and developed to the robust Kalman estimators (filter, predictor, smoother) with uncertain noise variances. The adaptive Kalman filters with unknown deterministic noise variances [9-11] were extended to the robust Kalman estimators with unknown uncertain noise variances. They may be applied to solve the weighted fusion robust Kalman filtering problems for multisensor uncertain systems [16]. They may also be applied to solve the robust Kalman filtering problems for systems with the model parameters uncertainties by introducing a virtual noise with uncertain noise variances to compensate model errors [1]. The extension of this work to the systems with both the uncertain model parameters and noise variances is in investigation.

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