

# On $A$ -Self-Adjoint, $A$ -Unitary Operators and Quasiaffinities

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**Abstract:** In this paper, we investigate properties of  $A$ -self-adjoint operators and other relations on Hilbert spaces. In this context,  $A$  is a self-adjoint and an invertible operator. More results on operator equivalences including similarity, unitary and metric equivalences are discussed. We also investigate conditions under which these classes of operators are self-adjoint and unitary. We finally locate their spectra.

**Keywords:**  $A$ -Self-Adjoint,  $A$ -Unitary, Hilbert Space, Metric Equivalence, Quasiaffinities

## 1. Introduction

Throughout this paper Hilbert spaces or subspaces will be denoted by capital letters,  $H$  and  $K$  respectively and  $T, A, B$  etc denote bounded linear operators where an operator means a bounded linear transformation.  $B(H)$  will denote the Banach algebra of bounded linear operators on  $H$ .  $B(H, K)$  denotes the set of bounded linear transformations from  $H$  to  $K$ , which is equipped with the (induced uniform) norm. If  $T \in B(H)$ , then  $T^*$  denotes the adjoint while  $\text{Ker}(T)$ ,  $\text{Ran}(T)$ ,  $\bar{M}$  and  $M^\perp$  stands for the kernel of  $T$ , range of  $T$ , closure of  $M$  and orthogonal complement of a closed subspace  $M$  of  $H$  respectively. For an operator  $T$ , we also denote by  $\sigma(T)$ ,  $\|T\|$  the spectrum and norm of  $T$  respectively.

A contraction on  $H$  is an operator  $T \in B(H)$  such that  $T^*T \leq I$  (i.e.  $\|Tx\| \leq \|x\| \forall x \in H$ ). A strict or proper

contraction is an operator  $T$  with  $T^*T < I$  (i.e.  $\sup_{0 \neq x} \frac{\|Tx\|}{\|x\|} < 1$ ). If  $T^*T = I$ , then  $T$  is called a non-strict

contraction (or an isometry). Many authors like Kubrusly [7] have extensively studied this class of operators.

An operator  $T \in B(H)$  is said to be positive if  $\langle Tx, x \rangle \geq 0 \forall x \in H$ . Suppose that  $A \in B(H)$  is a positive operator, then an operator  $T \in B(H)$  is called an  $A$ -contraction on  $H$  if  $T^*AT \leq A$ . If equality holds, that is  $T^*AT = A$ , then  $T$  is called an  $A$ -isometry, where  $A$  is a self adjoint and invertible operator.

In this research, we put more conditions on  $A$ . In particular, if  $A$  is a self adjoint and invertible operator, then we call such

an  $A$ -isometry an  $A$ -Unitary. Let  $T$  be a linear operator on a Hilbert space  $H$ .

We define the  $A$ -adjoint of  $T$  to be an operator  $S$  such that  $AS = T^*A$ . The existence of such an operator is not guaranteed. It may or may not exist. In fact a given  $T \in B(H)$  may admit many  $A$ -adjoints and if such an  $A$ -adjoint of  $T$  exists, we denote it as  $T^{[*]}$ . Thus  $AT^{[*]} = T^*A$ . We are making an assumption that  $A$  is invertible and so  $T^{[*]} = A^{-1}T^*A$ . It is also clear that  $A$ -adjoint of  $T$  is the adjoint of  $T$  if  $T = I$ . By [2],  $T$  admits an  $A$ -adjoint if and only if  $\text{Ran}(T^*A) \subset \text{Ran}(A)$ . In this case the operator  $A$  is acting as a signature operator on  $H$ .

Two operators  $T \in B(H)$  and  $S \in B(K)$  are similar (denoted  $T \approx S$ ) if there exists an operator  $X \in \mathcal{G}(H, K)$  where  $\mathcal{G}(H, K)$  is a Banach subalgebra of  $B(H, K)$  which is an invertible operator from  $H$  to  $K$  such that  $XT = SX$  (i.e.  $X^{-1}SX$  or  $S = XTX^{-1}$ ).  $T \in B(H)$  and  $S \in B(K)$  are unitarily equivalent (denoted  $T \cong S$ ), if there exists a unitary operator  $U \in \mathcal{G}(H, K)$  such that  $UT = SU$  (i.e.  $T = U^*SU$  or equivalently  $S = UTU^*$ ).

Two operators are considered the "same" if they are unitarily equivalent since they have the same, properties of invertibility, normality, spectral picture (norm, spectrum and spectral radius).

An operator  $X \in B(H, K)$  is quasi-invertible or a quasi-affinity if it is an injective operator with dense range (i.e.  $\text{Ker } X = \{0\}$  and  $\overline{\text{Ran}(X)} = K$ ; equivalently,  $\text{Ker } X = \{\bar{0}\}$

and,  $\text{Ker } X^* = \{\bar{0}\}$  thus  $X \in B(H, K)$  is quasi-invertible if and only if  $X^* \in B(K, H)$  is quasi-invertible).

An operator  $T \in B(H)$  is a *quasi-affine transform* of  $S \in B(K)$  if there exists a quasi-invertible  $X \in B(H, K)$  such that  $XT = SX$  (ie  $X$  intertwines  $T$  and  $S$ ).  $T$  is a *quasiaffine transform* of  $S$  if there exists a quasinvertible operator intertwining  $T$  to  $S$ .

Two operators  $A, B \in B(H)$  are said to be *almost similar* (a.s) (denoted by  $A \stackrel{a.s}{\sim} B$ ) if there exists an invertible operator  $N$  such that the following two conditions are satisfied:  $A^*A = N^{-1}(B^*B)N$  and  $A^* + A = N^{-1}(B^* + B)N$ .

Two operators  $A, B \in B(H)$  are said to be *metrically equivalent* (denoted by  $A \stackrel{m.e}{\sim} B$ ) if  $\|Ax\| = \|Bx\|$  (equivalently,  $|\langle Ax, Ax \rangle|_2^{\frac{1}{2}} = |\langle Bx, Bx \rangle|_2^{\frac{1}{2}}$  for all  $x \in H$ ) or  $A \stackrel{m.e}{\sim} B$  if  $A^*A = B^*B$ . This concept was introduced by Nzimbi et al ([8]).

Two linear operators  $T \in B(H)$  and  $S \in B(K)$  are said to be  $A$  – unitarily equivalent (denoted  $T \cong S$ ), if there exists an  $A$  – unitary operator  $U \in \mathcal{G}(H, K)$  such that  $TU = US$ .

We shall also define the following classes of operators in this paper:

An operator  $T \in B(H)$  is said to be an *involution* if  $T^2 = I$ .

An operator  $T \in B(H)$  is said to be *self-adjoint or Hermitian* if  $T^* = T$  (equivalently, if  $\langle Tx, x \rangle \in \mathbb{R}$ ).

An operator  $T \in B(H)$  is said to be *unitary* if  $T^*T = TT^* = I$  and *normal* if  $T^*T = TT^*$  (equivalently, if  $\|Tx\| = \|T^*x\| \forall x \in H$ ).

An operator  $T \in B(H)$  is said to be a *partial isometry* if  $T = TT^*T$  or equivalently, if  $T^*T$  is a projection.

An operator  $T \in B(H)$  is said to be *quasinormal* if  $T(T^*T) = (T^*T)T$  or equivalently if  $T$  commutes with  $(T^*T)$  that is  $[T, T^*T] = 0$ .

Let  $H$  and  $K$  be Hilbert spaces. An operator  $X \in B(H, K)$  is invertible if it is injective (one-to-one) and surjective (onto or has dense range); equivalently if  $\text{Ker}(X) = \{0\}$  and

$\text{Ran}(X) = K$ . We denote the class of invertible linear operators by  $\mathcal{G}(H, K)$ . The commutator of two operators  $A$  and  $B$ , denoted by  $[A, B]$  is defined by  $AB - BA$ . The self – commutator of an operator  $A$  is  $[A, A^*] = A^*A - AA^*$ .

Suppose  $A \in B(H)$  is a self-adjoint and invertible operator, not necessarily unique. An operator  $T \in B(H)$  is said to be  $A$  – self adjoint if  $T^* = ATA^{-1}$  (equivalently,  $T^{[*]} = T$ ),  $A$  – skew adjoint if  $T^* = -ATA^{-1}$  (equivalently,  $T^{[*]} = -T$ ),  $A$  – normal if  $A^{-1}T^*AT = TA^{-1}T^*A$  or equivalently,  $T^{[*]}T = TT^{[*]}$ ,  $A$  – unitary if  $T^*AT = A$  or equivalently,  $T^{[*]} = T^{-1}$ . Clearly, an  $A$ -isometry whose range is dense in  $H$  is an  $A$  – unitary.

## 2. Basic Results

We shall investigate operators in a Hilbert Space  $H$  that are not self-adjoint. It is well known that every self-adjoint operator has a real spectrum.

The following results will form a basis for our discussion throughout this paper.

**Theorem 2.1** [7, Theorem 2.1]. *An invertible operator  $T$  is a product of two self-adjoint operators if and only if  $\sigma(T) =$*

$\sigma(T^*)$ .

*Proof:* [See 7].

**Remark:** The product of two self-adjoint operators need not to have real spectrum. To justify our claim, we consider self-adjoint operators  $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The

product  $PQ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  has a purely imaginary

spectrum  $\{i, -i\}$ . Denoting by  $\mathfrak{S}_0$  the set of all invertible products of self-adjoint operators  $P$  and  $Q$  and by  $\mathfrak{S}$  the set of invertible operators that are similar to their adjoints, we see that  $\mathfrak{S}_0 \subseteq \mathfrak{S}$ . The above theorem asserts that  $\mathfrak{S} \subseteq \mathfrak{S}_0$  is also valid. By using the invariance of these two classes under similarity transformations, we notice that  $\mathfrak{S}$  is strictly larger than the class of operators that are similar to their adjoints.

We can give an example of a unilateral shift operator on  $H = l^2$  in this context.

**Theorem 2.2** [12]:  *$T \in B(H)$  is unitarily equivalent to its adjoint if and only if  $T$  is a product of a symmetry (self-adjoint or unitary involution) and a self-adjoint operator.*

**Theorem 2.3** [7, pp. 6]: *Two normal operators that are similar are unitarily equivalent.*

**Remark:** Any invertible normal operator which is similar to its adjoint can be expressed as a product of self-adjoint operators, that is, if  $T$  is normal and  $T \in \mathfrak{S}$ , then  $T \in \mathfrak{S}_0$ .

**Proposition 2.4** [17]: *If  $T \in B(H)$  is self-adjoint and injective, then  $T^{-1}$  is also self-adjoint.*

**Remark:** Just like other bounded linear operators, the  $A$ -self adjoint operation satisfies the following properties which can easily be shown using the definition of an  $A$  – self adjoint of  $T$ , that is,  $T^* = ATA^{-1}$ :

- $(T_1 + T_2)^{[*]} = T_1^{[*]} + T_2^{[*]}$
- $(T_1 T_2)^{[*]} = T_2^{[*]} T_1^{[*]}$
- $(T^{-1})^{[*]} = (T^{[*]})^{-1}$
- $(\alpha T)^{[*]} = \bar{\alpha} T^{[*]}$

## 3. $A$ -Self-Adjoint Operators

**Definition:** A Jordan algebra  $\mathbb{J}$  consists of a real vector space equipped with a bilinear product  $xy$  satisfying the commutative law and the Jordan identity:  $xy = yx$  and  $(x^2y)x = x^2(yx) \forall x, y \in \mathbb{R}$ . A Jordan algebra is formally real if  $\sum_{i=1}^n x_i^2 = 0 \Rightarrow x_1 = \dots = x_n = 0$ .

**Remark:** An associative algebra,  $\mathbb{J}'$  over a real Hilbert space  $H$  gives rise to a Jordan algebra  $\mathbb{J}$  under quasi-multiplication: the product  $xy = \frac{1}{2}(xy + yx)$  is commutative and satisfies the Jordan identity since  $4(x^2y)x = (x^2y + yx^2)x + x(x^2y + yx^2) = x^2yx + yx^3 + x^3y + xyx^2 = x^2(yx + yx) + x^2(yx + yx)4x^2(yx)$ .

We say that a Jordan algebra  $\mathbb{J}'$  is *special* if it can be realized as a Jordan subalgebra of some Jordan algebra  $\mathbb{J}$ .

**Example:** If  $\mathbb{J}_A$  is a set of Jordan operators, then the subspace of hermitian operator  $T_1^{[*]} = T_1$  is also closed under the Jordan product, since if  $T_1^{[*]} = T_1$  and  $T_2^{[*]} = T_2$ , then  $(T_1 T_2)^{[*]} = T_1^{[*]} T_2^{[*]} = T_2 T_1 = T_1 T_2$  forms a special algebra  $H(\mathbb{J}_A, [*)$ . These hermitian algebras are the archetypes of all Jordan algebras. We can easily check that hermitian matrices over  $\mathbb{R}$  or  $\mathbb{C}$  form special Jordan algebras that are formally

real.

We shall investigate the Jordan algebra  $\mathbb{J}_A$  of  $A$  – self adjoint Operators denoted by the set  $\mathbb{J}_A = \{T \in B(H): T^{[*]} = T\}$ . Note that just like many other algebras like the Lie algebra  $\mathbb{L}_A$ ,  $\mathbb{J}_A$  is an  $\mathbb{R}$ - linear subspace. That is, it is closed under real linear combinations.

We outline in the following results some conditions that guarantee an  $A$  – self adjoint to be self-adjoint.

**Proposition 3.1:** [7]. Every self –adjoint operator  $T$  is  $A$  – self adjoint.

**Remark:** The converse of the above proposition is not generally true. For consider the operators  $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . A quick calculation reveals that  $T$  is  $A$  – self adjoint but it is not self-adjoint. We note that  $A$  – self adjointness coincides with self-adjointness when  $A$  is an identity operator.

We now answer the question: when is an  $A$  – self adjoint operator self-adjoint? The results below give us answer the question.

**Lemma 3.2:** Let  $T \in B(H)$  be  $A$  – self adjoint operator. Then  $T$  is self-adjoint if and only if  $T$  and  $T^*$  commute with an involution.

*Proof:* Suppose  $T$  is  $A$  – self adjoint. Then  $T^* = ATA^{-1}$  for some invertible and self-adjoint operator  $A$ . Now suppose that the similarity transformation  $A$  is an involution. Then, clearly,  $ATA^{-1}T^* = A^{-1}T^*A = AT^*A^{-1}$ . This assertion proves that  $T = T^*$  and so  $T$  is self-adjoint.

**Theorem 3.3 [15]:** Let  $H$  and  $K$  be Hilbert spaces and let  $A \in B(H, K)$ . Then

- i.  $\text{Ker}(A) = \text{Ran}(A^*)^\perp$
- ii.  $\text{Ker}(A^*) = \text{Ran}(A)^\perp$
- iii.  $\overline{\text{Ran}}(A) = \text{Ker}(A^*)^\perp$
- iv.  $\overline{\text{Ran}}(A^*)^\perp = \text{Ker}(A)^\perp$

**Remark:** We note that if  $A \in B(H)$  is self-adjoint, then by iii above,  $\overline{\text{Ran}}(A) = \text{Ker}(A)^\perp$  and so  $H = \text{Ker}(A) \oplus \overline{\text{Ran}}(A)$ .

It has been proved in [7] that if  $T \in B(H)$  is an  $A$  – self adjoint, then its adjoint  $T^*$  is injective. This result together with the corollary to Theorem 4.12 [13] enables us identify the relationship between  $A$  – self adjoint operators and the quasi-affinity. (See Theorem 3.5 pp. 10, of [7]).

Evidently, if  $T \in B(H)$  is an  $A$  – self adjoint operator, then  $T$  and its adjoint,  $T^*$  are quasi-affinities. In fact  $T$  and  $T^*$  are left invertible, that is if there exists an operator  $S \in B(H)$  such that  $ST = I$  and  $ST^* = I$ .

We shall also give the relationship between metrically equivalent operators and  $A$  – unitarily equivalent operators for some given quasiaffinity:

**Theorem 3.4:** [10, Theorem 3.29 (ii)]: If  $A$  and  $B$  are metrically equivalent operators and  $A$  is self-adjoint, then  $A = |B|$ .

**Theorem 3.5** [9, Theorem 2.9 (Fuglede-Putnam-Rosenblum)]: Let  $A \in B(H)$  and  $B \in B(H)$ . If  $AX = XB$  holds for some operator  $X$ , then  $A^*X = XB^*$ .

**Theorem 3.6:** Let  $A, B \in B(H)$ . Suppose  $A$  and  $B$  are metrically equivalent operators,  $AA^* = BB^*$  and  $XB = AX, X^*B = AX^*$  for some quasiaffinity  $X$  which is  $A$  – unitary, then  $A$  and  $B$  are  $A$  – unitarily equivalent.

*Proof:* We first note that every unitary operator is  $A$  – unitary. We show that if  $A$  and  $B$  are metrically equivalent then they are unitarily equivalent.

Suppose  $A^*A = B^*B$ ,  $AA^* = BB^*$  and  $XB = AX, X^*B = AX^*$  for some quasiaffinity  $X$ . Suppose  $X = U|X|$  is the polar decomposition of  $X$ , where  $U$  is a partial isometry and  $|X| = \sqrt{X^*X}$  is positive.

Define  $W = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$  and  $S = \begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix}$  on  $H \oplus H$ . Since  $X$  is a quasiaffinity, so is  $W$ . Using  $XB = AX$  and  $X^*B = AX^*$  we have that  $S^*S = \begin{pmatrix} B^*B & 0 \\ 0 & A^*A \end{pmatrix} = \begin{pmatrix} A^*A & 0 \\ 0 & B^*B \end{pmatrix} = SS^*$  and  $SW = WS^*$  which means that  $S$  and  $S^*$  are quasisimilar normal operators. By the Fuglede-Putnam-Rosenblum Theorem above,  $S$  and  $S^*$  are unitarily equivalent meaning that there exists a unitary operator  $U$  such that  $S = U^*S^*U$  where  $U$  is a polar decomposition of  $X$ . That is  $\begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix} = U^* \begin{pmatrix} 0 & B \\ A^* & 0 \end{pmatrix} U$ , which shows that  $A = U^*BU$ .

**Question:** Is every part of an  $A$  – self adjoint operator  $T$  also  $A$  – self adjoint? This question can be answered if we decompose  $T$  as a direct sum  $T = T_1 \oplus T_2$  by specifying certain conditions on the direct summands of  $T$ . We summarize this in the following theorem:

**Theorem 3.7:** Every part of an  $A$  – self adjoint operator  $T$  is  $A$  – self adjoint.

*Proof:* Suppose  $T = T_1 \oplus T_2$  where  $T_1$  has a certain property  $P$  while  $T_2$  is devoid of property  $P$ . Then by definition of  $A$  – self adjointness we have  $T^* = T_1^* \oplus T_2^* = A(T_1 \oplus T_2)A^{-1} = AT_1A^{-1} \oplus AT_2A^{-1}$ . Thus,  $T_1 = AT_1A^{-1}$  and  $T_2 = AT_2A^{-1}$  as required.

**Remark:** It has been shown in [7] that if  $T \in B(H)$  is an  $A$  – self adjoint operator then  $T$  is unitary if  $T$  is an involution. In additional, the spectrum of  $T$  is either real or complex; if complex, then the eigen values come in complex conjugate pairs. (see [6]). This gives us a necessary and sufficient condition for  $A$  – self adjointness.

In general, such operators have a symmetric with respect to the real axis. Equality of spectra is a necessary condition for  $A$  – self adjointness. We summarize it in the following corollary:

**Corollary 3.8:** Let  $T \in B(H)$  is an  $A$  – self adjoint. Then

- a).  $\sigma_p(T) = \sigma_p(T^*)$
- b).  $\sigma_c(T) = \sigma_c(T^*)$
- c).  $\sigma_r(T) = \sigma_r(T^*)$

**Proof:** Since  $T$  is an  $A$  – self adjoint then by definition  $T^* = ATA^{-1}$ . Thus,  $T^*$  and  $T$  are similar and hence have the same spectrum. Therefore the above claims follow since  $\sigma(T)$  is the disjoint union of  $\sigma_p(T), \sigma_c(T)$  and  $\sigma_r(T)$ .

*Counter Example*

The backward shift operator  $T: l^2 \rightarrow l^2$  defined by  $T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$  is not  $A$  – self adjoint. Its adjoint (called the unilateral shift) is defined by  $T^*(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$ . We see (as an infinite matrix) that every  $\lambda \in \mathbb{C}$  with  $|\lambda| < 1$  (open unit disc centred at the origin) is in  $\sigma_p(T)$  and that  $\sigma_p(T^*) = \emptyset$ . Also,

$\{\lambda \in \mathbb{C}: |\lambda| < 1\} \subset \sigma_r(T^*)$ . Hence  $T$  is not  $A$ -self adjoint (for any  $A$  with the required properties) because the necessary condition for  $A$ -self adjointness is not satisfied i.e.  $\sigma(T) \neq \sigma(T^*)$ .

**Question:** Given that  $T \in B(H)$  is  $A$ -self adjoint, is  $AT$  –self adjoint? We provide the solution in the following theorem.

**Theorem 3.9:**  $T \in B(H)$  is  $A$ -self adjoint, if and only if is  $AT$  –self adjoint.

*Proof:*  $T \in B(H)$   $A$ -self adjoint implies that  $T^* = ATA^{-1}$ . We then have that  $T^*A = AT$ . Thus  $(AT)^* = T^*A^* = T^*A = AT$  (since  $A$  is self-adjoint).

Conversely, let  $AT$  be  $A$ -self adjoint. Then  $(AT)^* = AT$ . Post multiplying both sides of this equation by  $A^{-1}$  and using the definition we have  $T^* = ATA^{-1}$ . This completes the proof.

**Remark:** In view of the above theorem, we see that the mapping defined by  $\varphi: T \rightarrow AT$  is an isomorphism i.e. it establishes a one-to-one correspondence between the class of self-adjoint and  $A$ -self adjoint operators in the Hilbert space  $H$ . In fact if we let  $T \in B(H)$  to be  $A$ -self adjoint then we see that  $T$  is self-adjoint if  $A$  commutes with  $T$  i.e.  $AT = TA$ . Here  $T^* = ATA^{-1} = T$ . Then  $AT = TA$ .

## 4. $A$ -Self-Adjoint, Unitary Equivalence and $A$ -Unitarily Equivalence of Operators

It is well known that unitary equivalence is an equivalence relation. We give a condition which shows that unitary equivalence preserves  $A$ -self adjointness.

**Theorem 4.1:** Let  $S$  and  $T$  be bounded linear operators on a Hilbert space  $H$ . If  $T$  is  $A$ -self adjoint and  $T$  is unitarily equivalent to  $S$ , that is  $UT = SU$ , where  $U$  is a unitary operator, then  $S$  is  $UAU^*$ -self adjoint.

*Proof:* We have  $T^* = ATA^{-1}$  and  $T = U^*SU$  for some unitary operator  $U$ . Using these two equations we can simplify and re-write  $S^*$  in terms of operators  $U$ ,  $S$  and  $U^*$  only as:

$$S^* = UT^*U^* = U[ATA^{-1}]U^* = U\{A(U^*SU)A^{-1}\}U^* = (UAU^*)S(UAU^*)^* \text{ which establishes the claim.}$$

**Remark:** The above theorem shows that unitary equivalence preserves  $A$ -self adjointness if and only if  $UAU^* = A$ . That is, if the unitary operator  $U$  is  $A^*$ -unitary. We see that unlike self-adjointness, unitary equivalence does not preserve  $A$ -self adjointness.

The following results will enable us establish the relationship be  $A$ -unitarily equivalence and  $A$ -normal operators.

**Definition 4.2:** The automorphism group of  $A$ -unitary operators is the set  $\mathbb{G}_A = \{T \in B(H): T^{[*]} = T^{-1}\}$ .

**Theorem 4.3 [7].** Every unitary operator is  $A$ -unitary.

*Proof:* [7, pp. 21].

**Remark:**  $\mathbb{G}_A$  is a multiplicative group. If  $S \in \mathbb{G}_A$ , then  $ST \in \mathbb{G}_A$ . This follows from  $(ST)^{[*]} = T^{[*]}S^{[*]} = A^{-1}T^*S^*A = A^{-1}(AT^{-1}A^{-1}.AS^{-1}A^{-1})A = T^{-1}S^{-1} = (ST)^{-1}$ .

**Definition 4.4:** Two linear operators  $T \in B(H)$  and  $S \in B(K)$  are said to be  $A$  –unitarily equivalent (denoted  $T \cong S$ ), if there exists an  $A$  –unitary operator  $U \in \mathcal{G}(H, K)$  such that  $TU = US$ .

In a real Hilbert space of dimension  $n$ , an operator is called Lorentz if it is  $A$  –unitary where  $A = I_p \oplus -I_q$  where  $p, q \in \mathbb{N}$  and  $p + q = n$ . For instance if  $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , then  $T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  is Lorentz.

**Definition 4.5:** A conjugation is a conjugate-linear operator  $C: H \rightarrow H$  which is both involutory (i.e.,  $C^2 = I$ ) and isometric.

**Remark:** If we let  $A = A^* = A^{-1}$ , then  $A$  is a conjugation. Thus, this Jordan algebra  $\mathbb{J}_A$  will contain the invertible normal operators, operators defined by Hankel matrices, Toeplitz and the Volterra integration operator  $V(f)(t) = \int_0^t f(s)ds$  for a function  $f(s) \in L^2(0, 1)$  and  $t \in (0, 1)$ .

**Remark:** Every  $A$ -unitary operator  $T$  is invertible. We note that if  $T$  is  $A$ -unitary then  $T^*$  is also  $A$ -unitary. This follows from the fact that  $(T^{[*]})^* = (T^{-1})^* = (T^*)^{[*]} \Rightarrow (T^*)^{[*]} = (T^*)^{-1} \Rightarrow T^*$  is  $A$ -unitary.

**Theorem 4.6 [8]:** If  $T$  is a normal operator and  $S \in B(H)$  is unitarily equivalent to  $T$ , then  $S$  is normal.

*Proof:* [8].

**Theorem 4.7 [7]:** Every normal operator  $T$  is  $A$ -normal.

*Proof:* [7, pp. 30-31].

**Remark:** Not all  $A$ -normal operators are normal. For example, if  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} i & i \\ i & 0 \end{pmatrix}$  a quick

mathematical computation reveals that  $T^{[*]}T = TT^{[*]}$  and  $T^*T \neq TT^*$ . Therefore,  $T$  is  $A$ -normal but not normal.

We also see that  $A$ -self adjoint and  $A$ -unitary operators are special cases of  $A$ -normal operators.

**Corollary 4.8:** If  $T$  is an  $A$  – normal operator and  $S \in B(H)$  is  $A$ -unitarily equivalent to  $T$  then  $S$  is  $A$ -normal.

**Proof:** From Theorem 4.7 above, every unitary operator (w.l.o.g. letting  $A = I$ ) is  $A$  – unitary and using a similar argument, we see that every normal operator  $T$  is  $A$ -normal. It suffices to show that  $S$  is normal.

Now, suppose that  $SX = XT$ , that is  $S = XTX^*$  where  $X$  is  $A$  – unitary and  $T$  is  $A$ -normal.

Then  $S^*S = (XTX^*)^*(XTX^*) = XT^*X^*XTX^* = XT^*TX^* (Since X^*X = I) = XTT^*X^* (Since T is normal) = XT(XT)^* = SXX^*S^* (Since XT=TX and (XT)^* = X^*S^*) = SS^* (Since X^*X = I)$ . That is  $S$  is normal. Since every normal operator  $S$  is  $A$ -normal, it follows that  $S$  is  $A$ -normal as required.

Finally, we discuss some conditions that guarantee a product of  $A$ -self adjoint operators to be  $A$ -self adjoint:

**Theorem 4.9:** [7, Theorem 3.19 (ii)] If  $P$  and  $Q$  are  $A$ -self adjoint operators, then the product  $T = PQ$  is  $A$ -self adjoint if and only if  $[P, Q] = 0$ .

By the above Theorem, we note that  $\mathbb{J}_A$  is a linear space which is not closed under multiplication. However, it is closed with respect to the Jordan product given by the

equation  $\{P, Q\} = \frac{1}{2}\{PQ + QP\}$ .

**Corollary 4.10:** *An invertible operator  $T$  is a product of  $A$ -self adjoint operators  $P$  and  $Q$  if and only if  $T$  is  $A$ -self adjoint.*

*Proof:* Suppose  $T$  is invertible with  $T = PQ$  and  $P^* = APA^{-1}, Q^* = AQA^{-1}$ . Invertibility of  $T$  implies that  $I = TT^{-1} = (PQ)(PQ)^{-1} = PQQ^{-1}P^{-1}$  and  $0 \notin \sigma(T)$  implies that  $0 \notin \sigma(PQ)$ . Hence  $P$  and  $Q$  are invertible and so is  $QP$ . Clearly,  $T^* = (PQ)^* = Q^*P^* = (AQA^{-1})(APA^{-1}) = A(QP)A^{-1} = A(PQA^{-1})$  (Since  $[P, Q] = 0$ ). That is  $T^* = A(PQ)A^{-1} = ATA^{-1}$  which shows that  $T$  is  $A$ -self adjoint.

Conversely, suppose  $T$  is invertible and  $T$  is  $A$ -self adjoint. Since  $T$  is invertible, by the polar decomposition theorem,  $T$  has a unique polar decomposition  $T = UM$ , where  $U$  is unitary (and not necessarily self-adjoint) and  $M = (T^*T)^{1/2}$  is positive (hence self-adjoint) operator. We use  $A$ -self adjointness of  $T$  to show that  $U$ , must indeed, be self-adjoint.  $A$ -self adjoint of  $T$  implies that  $UM = A^{-1}(UM)^*A = A^{-1}MU^*A$ , for some invertible operator  $A$ .  $A$ -self adjoint of  $T = UM$  (invertible) implies that  $U$  is self adjoint. But every self adjoint operator is  $A$ -self adjoint. This completes the proof.

## Potential Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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