



Curvature of the Ellipsoid with Cartesian Coordinates

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To cite this article:

Sebahattin Bektas. Curvature of the Ellipsoid with Cartesian Coordinates. *Landscape Architecture and Regional Planning*.

Vol. 2, No. 2, 2017, pp. 61-66. doi: 10.11648/j.larp.20170202.13

Received: January 19, 2017; **Accepted:** February 4, 2017; **Published:** March 4, 2017

Abstract: This study aims to show how to obtain the curvature of the ellipsoid depending on azimuth angle. The curvature topic is quite popular at an interdisciplinary level. It can be to the friends of geometry, geodesy, satellite orbits in space, in studying all sorts of elliptical motions (e.g., planetary motions), curvature of surfaces and concerning eye-related radio-therapy treatment, for example the anterior surface of the cornea is often represented as ellipsoidal in form. On the calculation of the curvature, there is a famous Euler formula for rotating ellipsoid that everyone knows. Let θ be the angle, in the tangent plane, measured clockwise from the direction of minimum curvature κ_1 . Then the normal curvature $\kappa_n(\theta)$ in direction θ is given by $\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = \kappa_1 + (\kappa_2 - \kappa_1) \cos^2\theta$ I wonder how can a formula for a triaxial ellipsoid? So we started to work. And we finally found the formula for the triaxial ellipsoid.

Keywords: General Ellipsoid, Normal Section Curve, Principal Curvatures, Gaussian Curvature, Mean Curvature

1. Introduction

The curvature issue is very important in geodesy and also in ophthalmology. To make geodetic computations on the ellipsoid (rotational or triaxial) first we need to know the normal section curve that combines observation points. The normal section curve is also available from the intersection of the ellipsoid and a plane which contains normal of surface on the station point and passes destination point. Geodetic computational formulas are contain the curvature parameters. This current study aims to pave the way for our further study on triaxial ellipsoid work.

Considerable several numbers of relevant studies were found in the literature. Some of them, [11, 15, 2, 6, 8]. Concerning the study of Harris in 2006, I think he was made a mistake. Curvature calculation was produced based on the Cartesian coordinates. However, the curvature calculation should have been based on the surface parameters (u, v parameters) not on the Cartesian coordinates.

When we look at the literature, we see that the curvature calculation is usually given depending on the angle of parameters. However, in practice, the azimuth angle is used instead of the angle of parameters and azimuth angle can be easily calculated from the Cartesian coordinates. At this point the importance of our study appears. We give the curvature calculation depending on the angle of the ellipsoid azimuth

with a new formula. And I think it has not been previously in the related. On the other hand, we also notice the lack of numerical practical studies in the literature; and therefore, we have added a comprehensive numeric application to our work.

2. General Surfaces

A point on surface can be represented by means of its position vector relative to the origin of an orthogonal coordinate system:

$$X = (x, y, z)$$

Suppose each of the coordinates is a function of two parameters u, v

$$x = x(u, v)$$

$$y = y(u, v)$$

$$z = z(u, v)$$

2.1. Ellipsoid

An ellipsoid is a closed quadric surface that is analogue of an ellipse. Ellipsoid has three different axes ($a > b > c$) as shown in Fig. 1. Mathematical literature often uses “ellipsoid” in place of “Triaxial ellipsoid or general ellipsoid”. Scientific literature (particularly geodesy) often uses “ellipsoid” in place of “biaxial ellipsoid, rotational ellipsoid or ellipsoid revolution” ($a = b > c$). Older literature

uses 'spheroid' in place of rotational ellipsoid. The standard equation of an ellipsoid centered at the origin of a Cartesian coordinate system and aligned with the axes. General ellipsoid equation as below in [3-5].

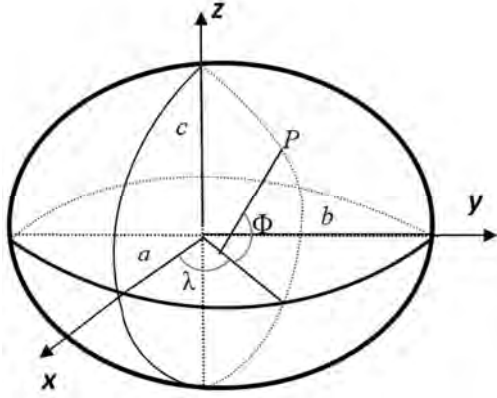


Figure 1. Triaxial Ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad (1)$$

Ellipsoid equation (u,v) Gauss Parametric form

$$\begin{aligned} x &= a \cos u \sin v \\ y &= b \sin u \sin v \end{aligned} \quad (2)$$

$$z = c \cos v$$

$$-\pi/2 \leq u \leq \pi/2, -\pi \leq v \leq \pi$$

Parametric coordinates calculated from Cartesian coordinates as below formula

$$v = \arccos \left(\frac{z}{c} \right) \quad (3)$$

$$u = \arctan \left(\frac{a.y}{b.x} \right) \quad (4)$$

$$H = \frac{a.b.c.[3(a^2 + b^2) + 2c^2 + (a^2 + b^2 - 2c^2)\cos 2v - 2(a^2 - b^2)\cos 2u.\sin^2 v]}{8[(a^2 b^2 \cos^2 v + c^2(b^2 \cos^2 u + a^2 \sin^2 u)\sin^2 v)^{3/2}} \quad (14)$$

The Gaussian curvature and Mean curvature can be calculated from Cartesian coordinates given below formulas [14, 17]

$$K = \frac{1}{\left(a.b.c.\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right) \right)^2} \quad (15)$$

$$H = \frac{|x^2 + y^2 + z^2 - a^2 - b^2 - c^2|}{2(a.b.c)^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{3/2}} \quad (16)$$

The parameter lines (u, v) and geodetic (planetographic) coordinates (Φ, λ) are orthogonal on rotational ellipsoid but are not orthogonal on triaxial ellipsoid.

In this parametrization, the coefficients of the first fundamental form are

$$E = [b^2 \cos^2 u + a^2 \sin^2 u] \sin^2 v \quad (5)$$

$$F = (b^2 - a^2) \cos u \sin u \cos v \sin v \quad (6)$$

$$G = [a^2 \cos^2 u + b^2 \sin^2 u] \cos^2 v + c^2 \sin^2 v \quad (7)$$

I. fundamental form

$$I = E.du^2 + 2.F.du.dv + G.dv^2 \quad (8)$$

and of the second fundamental form are

$$e = \frac{a.b.c.\sin^2 v}{\sqrt{(a.b.\cos v)^2 + c^2(b^2 \cos^2 u + a^2 \sin^2 u)\sin^2 v}} \quad (9)$$

$$f = 0 \quad (10)$$

$$g = \frac{a.b.c}{\sqrt{(a.b.\cos v)^2 + c^2(b^2 \cos^2 u + a^2 \sin^2 u)\sin^2 v}} \quad (11)$$

II. fundamental form

$$II = e.du^2 + 2.f.du.dv + g.dv^2 \quad (12)$$

Also in this parametrization, the Gaussian curvature is

$$K = \left[\frac{a.b.c}{(a.b.\cos v)^2 + c^2(b^2 \cos^2 u + a^2 \sin^2 u)\sin^2 v} \right]^2 \quad (13)$$

and the Mean curvature is

We will compute H and K in terms of the first and the second fundamental form.

$$K = \frac{e.g - f^2}{E.G - F^2} = \frac{II}{I} \quad (17)$$

$$H = \frac{G.e - 2F.f + E.g}{2(E.G - F^2)} \quad (18)$$

2.2. Principal Curvatures, Gaussian Curvature, Mean Curvature

We will now study how the normal curvature at a point varies when a unit tangent vector varies. In general, we will

see that the normal curvature has a minimum value κ_1 and a maximum value κ_2 . This was shown by Euler in 1760. The quantity

$$K = \kappa_1 \cdot \kappa_2 \text{ called the Gaussian curvature} \quad (19)$$

and the quantity

$$H = (\kappa_1 + \kappa_2)/2 \text{ called the mean curvature,} \quad (20)$$

play a very important role in the theory of surfaces.

$$R_1 = \frac{1}{\kappa_1} = \frac{1}{H - \sqrt{H^2 - K}} \text{ Maximum radii of curvature} \quad (21)$$

$$R_2 = \frac{1}{\kappa_2} = \frac{1}{H + \sqrt{H^2 - K}} \text{ Minimum radii of curvature} \quad (22)$$

The formula for the radius of curvature at arbitrary azimuth points up that the fact that the fundamental mathematical quantity is the inverse of these radii, which are simply called curvatures

2.3. Normal Section of a Surface

Let us construct a normal to a surface at a point P_o . Then the curve that is described on the surface by any plane passing through the normal (i.e. containing the normal) is called a normal section of the surface (Fig. 1). In other words a normal section is a plane section formed by a plane containing a normal to the surface [15, 12, 1].

2.4. Curvature of a Surface at a Point

Let us construct a unit normal \bar{n} and a tangent plane at given point P_o on surface and consider the curves that are formed on the surface by planes passing through P_o containing the normal i.e. the various normal sections passing through point P_o . Each normal section passing through P_o possesses a particular curvature at point P_o . We can specify a particular normal section by use of a polar coordinate system constructed on the tangent plane, origin at point P_o , polar axis as some arbitrarily chosen initial ray in the tangent plane, and an angle α measured clockwise from the polar axis to the plane of the normal section (Fig. 2). The curvature at point P_o in direction α is thus given as the function $\kappa_n(\alpha)$. For each value of α there is a curvature

associated with that particular normal section. This curvature $\kappa_n(\alpha)$ is called the normal curvature of the surface at point P_o in the direction α .

Then the normal curvature at point P_o is given by

$$\kappa_n(du, dv) = \frac{e \cdot du^2 + 2 \cdot f \cdot du \cdot dv + g \cdot dv^2}{E \cdot du^2 + 2 \cdot F \cdot du \cdot dv + G \cdot dv^2} \quad (23)$$

where E, F, G, e, f, g are the fundamental coefficients of the first and second order.

Formula (39) above can be re-written in the following way

$$\kappa_n\left(\frac{dv}{du}\right) = \frac{e + 2 \cdot f \cdot \left(\frac{dv}{du}\right) + g \cdot \left(\frac{dv}{du}\right)^2}{E + 2 \cdot F \cdot \left(\frac{dv}{du}\right) + G \cdot \left(\frac{dv}{du}\right)^2} \quad (24)$$

simply by dividing the numerator and denominator by du^2 . In this form it is obvious that κ_n is a function of the ratio dv/du . If we let $\cot \alpha = dv/du$ then (24) becomes

$$\kappa_n(\alpha) = \frac{e + 2 \cdot f \cdot \cot \alpha + g \cdot \cot^2 \alpha}{E + 2 \cdot F \cdot \cot \alpha + G \cdot \cot^2 \alpha} \quad (25)$$

where

$$\cot \alpha = \frac{E + F \tan \theta}{W \tan \theta} \quad (26)$$

A surface may be curved in many ways and consequently one might think that the dependence of the curvature κ on the angle α might be arbitrary. In fact this is not so. The following theorem is due to Euler.

2.5. Euler's Theorem

Let θ be the angle, in the tangent plane, measured clockwise from the direction of minimum curvature κ_1 . Then the normal curvature $\kappa_n(\theta)$ in direction θ is given by

$$\kappa_n(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta = \kappa_1 + (\kappa_2 - \kappa_1) \cos^2 \theta \quad (27)$$

$\kappa_n(\theta)$ curvature at azimuth θ

For spheroid (rotational ellipsoid):

$$N = R_1 = \frac{1}{\kappa_1} \text{ Radius of Curvature in Prime Vertical (Max. radii of curvature)} \quad (28)$$

$$M = R_2 = \frac{1}{\kappa_2} \text{ Radius of Curvature in Meridian (Min. radii of curvature)} \quad (29)$$

N, M can be easily calculated the latitude of point P_o as below

$$N = c_p / V, M = c_p / V^3, c_p = a^2 / c, V^2 = 1 + e_x^2 \cos^2 \Phi_o$$

$$\kappa_n(\theta) = \frac{N \cdot \cos^2 \theta + M \cdot \sin^2 \theta}{N \cdot M} \quad (30)$$

same as Eq. 26

$$\cot \alpha = \sqrt{\frac{E}{G}} \cot \theta \quad (31)$$

2.6. Computing the Principal Directions and Curvatures at a Point P_0

Given a point P_0 on a surface S , the directions at which the normal curvature at P_0 attains its minimum and maximum values can be computed as follows. Let the normal curvature at P_0 be given as

$$\kappa_n(\lambda) = \frac{e + 2.f.\lambda + g.\lambda^2}{E + 2.F.\lambda + G.\lambda^2} \quad (32)$$

where $\lambda = dv/du$. We wish to find those values of λ at which the function $\kappa_n(\lambda)$ has its minimum and maximum values. We are thus faced with a problem of finding the maxima and minima of a function. A necessary condition for the function $\kappa_n(\lambda)$ to have a maxima or minima at a point is that at that point $d\kappa_n(\lambda)/d\lambda = 0$. Using the usual formula for computing the derivative of a quotient we obtain

The directions corresponding to the minimum and maximum values of curvature are called the principal directions of the surface. The values κ_1 and κ_2 are called the principal curvatures of the surface [12, 1].

$$(E + 2F\lambda + G\lambda^2)(f + g\lambda) - (e + 2f\lambda + g\lambda^2)(F + G\lambda) = 0 \quad (33)$$

Upon expansion $f=0$ and rearrangement (33) becomes

$$(Fg)\lambda^2 + (Eg - Ge)\lambda - Fe = 0 \quad (34)$$

One can then solve (34) obtaining the two principal directions λ_1 and λ_2 . One can then substitute the two values λ_1 and λ_2 into (32) to obtain the principal curvatures κ_1 and κ_2 . The principal directions

$$r_{max} = \arctan(\lambda_1) \quad (35)$$

$$r_{min} = \arctan(\lambda_2) \quad (36)$$

2.7. The Curvature of the Ellipsoid with Cartesian Coordinates

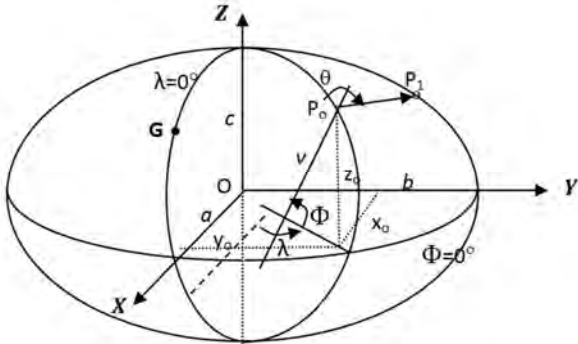


Figure 2. (X,Y,Z) Cartesian and (Φ, λ, h) Geodetic coordinates on Triaxial ellipsoid.

First, we need to find the azimuth angle between the two points known as the Cartesian coordinates. Let's assume that

θ is between azimuth angle of P_0 and P_1 points. For this we need the P_0 geodetic coordinates (Φ, λ) . We may obtain the P_0 geodetic coordinates (Φ, λ) from its Cartesian coordinates (x_0, y_0, z_0) . Formulas related the geodetic and Cartesian coordinates conversion on a triaxial ellipsoid were expressed on in [9, 15, 3]. For detailed information on this subject please refer to [3]. The following link can be used for the conversion of Cartesian coordinates to geodetic coordinates on triaxial ellipsoid [18].

$$\Delta x = x_1 - x_0 \quad \Delta y = y_1 - y_0 \quad \Delta z = z_1 - z_0$$

The azimuth angle of (P_0-P_1) from Cartesian coordinates [16].

$$\theta = (P_0-P_1) = \arctan$$

$$\left(\frac{-\Delta x \sin \lambda + \Delta y \cos \lambda}{-\Delta x \sin \Phi \cos \lambda - \Delta y \sin \lambda \sin \Phi + \Delta z \cos \Phi} \right) \quad (37)$$

P_0 geodetic coordinates (Φ, λ) calculated from its (x_0, y_0, z_0) Cartesian coordinates [18].

In order to calculation for the curvature, we need to add the reduction of the direction of r_{min} to the angle θ

Let's assume that r is the reduction of the direction of minimum curvature r_{min}

$$r = \arctan \left(\frac{W \tan(r_{min})}{E + F \tan(r_{min})} \right) \quad (38)$$

For spheroid r_{min} and r becomes zero

And we give a new formula for the curvature calculation depending on the θ angle of the azimuth on the triaxial ellipsoid

$$\kappa_n(\theta) = \kappa_1 + (\kappa_2 - \kappa_1) \cos^2(\theta - r) \quad \text{New Formula} \quad (39)$$

Curvature calculation can also be made as follows: First a plane's equation is determined which contains P_0 surface of normal and passes P_1 point. And then we find elliptical equation the intersection of the plane and the ellipsoid [10, 13, 7, 19]. The curvature can be calculated at point P_0 on the elliptical equation.

3. Numerical Example

Find the curvature of normal section curve at P_0 point which contains P_0 surface of normal and passes P_1 point on a triaxial ellipsoid

θ Angle is a azimuth angle P_0-P_1 direction and Cartesian coordinates of P_0 and P_1 point are given below

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} - 1 = 0 \quad (\text{Ellipsoid equation})$$

(a= 5, b= 4, c= 3) semi-axis

Cartesian coordinates (x,y,z)

P_0 (3.000 2.500 1.4981) P_1 (2.6189 2.4125 1.8047)

Geodetic coordinates [18]

$\Phi_0 = 40.194814370^\circ$ $\lambda_0 = 52.47573738^\circ$ $h_0 = 0$

u, v surface parameters on P_0 point Eq.(4)

$$u = 46.1691393 \quad v = 60.0413669$$

E, F, G, e, f, g are the Fundamental Coefficients of the First and Second Order evaluated at P point. Eqs. (5-12)

$$E = 15.52562500 \quad F = -1.94530956 \quad G = 11.82202800$$

$$e = 2.91030887 \quad f = 0 \quad g = 3.87718085$$

Gaussian Curvature, Mean Curvature

From fundamental coefficients of the first and second order.

$$K - \text{Gauss} = 0.06277139$$

$$H - \text{Mean Curve} = 0.26313232 \text{ (Eqs. 17-18)}$$

Main radii of curvatures

$$R_1 = \frac{1}{\kappa_1} = \frac{1}{H - \sqrt{H^2 - K}} = 5.4730575$$

maximum radii of curvature (Eq. 21)

$$R_2 = \frac{1}{\kappa_2} = \frac{1}{H + \sqrt{H^2 - K}} = 2.9107724 \quad \text{minimum radii of}$$

curvature Eq.(22)

Principal Curvatures

$$\kappa_1 = 0.182713 \quad \text{minimum curvature}$$

$$\kappa_2 = 0.343551 \quad \text{maximum curvature}$$

Principal directions Eq.(34)

$$-7.54231697 \lambda^2 + 25.7899 \lambda + 5.66145 = 0$$

$$\lambda_1 = 3.62635252$$

$$\lambda_2 = -0.20699173$$

$$r_{\max} = \arctan(\lambda_1) = 74.583317^\circ$$

$$r_{\min} = \arctan(\lambda_2) = -11.694599^\circ$$

Azimuth angle (P_0 - P_1) from Eq.(37)

$$\theta = (P_0 - P_1) = 30.136633^\circ$$

$$r = \arctan\left(\frac{W \tan(r_{\min})}{E + F \tan(r_{\min})}\right) \Rightarrow r = -9.88360428^\circ$$

$$\theta - r = 30.136633^\circ - (-9.88360428^\circ) = 40.02023728^\circ$$

and curvature

$$\kappa_n(\theta) = \kappa_1 + (\kappa_2 - \kappa_1) \cos^2(\theta - r) = 0.277041$$

Control

Equation of plane which contains P_0 surface of normal and passes P_1 point

$$0.3125x - 0.50134y + 0.245316z - 0.051655 = 0$$

Intersection ellipse's equation [19]

$$\frac{x^2}{\eta^2} + \frac{y^2}{\xi^2} = 1 \Rightarrow y = \xi \sqrt{1 - \frac{x^2}{\eta^2}}$$

$$\eta = 4.65998 \quad \xi = 3.11078$$

Transformed coordinates of P_0 point in intersection's plane

$$x_o = -3.7331 \quad y_o = -1.8619$$

$$y_o' = 0.89347 \quad y_o'' = -0.66809$$

and curvature

k

$$k = \frac{1}{R} = \frac{y_o''}{(1 + y_o'^2)^{3/2}} = 0.2770411$$

4. Conclusion

This study aims to show how to obtain the curvature of the ellipsoid depending on azimuth angle. We have developed an algorithm to obtain the curvature of the ellipsoid depending on azimuth angle. The efficiency of the new approaches is demonstrated through a numerical example. The presented algorithm can be applied easily for spheroid, sphere and also other quadratic surface, such as paraboloid and hyperboloid. Today, backward and forward problem between the two points on the triaxial ellipsoid with geodetic coordinates could not be a clear solution. Our future work will be on this unsolvable problem. I hope, the result of this study will contribute to the solution of the above problem.

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