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About Calculation the Resistance of Two-dimensional Infinite Grid Systems

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Abstract: The paper considers the problem of calculating the resistance between nodes of infinite grid resistor systems with square and triangular cells. There has long been a question about the resistance between the nearest nodes of an infinite grid of resistances with square cells with the same resistance *r*. Here, earlier, by the method of symmetry and superposition, a result was obtained *r*/2 that is striking in its simplicity. However, this result is only approximate, although many physicists consider this result to be accurate. New examples are presented proving what the results obtained earlier by the superposition and symmetry method is only approximate. The result *r*/2 gives only the lower limit of the correct resistance value. In our work, the correctness of using the equivalent resistance method to calculate the resistance between nearest nodes of infinite grid systems is proved. Using this method, for the resistance between the nearest nodes of an infinite grid of resistances with square cells, a result is obtained about 0.5216 *r* that only slightly differs from *r*/2. The results differ from the previously obtained values by about 10%. The resistance between the diagonal points of an infinite grid of identical resistors *r* with square cells is calculated. For the value of this resistance, a value founded about 0.7071 *r* that differs from the value 2*r*/*π* obtained previously by the superposition and symmetry method.

Keywords: Calculation of Resistance, Infinite Two-Dimensional Grid of Resistances, Equivalent Resistance Method

1. Introduction

The calculation of the resistance of complex resistor compounds has always attracted the attention of physicists. Many different original methods for calculating endless circuits of resistance have been developed [1–7]. The tasks of finding the resistance of infinite grid systems also been included in Irodov's book [8] and were considered in our works [9, 10]. The problems of calculating the resistances of infinite resistor grids, using graphene [11, 12] and thin films [12] in connection with the development of nanotechnologies have become especially urgent.

Let us dwell in more detail on one problem reviewed in the works [2–8]. «There is a boundless wire grid with square cells (Figure 1). The resistance of each conductor between neighboring grid nodes is *r*. Find the resistance R_{AB} of this grid between two adjacent grids modes *A* and *B*».

This task first appeared in Irodov's problem book of 1979 yeas, where the method of « symmetry & superposition » was applied and the result was obtained:

$$R_{AB}=\frac{r}{2}. \tag{1}$$

The same result (1) was also obtained in the works [2–4]. This opinion was especially strengthened in connection with work [4], as well as in works [14–16]. In this work, complex mathematics made it possible to obtain a general formula for the resistance between any points of the grid. However, here results is also based on the method the «symmetry & superposition», therefore, for the resistance between the nearest point of grid, this formula gives the same result $r/2$.

Many physicists consider this result to be accurate. Now this is a general misconception, which is very difficult to overcome. Of course, the result $r/2$ for resistances fascinates with its simplicity, but it is only approximate. Moreover, it gives only a lower bound for estimating the magnitude of the resistance.

In Figure 2 shows three cases of connecting voltage to the points of the grid. In case *a*) the voltage $+U/2$ is supplied only to point *A*, and in case *b*) the voltage $-U/2$ is supplied only to point *B*. In these cases, the current distribution is symmetrical and they differ only in the direction of the currents. In this case, points with a potential equal to zero are at infinity.

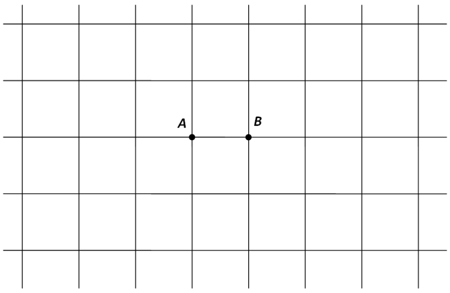

Figure 1. *Infinite wire grid with square cells.*

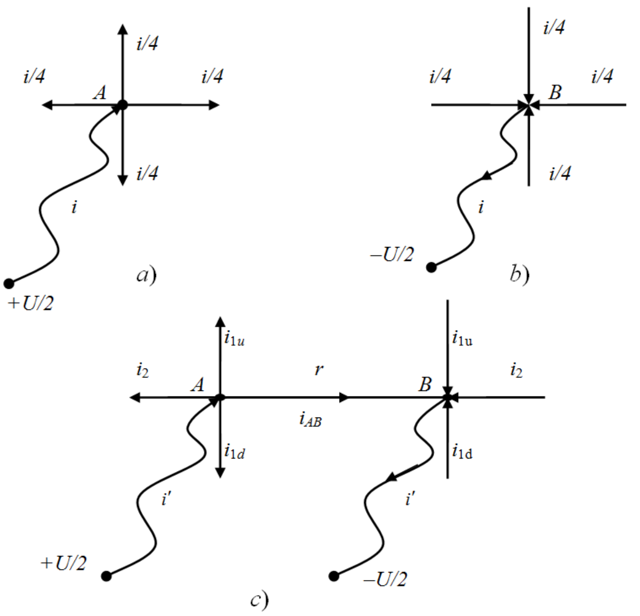

Figure 2. *a) The potential $+U/2$ only at point A (zero at infinity); b) The potential $-U/2$ only at point B (zero at infinity); c) The potential $+U/2$ at point A and the potential $-U/2$ at point B.*

In case c) voltage $+U/2$ is supplied to point *A*, and voltage $-U/2$ to point *B*. Here *A* and *B* are the nearest points of the grid, between which the resistance r is located. Current $i' \neq i$ approaches point *A*, and the same current i' emerges from point *B*. The current distribution at points *A* and *B* is shown, which corresponds to the symmetry of the problem. Now a line with a potential equal to zero passes in the middle between points *A* and *B*. A current i_{AB} flows from *A* to *B*. Then according to Ohm's law:

$$U = i' R_{AB} = i_{AB} r \, . \tag{2}$$

Here R_{AB} is the desired grid resistance between the points *A* and *B*. If we make two assumptions that $i' = i$ and $i_{AB} = i/2$, then from formula (2) we get result (1): $R_{AB} = \dfrac{r}{2}$. In this case, if $i_{AB} = i/2$, $i_{1u} = i_{1d} = i_1 = i/4$, then according to the Kirchhoff rule: $i_2 = 0$.

Symmetry in case c) has changed. So the distribution of currents at points *A* and *B* has changed. Note that a line with a potential equal to zero runs in the middle between the points *A* and *B* and goes to infinity. Since the potential difference between points *A* and *B* and infinity is not equal to zero, the currents $i_2 \neq 0$ and $i_{AB} > i'/2$. These three factors indicate that the result $R_{AB} = r/2$ is an approximation, more precisely even $R_{AB} > r/2$. In our work [6] we used the equivalent resistance method and obtained the following result:

$$R_{AB} = \frac{2\left(\sqrt{2}-1\right) + \sqrt{2\sqrt{2}-1}}{2\sqrt{2} + \sqrt{2\sqrt{2}-1}} r \cong 0.521602 \, r \, , \tag{3}$$

and the following values of the currents: $i_{AB} \cong 0.522 \, i'$, $i_{1u} = i_{1d} = i_1 \cong 0.207 \, i'$, $i_2 \cong 0.064 \, i'$. It is easy to verify that in this case:

$$i' = i_{AB} + i_{1u} + i_{1d} + i_2 \, .$$

So, here everything is in order with the Kirchhoff rule in the points *A* and *B*.

But the result, which we obtained $R_{AB} \cong 0.522 \, r$, is just only slightly superior to the result $r/2$, which confirms its correctness. The arguments we have presented, we think, prove that the result $r/2$ is only approximate.

2. Calculation of the Resistance Between Nearest Points of the Infinite Grid with Square Cells

We give below an alternative solution to this problem. First, we divide the entire plane into two identical half-planes, cutting the grid along a straight line passing through points *A* and *B*, to the left of point *A* and to the right of *B*. For this, each of the resistances *r*, lying to the left of point *A* and to the right of *B*, is replaced by two parallel-connected resistance of $2r$ each. We obtain the following picture, shown in Figure 3. In Figure 4 shows an equivalent scheme of a cut mesh, where the resistance of the half-planes obtained as a result of cutting is denoted by *R*:

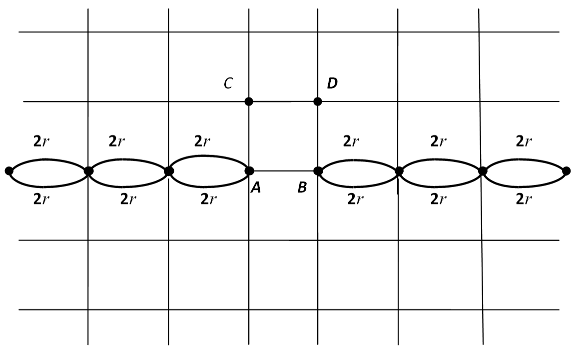

Figure 3. *Infinite wire grid, divided into two half-planes.*

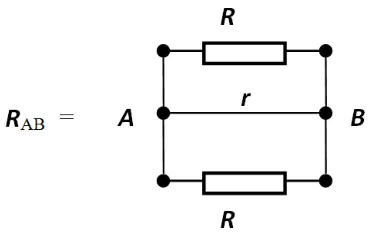

Figure 4. Equivalent scheme of unlimited wire mesh.

Then the resistance between adjacent nodes A and B of an infinite two-dimensional grid with square cells is equal to:

$$R_{AB} = \frac{rR}{2r+R}. \tag{4}$$

To do this, we similarly cut the grid along a straight line passing through points C and D. The resulting picture is shown in Figure 5, where you can see two infinite half-planes with resistance R, located below points A, B and above points C, D, and also two identical infinite chains going to the left of points A and C and to the right of points B and D. We denote the resistance of such an infinite chain by r^*. Then, comparing the schemes in Figure 4, we can draw up the equivalent circuit depicted in Figure 5.

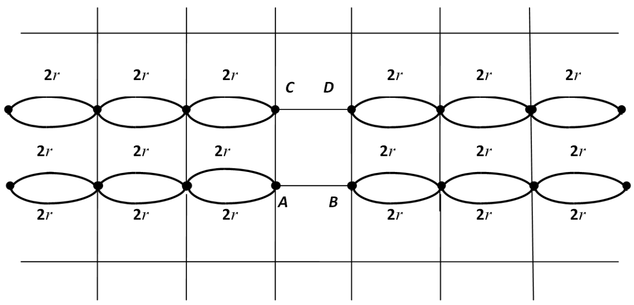

Figure 5. Infinite wire mesh, cut along two straight lines.

Using the equivalent circuit in Figure 6, we write the equation for determining the resistance of the half-plane R:

$$R = 2r^* + \frac{rR}{r+R}. \tag{5}$$

Solving this quadratic equation with respect to R, we find

$$R = r^* + \sqrt{r^{*2} + 2r^*r}. \tag{6}$$

The resistance of an infinite periodic chain r^* is found by the standard method, solving equation

$$r^* = \frac{(4r + r^*)r}{5r + r^*}. \tag{7}$$

Location

$$r^* = 2r\left(\sqrt{2} - 1\right). \tag{8}$$

Substituting (8) into (6), we obtain

$$R = \left(-2 + 2\sqrt{2} + \sqrt{2\sqrt{2} - 1}\right) r. \tag{9}$$

Substitution of this value of R into (4) leads to the final result:

$$R_{AB} = \frac{-2 + 2\sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2\sqrt{2} + \sqrt{2\sqrt{2} - 1}} r \cong 0.52160212\, r. \tag{10}$$

This result, although not very strong, is still different from the result of 0.5 r obtained in the approximation of the principles of symmetry and superposition. In this connection it is interesting to look at the distribution of currents at point A. If current i' approaches the point A, then it will be distributed as follows: according to the resistance going up and down from point A, the same currents will go $i_{1u} = i_{1d} = i_1 \cong 0{,}207\, i'$, according to the resistance between nodes A and B, there will be a current $i_{AB} \cong 0{,}522\, i'$, and in the opposite direction there will be a current $i_2 \cong 0{,}064\, i'$.

It's a pity, the beauty is gone, the symmetry has disappeared, and everything has become very prosaic. Well, in life it often happens that beauty deceives us and then it is difficult to get rid of beautiful illusions.

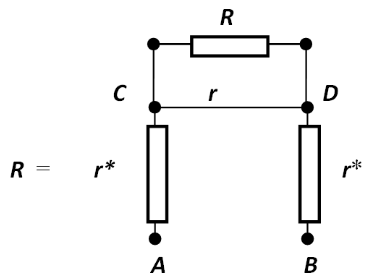

Figure 6. An equivalent circuit for the resistance of the half-plane R in the case of a square grid.

3. Calculation of the Resistance Between the Diagonal Points of the Infinite Grid with Square Cells

Here we give a solution to the problem of an infinite grid with square cells by the same resistances r, as shown in Figure 7. Suppose it is necessary to find the resistance R_{AC} of the lattice between the two points A and C. The solution of the problem by the method of superposition & symmetry in [4] leads to the following result:

$$R_{AC} = \frac{2r}{\pi} \cong 0.636620\, r. \tag{11}$$

Figure 3 shows a part of the grid, and the dashed lines show the directions along which it is necessary to make cuts. First, we cut along the rays issuing from the nodes A and C, thus breaking the entire grid into two half-planes with the same resistances R. As a result, we obtain for the resistance formula:

$$R_{AC} = \frac{R}{2}. \quad (12)$$

Now you need to find the half-plane resistance *R*. To do this, we draw a second section from the points A_1 and C_1, as shown in Figure 7. To determine *R,* we construct the following equivalent circuit shown in Figure 8.

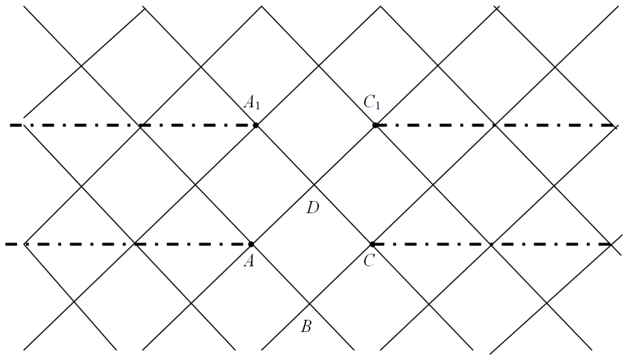

Figure 7. *Infinite grid with square cells and cutting line directions.*

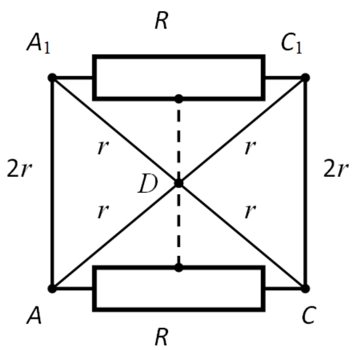

Figure 8. *Equivalent circuit for calculation of the half-plane resistance* R.

In Figure 4 shows a dashed vertical line passing through the point D and dividing the circuit into two symmetrical parts. Connecting the point D with the middle of the resistance R, we calculate the resistance of the resulting compound. First, find the resistance of the parallel connection $\frac{R}{2}$ and r get:

$$r' = \frac{\frac{R}{2} r}{\frac{R}{2} + r} = \frac{Rr}{R+2r}. \quad (13)$$

Then, connecting in series r' with $2r$, we find:

$$r'' = \frac{Rr}{R+2r} + 2r = \frac{3Rr+4r^2}{R+2r}. \quad (14)$$

Now we make up the equation to determine the half-plane resistance *R*:

$$\frac{R}{2} = \frac{r'' r}{r'' + r}. \quad (15)$$

Substituting (14) into (15), we obtain:

$$\frac{R}{2} = \frac{3Rr^2 + 4r^3}{4Rr + 6r^2}. \quad (16)$$

Here will we find

$$R = \sqrt{2}\, r. \quad (17)$$

Substituting (17) into (12), we obtain the desired resistance between the diagonal points:

$$R_{AC} = \frac{R}{2} = \frac{r}{\sqrt{2}} \cong 0.707107\, r. \quad (18)$$

It is easy to verify that the difference with the result (11) obtained by the method of «superposition & symmetry» is 10%.

4. Calculation of the Resistance Between Points of the Infinite Grid with Triangles Cells

Here we also give a solution to the problem of an infinite net by the same resistances *r* forming regular triangles, as shown in Figure 9. Suppose it is necessary to find the resistance R_{AB} of the lattice between the two nearest nodes *A* and *B*. Using for the solution of the problem the principles of symmetry and superposition, by analogy with a square grid, it is easy to obtain the following simple result of $R_{AB} = r/3$, which is also only approximate.

In Figure 6 shows the part of the grid and the bold lines show the directions along which it is necessary to make cuts in order to break it into the same half-planes and endless chains. First, we cut along the rays issuing from the nodes *A* and *B*, thus breaking the entire grid into two half-planes with the same resistances *R*. As a result, just as in the case of a rectangular grid, we obtain the equivalent circuit shown in Figure 9, and formula (4) for the resistance of an infinite grid.

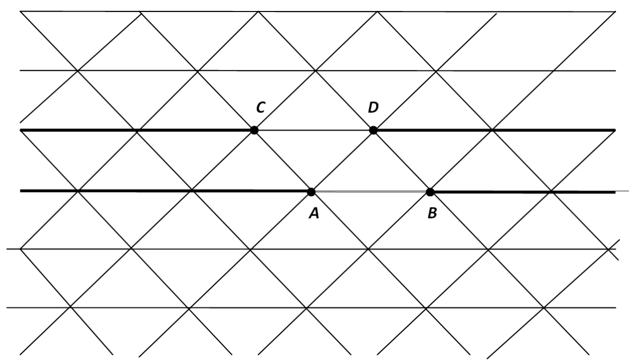

Figure 9. *Infinite grid of regular triangles.*

On Figure 9 each side of the triangle has a resistance *r*. The thick lines indicate the directions of the sections. To determine the resistance of the half-plane R, we cut once more the lattice along the rays emanating from the nodes C and D. We obtain the equivalent circuit shown in Figure 10 and, respectively, equation (19):

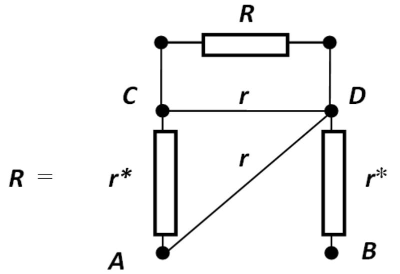

Figure 10. An equivalent circuit for the resistance of the half-plane R in the case of a triangular grid.

$$\frac{\left(\frac{R\,r}{R+r}+r*\right)r}{\frac{R\,r}{R+r}+r*+r}+r^*=R\,. \quad (19)$$

After simple transformations, we obtain the following quadratic equation for the determination of R:

$$R^2-r*R-r\,r^*=0\,. \quad (20)$$

Location

$$R=\frac{r*+\sqrt{r*^2+4r\,r*}}{2}\,. \quad (21)$$

Here, the resistance of the infinite chain r^* is found from the equivalent circuit shown in Figure 11, which leads to the equation (21).

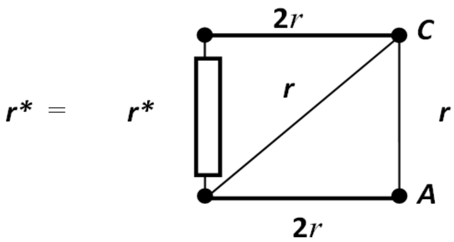

Figure 11. Equivalent circuit for calculation of resistance r.*

Infinite chain in the case of a triangular grid:

$$\frac{\frac{(r*+2r)\,r}{r*+3r}+2r}{\frac{r*+2r}{r*+3r}+3}=r*\,. \quad (22)$$

Solving this equation, we find

$$r^*=\left(\sqrt{3}-1\right)r\,. \quad (23)$$

Substituting this value of r^* into formula (10), we obtain for R the expression:

$$R=\frac{\sqrt{3}+\sqrt{2\sqrt{3}}-1}{2}r\,. \quad (24)$$

Substitution of this expression in (4) leads to the final result:

$$R_{AB}=\frac{\sqrt{3}+\sqrt{2\sqrt{3}}-1}{3+\sqrt{3}+\sqrt{2\sqrt{3}}}r\cong 0.39331989\,r\,. \quad (25)$$

Thus, in this paper we develop a method for calculating infinite networks of resistances, which makes it possible to find the exact resistance between the nearest nodes of such networks. It is shown that the method «symmetry & superposition» used in [2-5], based on the principles of superposition and symmetry, gives only an approximate underestimated result for this resistance. And for a grid with square cells, the difference of results does not exceed 5%, and for a network with triangular elements it approaches 20%.

5. Conclusion

Thus, in work [6] and this article we have developed a method for calculating infinite resistance grids, which allows us to find the exact resistance between the nodes of such grids. It is shown that the calculation method used in [2–5], based on the principles of symmetry & superposition gives only an approximate underestimated result for this resistance. As we have shown in our works for the grid with square cells, the difference in results does not exceed 10%, and for the grid with triangular elements it approaches 15%.

The method of calculating the resistances of the infinite grids, which uses the principles of symmetry & superposition, is quite good, and its simplicity makes it very attractive for an approximate evaluation of the resistances of various infinite configurations of resistances. So, for example, for resistance R_{AB} between two nearest points of an infinite grid with hexagonal cells (a configuration of graphene) we get value $R_{AB}=2r/3$, and for resistance R_{AB} between two nearest points of an infinite 3D grid with cubic cells get value $R_{AB}=r/3$.

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