

# The Fractional Half Dirac Operator Over Sobolev Spaces

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**Abstract:** The classical Dirac operator of a regular order reduces smoothness of functions by the amount of the order and that property is valid when the order is fractional. What is interesting is that fractional order derivatives offer more practical framework on studying physical phenomena such as motion in viscous or resistive media. This article introduces a fractional half order Dirac operator and develops theoretical results in Sobolev spaces of an inner product which permit orthogonality. Using domain and boundary integrals, the author establishes that solutions to first and second order boundary value problems of the half order Dirac operator decompose in a unique way as an orthogonal sum of functions, one that evolves from the interior domain data and the other from the boundary trace value data. These results extend successfully classical decomposition results of spaces by a regular order Dirac operator to results of a fractional half order. At the end we demonstrate the practical use of Dirac operators of fractional orders and orthogonal decompositions through motions in electromagnetic fields and viscous media where a solution vector of a boundary value problem of fractional half order Dirac equation splits as an orthogonal sum of solution parts called irrotational and solenoidal.

**Keywords:** Fractional Half Dirac Operator, Orthogonal Sum, Boundary Value Problems (BVPs), Hilbert Spaces, Sobolev Spaces, Caputo Fractional Derivative, Riemann-Liouville Fractional Derivative

## 1. Introduction

In this paper we use the following notations.  $(\oplus)$  : for orthogonal sum of subspaces of a generalized Hilbert/Sobolev space with an inner product and  $(\uplus)$  : for orthogonal sum of elements from the orthogonal subspaces,  $(\widehat{\phantom{x}})$  : for Fourier transform and  $(\overline{\phantom{x}})$  : conjugation.

We introduce the fractional Dirac operator of a half order:

*Definition 1.1.*

$$D^{\frac{1}{2}} := \sum_{j=1}^n e_j \frac{\partial^{\frac{1}{2}}}{\partial x_j^{\frac{1}{2}}} := \sum_{j=1}^n e_j \partial x_j^{\frac{1}{2}} \quad (1)$$

in a Clifford/hypercomplex algebra with basis vectors,  $e_1, e_2, e_3, \dots, e_n$  and multiplicative properties:

$$e_i^2 = -1 \text{ and } e_i e_j + e_j e_i = -\delta_{ij}$$

where  $\delta$  is the Kronecker delta function.

We develop some functional theoretical results based on the fractional Dirac operator defined above (1) and see the application of the theory in an applied fields [1, 7, 8, 16].

Note that

$$\left(D^{\frac{1}{2}}\right)^2 = \sum_{j=1}^n e_j^2 \frac{\partial}{\partial x_j} + \sum_{j < k} e_j^2 \{e_j, e_k\} = - \sum_{k=1}^n \frac{\partial}{\partial x_k} \neq D$$

where  $D$  is the regular vector Dirac operator given below, but  $\left(D^{\frac{1}{2}}\right)^2$  is a scalar partial differential operator.

*Definition 1.2.* The Dirac operator is defined by

$$D = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j} \quad (2)$$

a partial differential operator of first order in a hyper complex or Clifford analysis setting

There is a good amount of work done regarding the Dirac operator, its variants and Clifford analysis in general. See [2, 3, 5, 6, 9, 10, 15, 17, 18] Our work focuses on the fractional

half order Dirac derivative  $D$ . To have some observation on fractional order derivatives, the regular ordinary derivative  $\frac{d}{dx}f(x)$  of a differentiable function  $f$  makes the function to loose smoothness or regularity by 1, while the half order derivative makes the function to loose  $\frac{1}{2}$ -regularity exponent. Lose of regularity creates discontinuity on functions that are rough on their graphs. The lose of half order rate of change different from the regular lose of 1 in differentiation, has physical meanings in applied sciences. For instance, if a particle moves in a media of resistance, then the half order rate of change of the position displays the reduced speed in the media due to the resistance the object encounters, such as viscosity, wave fronts or others, see the versatile places where fractional rates of change appear and used.

Let us consider the half order derivative  $D^{\frac{1}{2}} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$  of  $x^\lambda$ . Although there are few more fractional derivatives, we consider what are called the Caputo and Riemann-Liouville fractional derivatives.

The Caputo fractional half derivative of a function  $f$  defined on an interval  $(0, 1) \subseteq \mathbb{R}^1$  at  $x \in (0, 1)$

$${}_0^C D_x^{\frac{1}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{f'(t)}{(x-t)^{\frac{1}{2}}} dt \quad (3)$$

Clearly when  $f(x) = c$ — non zero constant, the *Caputo*

Note here that

$${}^{RL} D_x^{\frac{1}{2}} ({}^{RL} D_x^{\frac{1}{2}}(c)) = {}^{RL} D_x^{\frac{1}{2}}(c \frac{1}{\sqrt{\pi x}}) = \frac{c}{\pi} \frac{d}{dx} \int_0^x \frac{1}{\sqrt{tx-t^2}} dt, \because \int_0^x \frac{1}{\sqrt{tx-t^2}} dt = \pi = 0$$

which signifies:

$${}^{RL} D_x^{\frac{1}{2}} ({}^{RL} D_x^{\frac{1}{2}}(f(x))) = {}^{RL} D_x f(x)$$

which is the regular first order derivative of  $f(x)$ , i.e.,  ${}^{RL} D_x f(x) = f'(x)$  to be shown next.

*Corollary 1.1.* For  $\alpha = 1$ , the Reimann-Liouville first order derivative of a function  $f$  is the regular first order derivative. That is  ${}^{RL} D^1 f(x) = \frac{d}{dx} f(x)$ .

*Proof.* For  $\alpha \in (0, 1)$  :

$${}^{RL} D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt$$

Then

$${}^{RL} D_x^1 f(x) := \lim_{\alpha \rightarrow 1^-} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt = \frac{d}{dx} \lim_{\alpha \rightarrow 1^-} \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt$$

But

$$\int_0^x \frac{f(t)}{(x-t)^\alpha} dt = \frac{f(0)x^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha} \int_0^x \frac{f'(t)}{(x-t)^{\alpha-1}} dt$$

fractional half derivative  $D_x^{\frac{1}{2}} f(x) = 0$ .

When  $f(x) = x^\lambda$ , we have:

$${}_0^C D_x^{\frac{1}{2}}(x^\lambda) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{\lambda t^{\lambda-1}}{(x-t)^{\frac{1}{2}}} dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})} x^{\lambda-\frac{1}{2}}$$

When  $\alpha = \frac{1}{2}$  and  $\lambda = \frac{1}{2}$  we have

$${}_0^C D_x^{\frac{1}{2}}(x^{\frac{1}{2}}) = \frac{\sqrt{\pi}}{2}$$

One can observe that the *Caputo* fractional derivative and the regular derivative of a non-zero constant coincides.

*Definition 1.3.* The  $\alpha^{th}$ -order *Riemann-Liouville* fractional derivative: for  $0 \leq \alpha < 1$ ,

$${}^{RL} D_x^\alpha f(x) = \frac{d}{dx} \int_0^x \frac{f(\tau)}{(x-\tau)^\alpha} d\tau \quad (4)$$

Note that the constant function  $f(x) = c$  behaves differently in this case.

*Riemann-Liouville*  $\frac{1}{2}^{th}$ -derivative:

$${}^{RL} D_x^{\frac{1}{2}}(c) = \frac{d}{dx} \int_0^x \frac{c}{(x-\tau)^\alpha} d\tau = c \frac{x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = c \frac{1}{\sqrt{\pi x}}$$

Therefore,

$$\begin{aligned} & \lim_{\alpha \rightarrow 1^-} \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(0)x^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha} \int_0^x \frac{f'(t)}{(x-t)^{\alpha-1}} dt \right) \\ &= \lim_{\alpha \rightarrow 1^-} \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \left( f(0)x^{1-\alpha} + \int_0^x \frac{f'(t)}{(x-t)^{\alpha-1}} dt \right) \\ &= \lim_{\alpha \rightarrow 1^-} \frac{1}{\Gamma(2-\alpha)} \left( f(0)x^{1-\alpha} + \int_0^x \frac{f'(t)}{(x-t)^{\alpha-1}} dt \right) \\ &= f(0) + \int_0^x f'(t) dt = f(x), \quad \because \Gamma(1) = 1 \end{aligned}$$

Hence,

$${}^{RL}D_x^1 f(x) = \frac{d}{dx} f(x)$$

Clearly, both derivatives, the Caputo fractional derivative and the Riemann-Liouville derivative of a constant offers two contrasting results. One, constants have invariant rate of change, which is zero and the other one says, it behaves like a non constant if the object is in resistive media such as viscosity.

In this work, we study orthogonally decomposable solutions to boundary value problems of a fractional half or twice of a half order in an inner product Sobolev space. We establish that solutions to BVPs to be orthogonal sums ( $\oplus$ ) of functions that evolve from the trace values on the boundary and the values of the differential equation in the domain. In [5, 9–11] we successfully developed regular decomposition results of Hilbert and Sobolev spaces with an inner product. We established properties of the inner product that are considered and that of functions in the respective spaces.

**Definition 1.4.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a boundary  $\partial\Omega$ . We define the fractional Sobolev space of regularity  $\frac{1}{2}$ , denoted by  $W^{\frac{1}{2},2}(\Omega)$  as

$$W^{\frac{1}{2},2}(\Omega) := \{f \in \mathcal{L}^2(\Omega) : D^{\frac{1}{2}}f \in \mathcal{L}^2(\Omega)\} \quad (5)$$

*Proof.* From integration by parts:

$$\underbrace{\int_{\Omega} u(y) D_y^{\frac{1}{2}} \phi_{\frac{1}{2}}(y-x) dy}_{\|u(x)\|} = \underbrace{\int_{\partial\Omega} u(y) \phi_{\frac{1}{2}}(y-x) \nu_{\frac{1}{2}}(y) d\partial\Omega_y}_{\|\zeta_{\partial\Omega}^{\frac{1}{2}} \tau(u)\|} - \underbrace{\int_{\Omega} D^{\frac{1}{2}} u(y) \phi_{\frac{1}{2}}(y-x) d\Omega_y}_{\|\zeta_{\Omega}^{\frac{1}{2}}(f)\|}$$

That is

$$u(x) = \zeta_{\partial\Omega}^{\frac{1}{2}}(g) + \zeta_{\Omega}^{\frac{1}{2}}(f) \quad (8)$$

From the orthogonal decomposition

$$W^{\frac{1}{2},2}(\Omega) = \text{Ker} D^{\frac{1}{2}} \cap W^{\frac{1}{2},2}(\Omega) \oplus D^{\frac{1}{2}} \left( W_0^{1,2}(\Omega) \right) \quad (9)$$

taking the Dirac half derivative  $D^{\frac{1}{2}}$  of both sides of the above integral equation, we have  $D^{\frac{1}{2}} \zeta_{\partial\Omega}^{\frac{1}{2}}(g) = 0$

where  $D^{\frac{1}{2}}f$  is the fractional  $\frac{1}{2}$ -Caputo derivative of  $f$ .

**Proposition 1.1.** For  $f, g \in \mathcal{L}^2(\Omega)$ , the BVP:

$$\begin{cases} D^{\frac{1}{2}}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (6)$$

has a solution  $u$  in the Sobolev space  $W^{\frac{1}{2},2}(\Omega)$  of an inner product  $\langle \cdot, \cdot \rangle_{W^{\frac{1}{2},2}(\Omega)}$  such that  $u$  is an orthogonal sum of its parts, i.e.,

$$u = [u]_f \oplus [u]_g \quad (7)$$

with  $[u]_f$  be the part of the solution that evolves from  $f$  and  $[u]_g$  the part that evolves from  $g$  with properties:

$$(i) \quad \langle [u]_f, [u]_g \rangle_{W^{\frac{1}{2},2}(\Omega)} = 0$$

$$(ii) \quad \|u\|_{W^{\frac{1}{2},2}(\Omega)}^2 = \|[u]_f\|_{W^{\frac{1}{2},2}(\Omega)}^2 + \|[u]_g\|_{W^{\frac{1}{2},2}(\Omega)}^2$$

*Example 1.1.* Consider  $f(x) = \begin{cases} x-1, & 1 \leq x \leq 2 \\ 0, & 0 \leq x \leq 1 \end{cases}$

Then

$$g(x) = \begin{cases} \frac{2\sqrt{x-1}}{\sqrt{\pi}}, & 1 < x < 2 \\ 0, & 0 < x < 1 \end{cases}$$

is the  $\frac{1}{2}$ -derivative of  $f$  in the Riemann-Liouville sense. Indeed,

$$\begin{aligned} {}^{RL}D^{\frac{1}{2}}f(x) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\frac{1}{2}}} dt \\ &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_1^x \frac{t-1}{(x-t)^{\frac{1}{2}}} dt \\ &= \frac{1}{\sqrt{\pi}} 2\sqrt{x-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} {}^{RL}D^{\frac{1}{2}}f(x) &= \begin{cases} \frac{2\sqrt{x-1}}{\sqrt{\pi}}, & 1 < x < 2 \\ 0, & 0 < x < 1 \end{cases} \\ &= g(x) \end{aligned}$$

Also  $g$  is the weekly  $\frac{1}{2}$ -derivative of  $f$  as well. That is  $\forall \phi \in C_0^\infty([0, 2])$ ,

$$\int_0^2 g(x)\phi(x) dx = \int_0^2 f(x) {}^{RL}D^{\frac{1}{2}}\phi(x) dx.$$

Now

$${}_x^{RL}D^{\frac{1}{2}}\phi(x) = -\frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_x^2 \frac{\phi(t)}{\sqrt{t-x}} dt$$

yields,

$$\begin{aligned} \int_0^2 g(x)\phi(x) dx &= \int_1^2 \frac{2\sqrt{x-1}}{\sqrt{\pi}} \phi(x) dx \\ &= \int_1^2 f(x) \left( {}^{RL}D^{\frac{1}{2}}\phi(x) \right) dx \end{aligned}$$

Therefore,  $g(x)$  is the week  $\frac{1}{2}$ -derivative of  $f$ .

If we continue taking the Riemann-Liouville  $\frac{1}{2}$ -derivative of  $g$ ,

$$\begin{aligned} {}^{RL}D^{\frac{1}{2}}g(x) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \frac{2\sqrt{t-1}}{\sqrt{\pi}\sqrt{x-t}} dt \\ &= \frac{2}{\pi} \frac{d}{dx} \left( \frac{\pi}{2} (x-1) \right) \\ &= 1. \end{aligned}$$

That is

$${}^{RL}D^{\frac{1}{2}} \left( {}^{RL}D^{\frac{1}{2}}(f(x)) \right) = 1 = {}^{RL}D^1 f(x)$$

which is to be shown the regular ordinary derivative of first order.

Note here that  $f$  is not smooth and hence not differentiable on the given interval in a regular sense and  ${}^{RL}D^{\frac{1}{2}}(1) \neq 0$ . Therefore the only function that has a zero derivative in the Riemann-Liouville sense is the zero function.

We introduce the fractional half Dirac operator in the Caputo sense.

*Definition 1.5.* The Dirac operator of order  $\frac{1}{2}$  (Caputo sense) is defined to be

$$D^{\frac{1}{2}} := \sum_{j=1}^n e_j \frac{\partial^{\frac{1}{2}}}{\partial x_j^{\frac{1}{2}}} =: \sum_{j=1}^n e_j \partial_{x_j}^{\frac{1}{2}} \quad (10)$$

The fundamental solution to  $D^{\frac{1}{2}}$  is given by

$$\psi_{\frac{1}{2}}(y-x) = \frac{y-x}{\omega_n \|y-x\|^{n+\frac{1}{2}}}$$

That is

$$D^{\frac{1}{2}}\psi_{\frac{1}{2}}(y-x) = \delta(y-x)$$

*Definition 1.6.* where  $\delta$  is the Kronecker delta function.

*Example 1.2.* Let  $f(x) = x = \sum_{i=1}^n e_i x_i$ . Then the fractional  $\frac{1}{2}$ -Caputo derivative of  $f$  is

$$\begin{aligned} D^{\frac{1}{2}}f(x) &= \sum_{i=1}^n e_i \left( \frac{1}{\sqrt{\pi}} \int_0^{x_i} \frac{\partial f_i(x_1, x_2, \dots, t_i, \dots, x_n)}{\sqrt{x_i-t}} dt_i \right) \\ &= \frac{2}{\sqrt{\pi}} \sum_{i=1}^n e_i \sqrt{x_i} \end{aligned}$$

*Example 1.3.* What happens, when we differentiate  $D^{\frac{1}{2}}f(x)$  with half order in the Caputo sense again?

$$\begin{aligned} D^{\frac{1}{2}}(D^{\frac{1}{2}}f(x)) &= D^{\frac{1}{2}} \left( \sum_{i=1}^n e_i \frac{2}{\sqrt{\pi}} \sqrt{x_i} \right) \\ &= \sum_{i=1}^n e_i D_{x_i}^{\frac{1}{2}} \left( 2\sqrt{\frac{x_i}{\pi}} \right) \\ &= \sum_{i=1}^n e_i = \mathbf{1} \end{aligned}$$

Note here that

$$\mathbf{1} = \sum_{i=1}^n e_i = D(\mathbf{x})$$

with  $\mathbf{x} = \sum_{i=1}^n e_i x_i$ , the regular vector Dirac derivative of first order. But again,

$$D^{\frac{1}{2}}(\mathbf{1}) = 0$$

Therefore, we see

$$D^{\frac{1}{2}} \left( D^{\frac{1}{2}} \left( D^{\frac{1}{2}} (\mathbf{x}) \right) \right) = 0 = D^{1+\frac{1}{2}} (\mathbf{x}) = \sum_{j=1}^n e_j \frac{\partial^{1+\frac{1}{2}}}{\partial x_j^{1+\frac{1}{2}}} (\mathbf{x}) = \sum_{j=1}^n e_j \frac{\partial^{\frac{1}{2}}}{\partial x_j^{\frac{1}{2}}} (\mathbf{1}) = D^{\frac{1}{2}} (\mathbf{1})$$

We introduce two integral fractional transforms  $\zeta_{\Omega}^{\frac{1}{2}}$  (Teodorescu transform) and  $\zeta_{\partial\Omega}^{\frac{1}{2}}$  (Feuter transform), which are domain and boundary integrals respectively.

**Definition 1.7.** For  $f \in \mathcal{L}^2(\Omega)$

$$\zeta_{\Omega}^{\frac{1}{2}} f(x) = \int_{\Omega} \psi_{\frac{1}{2}}(y-x) f(y) d\Omega_y : \text{domain fractional integral} \quad (11)$$

and

$$\zeta_{\partial\Omega}^{\frac{1}{2}} f(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) f(y) d\partial\Omega_y : \text{boundary fractional integral} \quad (12)$$

where  $v_{\frac{1}{2}}(y)$  is the outward fractional half unit normal vector at  $y \in \partial\Omega$  and  $\psi_{\frac{1}{2}}(y-x)$  is the fundamental solution to the Caputo  $D^{\frac{1}{2}}$ .

One can show that  $\zeta_{\Omega}^{\frac{1}{2}}$  is the right algebraic inverse of the half Dirac operator  $D^{\frac{1}{2}}$  and  $\zeta_{\partial\Omega}^{\frac{1}{2}}$  is in its kernel. That is

$$D^{\frac{1}{2}} \left( \zeta_{\Omega}^{\frac{1}{2}} (f) \right) = f \quad \text{and} \quad D^{\frac{1}{2}} \left( \zeta_{\partial\Omega}^{\frac{1}{2}} (f) \right) = 0$$

Let  $f \in W^{\frac{1}{2},2}(\Omega)$ , and  $\psi_{\frac{1}{2}}(y-x)$  be the fundamental solution to the Caputo fractional half Dirac operator  $D^{\frac{1}{2}}$ . Then the method of integration by parts yields:

$$\begin{aligned} \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} f(y) d\Omega_y &= \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) f(y) d\partial\Omega_y - \int_{\Omega} D^{\frac{1}{2}} \psi_{\frac{1}{2}}(y-x) f(y) d\Omega_y \\ &= \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) f(y) d\partial\Omega_y - \int_{\Omega} \delta(y-x) f(y) d\Omega_y \\ &= \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) f(y) d\partial\Omega_y - f(x) \end{aligned}$$

Therefore we can write the following:

**Proposition 1.2.** For  $f \in W^{\frac{1}{2},2}(\Omega)$ ,

$$f(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) f(y) d\partial\Omega_y - \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} f(y) d\Omega_y$$

**Proposition 1.3.** Let  $f, g \in \mathcal{L}^2(\Omega)$ , then the BVP:

$$\begin{cases} D^{\frac{1}{2}} u = f, & \text{in } \Omega \\ \text{tru} = g, & \text{on } \partial\Omega \end{cases}$$

has a solution given by

$$\begin{aligned} u(x) &= \zeta_{\partial\Omega}^{\frac{1}{2}} g(x) - \zeta_{\Omega}^{\frac{1}{2}} f(x) \\ &= \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y - \int_{\Omega} \psi_{\frac{1}{2}}(y-x) f(y) d\Omega_y \end{aligned}$$

**Proof.** The proof follows from the argument of integration by parts and taking the trace value of  $u$  on the boundary  $\partial\Omega$  and the fractional  $\frac{1}{2}$  derivative of  $u$  in the domain  $\Omega$ .

**Lemma 1.1.** For  $g \in \mathcal{L}^2(\Omega)$ , the function  $\phi(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y$  is  $\frac{1}{2}$ -monogenic or  $\frac{1}{2}$ -analytic over  $\Omega$ .

**Proof.** We need to show that for  $x \in \Omega \setminus \partial\Omega$ :

$$D^{\frac{1}{2}} \phi(x) = D^{\frac{1}{2}} \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y = \int_{\partial\Omega} D^{\frac{1}{2}} \left( \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) \right) d\partial\Omega_y = 0$$

Differently, we have to show that

$$D_x^{\frac{1}{2}} \left( \psi_{\frac{1}{2}}(y-x) \right) = 0 \text{ in the above integral.}$$

But  $\psi_{\frac{1}{2}}(x-y)$  is the fundamental solution to the fractional  $\frac{1}{2}$ -Dirac operator  $D_x^{\frac{1}{2}}$ . Therefore,

$$D_x^{\frac{1}{2}} \left( \psi_{\frac{1}{2}}(y-x) \right) = \delta(y-x) = \begin{cases} 0, & x \neq y \\ \infty, & x = y \end{cases}$$

But  $\phi$  is defined at points  $x \in \Omega \setminus \partial\Omega$  and point  $y \in \partial\Omega$ . Therefore  $\|y-x\| > 0$  and hence  $x \neq y$  which implies  $\delta(y-x) = 0$ .

$$\begin{aligned} \therefore \quad & \int_{\partial\Omega} D_x^{\frac{1}{2}} \left( \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) \right) d\partial\Omega_y = 0 \\ \implies & D^{\frac{1}{2}} \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y = \int_{\partial\Omega} D_x^{\frac{1}{2}} \left( \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) \right) d\partial\Omega_y = 0 \end{aligned}$$

Hence,

$$\phi(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \in \text{Ker} D^{\frac{1}{2}}(\Omega)$$

**Corollary 1.2.** For a function  $u$  whose trace  $u|_{\partial\Omega} = 0$  and we have  $D^{\frac{1}{2}}u = f$ , in  $\Omega$ , the solution is given by

$$u(x) = - \int_{\Omega} \psi_{\frac{1}{2}}(y-x) f(y) d\Omega_y$$

Likewise a function  $u$  which is  $\frac{1}{2}$ -monogenic on  $\Omega$  and trace value  $g$  on  $\partial\Omega$  is given by

$$u(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y$$

**Remark 1.1.** Note that the solution  $u$  for both cases is in the fractional Sobolev space  $W^{\frac{1}{2},2}(\Omega)$ .

**Lemma 1.2.** The initial value problem:

$$\begin{cases} D^{\frac{1}{2}}y(x) = y(x) \\ y(x_0) = y_0 \end{cases}$$

has a trivial solution  $y \equiv 0$  in  $\mathbb{R}^n$  for  $n \geq 1$ .

*Proof.* Taking the  $D^{\frac{1}{2}}$  of both sides, we have

$$\underbrace{D^{\frac{1}{2}} \left( D^{\frac{1}{2}} y(x) \right)}_{\parallel D y(x)} = D^{\frac{1}{2}} y(x) = y(x)$$

That is

$$Dy(x) = y(x)$$

where  $D = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$

Now from

$$\mathcal{F} \left( \frac{\partial y}{\partial x_j} \right) (\xi) = i \xi_j \widehat{f}(\xi)$$

where  $\mathcal{F}$  or  $\widehat{(\cdot)}$  is the Fourier transform, and taking the Fourier transform of both sides of

$$Dy(x) = y(x)$$

$$\begin{aligned} \mathcal{F}(Dy)(\xi) &= \sum_{j=1}^n e_j \mathcal{F} \left( \frac{\partial y}{\partial x_j} \right) (\xi) = i \sum_{j=1}^n e_j \xi_j \widehat{y}(\xi) \\ &= \mathcal{F}(y)(\xi) = \widehat{y}(\xi) \\ \implies i(e \cdot \xi) \widehat{y}(\xi) &= \widehat{y}(\xi) \\ \implies (i(e \cdot \xi) - I) \widehat{y}(\xi) &= 0. \end{aligned}$$

Now  $i(e \cdot \xi) - I$  has eigen values  $\pm i \| \xi \| - 1$  which will never be zero.

Therefore

$$\widehat{y}(\xi) = 0 \implies y \equiv 0$$

## 2. Orthogonality and Orthogonal Decompositions

Let  $d$  be a positive integer and  $\Omega$  be a smooth and bounded domain in  $\mathbb{R}^d$  with a non empty boundary  $\partial\Omega$  and  $p = 2$

**Definition 2.1.** (Weak Fractional Derivative) For a function  $f$ , we say  $g$  is the fractional weak generalized  $\frac{1}{2}^{th}$  order derivative of  $f$  over  $\Omega$  written as

$$\begin{aligned} g &= D^{\frac{1}{2}} f \quad \text{if} \quad \int_{\Omega} g(x) \psi(x) d\Omega_x \\ &= \int_{\Omega} f(x) (-\Delta)^{\frac{1}{4}} \psi(x) d\Omega_x, \forall \psi \in C_0^\infty(\Omega). \end{aligned} \quad (13)$$

Clearly a function that is differentiable in the ordinary sense is also weakly differentiable but not the converse. That is a weakly differentiable function may not be differentiable in ordinary sense.

**Example 2.1.** Consider the function

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ x-1, & 1 \leq x \leq 2 \end{cases}.$$

Then  $f$  is continuous on  $[0, 1]$  but not differentiable in the regular sense as the derivative

$$g(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

is discontinuous at  $x = 1$ . However  $g$  is the first order weakly or generalized derivative of  $f$  on  $[0, 1]$  since

$$\begin{aligned} \int_{[0,2]} f(x)\phi'(x)dx &= \int_{[1,2]} x\phi'(x)dx - \int_{[1,2]} \phi'(x)dx \\ &= x\phi(x) \Big|_1^2 - \int_{[1,2]} \phi(x)dx - \phi(2) + \phi(1) \\ &= - \int_{[1,2]} \phi(x)dx \\ &= - \int_{[0,2]} g(x)\phi(x)dx \\ &= - \int_{[0,2]} f'(x)\phi(x)dx \end{aligned}$$

$\forall \phi \in C_0^\infty([0, 2])$ .

**Definition 2.2.** For  $1 < p < \infty$ ,  $k \in \mathbb{N} \cup \{0\}$ , the Sobolev space  $W^{k,p}(\Omega)$  is defined as the set of all functions  $f$  in  $\mathcal{L}^p(\Omega)$  such that the  $\alpha^{\text{th}}$  order weak derivative  $D^\alpha f \in \mathcal{L}^p(\Omega)$  for  $0 \leq |\alpha| \leq k$ .

**Definition 2.3.** When  $k = 1/2$ ,  $p = 2$ , we define an inner product:

$$\langle f, g \rangle_{W^{\frac{1}{2},2}(\Omega)} := \int_{\Omega} f(x)g(x)d\Omega_x + \int_{\Omega} \int_{\Omega} \frac{(f(x) - f(y))(g(x) - g(y))}{\|x - y\|^{n+1}} d\Omega_x d\Omega_y. \quad (17)$$

and for  $f \in W^{\frac{1}{2},2}(\Omega)$ , the  $W^{\frac{1}{2},2}(\Omega)$  norm is defined by

$$\|f\|_{W^{\frac{1}{2},2}(\Omega)}^2 := \int_{\Omega} f^2(x)d\Omega_x + \int_{\Omega} \int_{\Omega} \frac{(f(x) - f(y))^2}{\|x - y\|^{n+1}} d\Omega_x d\Omega_y$$

$$\therefore \|f\|_{W^{\frac{1}{2},2}(\Omega)} = \sqrt{\langle f, f \rangle_{W^{\frac{1}{2},2}(\Omega)}} = \sqrt{\int_{\Omega} f^2(x)d\Omega_x + \int_{\Omega} \int_{\Omega} \frac{(f(x) - f(y))^2}{\|x - y\|^{n+1}} d\Omega_x d\Omega_y}$$

**Definition 2.4.** We say two functions  $f, g \in W^{\frac{1}{2},2}(\Omega)$  orthogonal with respect to the inner product defined by 17 if

$$\langle f, g \rangle_{W^{\frac{1}{2},2}(\Omega)} = 0. \quad (18)$$

For more orthogonal functions of integer regularity, see [2].

We investigate orthogonal decompositions of function spaces with inner product  $\langle, \rangle$  so that any function in the

As I mentioned in the preliminary, these function spaces are ideal spaces to search for solutions to problems of reality unlike regular function spaces such as  $C^k$ , where continuity to a particular order is required.

The particular square integrable Sobolev space (or Hilbert space of higher regularity)  $W^{k,2}(\Omega)$  where  $p = 2$ , and  $k \geq 1$  is an inner product space with inner product

$$\langle f, g \rangle_{W^{k,2}(\Omega)} = \int_{\Omega} \left( \sum_{0 \leq |\alpha| \leq k} D^\alpha f(x) D^\alpha g(x) \right) d\Omega_x \quad (14)$$

with norm given by

$$\|f\|_{W^{k,2}(\Omega)} = \left( \langle f, f \rangle_{W^{k,2}(\Omega)} \right)^{\frac{1}{2}}$$

Therefore there is a distance or metric defined in terms of this norm given by

$$\rho_{W^{k,2}(\Omega)}(f, g) := \|f - g\|_{W^{k,2}(\Omega)}. \quad (15)$$

When  $p = 2$  and  $k = 0$ , we have the usual Hilbert space

$$W^{0,2}(\Omega) = \mathcal{L}^2(\Omega)$$

with inner product

$$\langle f, g \rangle_{W^{0,2}(\Omega)} = \langle f, g \rangle_{\mathcal{L}^2(\Omega)} = \int_{\Omega} f(x)g(x)d\Omega_x \quad (16)$$

Now we define the fractional  $\frac{1}{2}$ -regularity order inner product in a Hilbert space

space is an orthogonal sum of component functions from the orthogonal parts of the space. The spaces we considered in earlier studies are the Sobolev spaces  $W^{k-1,2}(\Omega)$ , the space for  $p = 2$  and  $k \in \mathbb{N}$  and established several properties in [3–6, 9, 10]

We consider now fractional decompositions of Hilbert spaces of fractional regularities:

*Proposition 2.1.* The space  $\mathcal{L}^2(\Omega)$  has an orthogonal decomposition in terms of the Caputo fractional  $D^{\frac{1}{2}}$  :

$$\mathcal{L}^2(\Omega) = A^{\frac{1}{2},2}(\Omega) \oplus D^{\frac{1}{2}}\left(W_0^{\frac{1}{2},2}(\Omega)\right) \quad (19)$$

where

$$A^{\frac{1}{2},2}(\Omega) = \text{Ker} D^{\frac{1}{2}} \cap \mathcal{L}^2(\Omega) = \left\{ g(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) h(y) d\partial\Omega_y : h \in \mathcal{L}^2(\Omega) \right\}$$

and

$$W_0^{\frac{1}{2},2}(\Omega) = \left\{ f \in W^{\frac{1}{2},2}(\Omega) : \left( f|_{\partial\Omega}, D^{\frac{1}{2}} f|_{\partial\Omega} \right) = (0, 0) \right\}$$

so that

$$\forall f \in \mathcal{L}^2(\Omega), \exists g \in A^{\frac{1}{2},2}(\Omega) \text{ and } h \in D^{\frac{1}{2}}\left(W_0^{\frac{1}{2},2}(\Omega)\right) \\ f = g \uplus h.$$

*Proof.* Let  $f \in \mathcal{L}^2(\Omega)$ . Then consider  $g(x) = \frac{1}{\mu(\Omega)} \int_{\Omega} f(x) d\Omega_x$  which is a constant. Then  $g \in \text{Ker} D^{\frac{1}{2}}(\Omega) \cap \mathcal{L}^2(\Omega)$ .

Now consider  $h = f - g$ . Claim that  $h \in D^{\frac{1}{2}}\left(W_0^{\frac{1}{2},2}(\Omega)\right)$

Therefore

$$\exists \eta \in W_0^{\frac{1}{2},2}(\Omega) \ni h(x) = D^{\frac{1}{2}} \eta(x) = (-\Delta)^{\frac{1}{4}} \eta(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi^{n+1}}} P.V. \int_{\Omega} \frac{\eta(x) - \eta(y)}{\|x - y\|^{n+1}} d\Omega_y$$

Then

$$\langle g(x), h(x) \rangle_{\mathcal{L}^2(\Omega)} = \left\langle g(x), D^{\frac{1}{2}} \eta(x) \right\rangle_{\mathcal{L}^2(\Omega)} = \left\langle D^{\frac{1}{2}} g(x), h(x) \right\rangle_{\mathcal{L}^2(\Omega)} = \langle 0, h(x) \rangle_{\mathcal{L}^2(\Omega)} = 0, \because g \text{ is a constant}$$

Hence,

$$f = g + h \text{ with property : } \langle g, h \rangle_{\mathcal{L}^2(\Omega)} = 0 \implies f = g \uplus h$$

*Remark 2.1.* Constants are also in  $\text{Ker} D^{\frac{1}{2}}(\Omega) \cap \mathcal{L}^2(\Omega)$  as long as  $\Omega$  is bounded, since for  $\int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) d\partial\Omega_y = c \int_{\partial\Omega} \underbrace{\psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y)}_{\|1\|} d\partial\Omega_y = c$

*Lemma 2.1.* For  $\psi_{\frac{1}{2}}(y-x)$ , the fundamental solution to  $D^{\frac{1}{2}}$ , we have

$$\int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) d\partial\Omega_y = 1$$

*Proof.* For  $y \in \partial\Omega$ , and  $x \in \Omega \setminus \partial\Omega$  consider a small ball  $B_{\delta}(x) \subseteq \Omega$  of small radius  $\delta = \|x - y\| > 0$ . Consider the normal vector  $\nu_{\frac{1}{2}}(y) = \frac{y-x}{\delta}$ .

Then

$$\begin{aligned} \int_{\partial B_{\delta}(x)} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) d\partial\Omega_y &= \int_{\partial B_{\delta}(x)} C_{n,\frac{1}{2}} \frac{1}{\delta^{n-\frac{1}{2}}} \cdot \delta^{n-1} d\omega_n \\ &= C_{n,\frac{1}{2}} \int_{\partial B_1(0)} \delta^{-\frac{1}{2}} d\omega_n = C_{n,\frac{1}{2}} \delta^{-\frac{1}{2}} \omega_n = 1 \end{aligned}$$

$\therefore C_{n,\frac{1}{2}} = \frac{\sqrt{\delta}}{\omega_n}$ , where  $\omega_n :=$  surface area of a unit sphere in  $\mathbb{R}^n$ .

*Corollary 2.1.*  $\text{Ker} D^{\frac{1}{2}}(\Omega) = \left\{ \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y, g \in \mathcal{L}^2(\partial\Omega) \right\}$ .

*Proof.* Clearly, for  $g \in \mathcal{L}^2(\partial\Omega)$ , let  $u$  be its smooth extension to  $\Omega$ . Then

$$u(x) = \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y + \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} u(y) d\Omega_y$$

Then differentiating both sides with half order:

$$\begin{aligned} D^{\frac{1}{2}} u(x) &= D^{\frac{1}{2}} \left\{ \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y + \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} u(y) d\Omega_y \right\} \\ &= D^{\frac{1}{2}} \left\{ \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y \right\} + D^{\frac{1}{2}} \left\{ \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} u(y) d\Omega_y \right\} \end{aligned}$$

But

$$D^{\frac{1}{2}} \left\{ \int_{\Omega} \psi_{\frac{1}{2}}(y-x) D^{\frac{1}{2}} u(y) d\Omega_y \right\} = D^{\frac{1}{2}} \xi_{\Omega}^{\frac{1}{2}} (D^{\frac{1}{2}} u(x)) = D^{\frac{1}{2}} u(x)$$

since  $D^{\frac{1}{2}} \xi_{\Omega}^{\frac{1}{2}}$  is the identity operator.

Hence,

$$\begin{aligned} D^{\frac{1}{2}} u(x) &= D^{\frac{1}{2}} \left\{ \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y \right\} + D^{\frac{1}{2}} u(x) \\ \implies D^{\frac{1}{2}} \left\{ \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y \right\} &= 0, \text{ i.e., } \int_{\partial\Omega} \psi_{\frac{1}{2}}(y-x) g(y) v_{\frac{1}{2}}(y) d\partial\Omega_y \in \ker D^{\frac{1}{2}}(\Omega). \end{aligned}$$

This result includes constants as well, i.e.,  $g(x) = c$

*Example 2.2.* Consider  $f(x) = \|x\|^2 = \sum_{i=1}^n x_i^2 : \Omega := \Pi_{i=1}^n [-1, 1] \longrightarrow \mathbb{R}$ .

Then we have:

$$g(x) = \frac{1}{|\Omega|} \int_{\Omega} f(x) d\Omega_x = \frac{1}{2^n} \int_{-1}^1 \dots \int_{-1}^1 \sum_{i=1}^n x_i^2 \underbrace{dx_1 dx_2 \dots dx_n}_{d\Omega_x} = \frac{1}{2^n} \left( n \cdot \frac{2^n}{3} \right) = \frac{n}{3}$$

Set  $h(x) = \|x\|^2 - \frac{n}{3}$ . Then  $f(x) = g(x) + h(x)$  and

$$\langle g(x), h(x) \rangle_{\mathcal{L}^2(\Omega)} = \int_{\Omega} g(x) h(x) d\Omega_x = \frac{n}{3} \int_{\Omega} \left( \|x\|^2 - \frac{n}{3} \right) d\Omega_x = \frac{n}{3} \left( \frac{n2^n}{3} - \frac{n2^n}{3} \right) = 0$$

Hence

$$f(x) = g(x) \uplus h(x)$$

Also the parallelogram law holds:

$$\|f\|_{\mathcal{L}^2(\Omega)} = \left( \|g\|_{\mathcal{L}^2(\Omega)}^2 + \|h\|_{\mathcal{L}^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

These decompositions enable us to give the following main results of our research.

### 3. Boundary Value Problems and Decompositions

In this section we consider first and second order boundary value problems of the half Dirac operator and see how the solutions are simply the orthogonal sums of parts that evolve from boundary values of the solution and interior values of the derivative of the solution to the given order. We start with the first order Cauchy problem of the half Dirac operator.

*Proposition 3.1.* Let  $f \in \mathcal{L}^2(\Omega), g \in \mathcal{L}^2(\partial\Omega)$  The first

Caputo fractional 1/2 order Cauchy problem

$$\begin{cases} D^{\frac{1}{2}} u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (20)$$

has a solution  $u \in W^{\frac{1}{2},2}(\Omega)$  such that

$$u = [u]_f \uplus [u]_g$$

where  $[u]_f$  is the part of the solution that evolves from  $f$  and  $[u]_g$  is the part of the solution that evolves from  $g$  with the following properties

$$\begin{aligned}
(i) \quad \langle [u]_f, [u]_g \rangle_{W^{\frac{1}{2},2}(\Omega)} &= 0 & (iii) \quad \langle u, [u]_f \rangle_{W^{\frac{1}{2},2}(\Omega)} &= \| [u]_f \|_{W^{\frac{1}{2},2}(\Omega)}^2 \\
(ii) \quad \| u \|_{W^{\frac{1}{2},2}(\Omega)}^2 &= \| [u]_f \|_{W^{\frac{1}{2},2}(\Omega)}^2 + \| [u]_g \|_{W^{\frac{1}{2},2}(\Omega)}^2 & (iv) \quad \langle u, [u]_g \rangle_{W^{\frac{1}{2},2}(\Omega)} &= \| [u]_g \|_{W^{\frac{1}{2},2}(\Omega)}^2
\end{aligned}$$

*Proof.* For two functions  $f, g \in C^1(\Omega)$ , integration by parts provide

$$\int_{\Omega} f(y-x) D_y^{\frac{1}{2}} g(y) d\Omega_y = \int_{\partial\Omega} f(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y - \int_{\Omega} D_y^{\frac{1}{2}} f(y-x) g(y) d\Omega_y.$$

Then taking  $f = \Gamma_{\frac{1}{2}}$  and  $g = u$ , we have

$$\int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) D_y^{\frac{1}{2}} u(y) d\Omega_y = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) u(y) d\partial\Omega_y - \int_{\Omega} D_y^{\frac{1}{2}} \Gamma_{\frac{1}{2}}(y-x) u(y) d\Omega_y$$

where  $\Gamma_{\frac{1}{2}}$  is the fundamental solution to the fractional half Dirac operator  $D^{\frac{1}{2}}$ .

But

$$D_y^{\frac{1}{2}} \Gamma_{\frac{1}{2}}(y-x) = \delta(y-x)$$

the Kronecker delta function and thus,

$$\begin{aligned}
\int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) D_y^{\frac{1}{2}} u(y) d\Omega_y &= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) u(y) d\partial\Omega_y - \int_{\Omega} \delta(y-x) u(y) d\Omega_y \\
&= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) u(y) d\partial\Omega_y - u(x)
\end{aligned}$$

i.e.,

$$u(x) = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) u(y) d\partial\Omega_y - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) D_y^{\frac{1}{2}} u(y) d\Omega_y \quad (21)$$

which provides the integral representation of the solution  $u$  to the BVP given by

$$u(x) = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y + \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right).$$

We will see that this sum is in fact an orthogonal sum  $\oplus$ .

Because the solution

$$u \in W^{\frac{1}{2},2}(\Omega) = A^{\frac{1}{2},2}(\Omega) \oplus D^{\frac{1}{2}} \left( W_0^{\frac{1}{2},2}(\Omega) \right)$$

has a unique decomposition as sum of components from the two sub spaces,  $A^{\frac{1}{2},2}(\Omega)$  and  $D^{\frac{1}{2}} \left( W_0^{\frac{3}{2},2}(\Omega) \right)$  and the first integral

$$\int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y$$

is monogenic over  $\Omega$  and hence an element of

$$\text{Ker} D^{\frac{1}{2}} \cap \mathcal{L}^2(\Omega) = A^{\frac{1}{2},2}(\Omega).$$

We need to verify that the second integral

$$\left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \in D^{\frac{1}{2}} \left( W_0^{\frac{1}{2},2}(\Omega) \right)$$

as well. That is, there is a function  $\xi \in W_0^{\frac{3}{2},2}(\Omega)$  so that

$$-\int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y = D^{\frac{1}{2}} \xi(x) \quad \text{with} \quad \xi|_{\partial\Omega} = 0.$$

Clearly from the fact that

$$u(x) = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y + \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \quad \text{and} \quad u|_{\partial\Omega} = g$$

we have, the trace  $\tau$  of  $u$  on  $\partial\Omega$

$$\begin{aligned} \tau u|_{\partial\Omega} &= \tau \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y + \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \right) \Big|_{\partial\Omega} \\ &= \tau \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \right) \Big|_{\partial\Omega} + \tau \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \Big|_{\partial\Omega} \\ &= g \\ \implies \tau \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) &= \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \Big|_{\partial\Omega} = 0. \end{aligned}$$

Thus  $\xi|_{\partial\Omega} = 0$ . In an analogous manner of the integral representation of the solution  $u$ , we get the integral representation of  $\xi$  as

$$\begin{aligned} \xi(x) &= - \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \int_{\Omega} \Gamma_{\frac{1}{2}}(y-z) f(y) d\Omega_y d\Omega_z \\ &= - \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(y) d\Omega_y d\Omega_z \in W_0^{1,2}(\Omega). \end{aligned}$$

Therefore, there is a function:

$$\xi(x) = - \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(y) d\Omega_y d\Omega_z \in W_0^{1,2}(\Omega)$$

so that

$$\begin{aligned} D_x^{\frac{1}{2}} \xi(x) &= D_x^{\frac{1}{2}} \left( - \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(y) d\Omega_y d\Omega_z \right) \\ &= - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y. \end{aligned}$$

Hence

$$\begin{aligned} u(x) &= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \uplus \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right) \\ &= [u]_g \uplus [u]_f \end{aligned}$$

with

$$[u]_g = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \quad \text{and} \quad [u]_f = - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y.$$

(i) Then since  $W^{\frac{1}{2},2}(\Omega)$  is an inner product space of an orthogonal decomposition we have

$$\langle [u]_g, [u]_f \rangle = \left\langle \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y, - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right\rangle_{W^{\frac{1}{2},2}(\Omega)} = 0.$$

(ii) From the fact that the sum is an orthogonal sum, the components obey the parallelogram law

$$\begin{aligned} \|u\|_{W^{\frac{1}{2},2}(\Omega)}^2 &= \left\| \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \right\|_{W^{\frac{1}{2},2}(\Omega)}^2 + \left\| \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right\|_{W^{\frac{1}{2},2}(\Omega)}^2 \\ &= \| [u]_g \|_{W^{\frac{1}{2},2}(\Omega)}^2 + \| [u]_f \|_{W^{\frac{1}{2},2}(\Omega)}^2 \end{aligned}$$

(iii) The third result follows from the fact that

$$\begin{aligned} \langle u, [u]_f \rangle_{W^{\frac{1}{2},2}(\Omega)} &= \left\langle \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y, - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right\rangle_{W^{\frac{1}{2},2}(\Omega)} \\ &= \left\langle - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y, - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right\rangle_{W^{\frac{1}{2},2}(\Omega)} \\ &= \left\| - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right\|_{W^{\frac{1}{2},2}(\Omega)}^2 \end{aligned}$$

and likewise (iv) follows from

$$\begin{aligned} \langle u, [u]_g \rangle_{W^{\frac{1}{2},2}(\Omega)} &= \left\langle \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y, \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \right\rangle_{W^{\frac{1}{2},2}(\Omega)} \\ &= \left\| \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g(y) d\partial\Omega_y \right\|_{W^{\frac{1}{2},2}(\Omega)}^2 \end{aligned}$$

The next result is for second order BVP of the half Dirac operator.

*Proposition 3.2.* Let  $f \in \mathcal{L}^2(\Omega)$  and  $g_1 \in W^{\frac{3}{2},2}(\partial\Omega), g_2 \in W^{\frac{1}{2},2}(\partial\Omega)$ , then the second order BVP interns of  $D^{\frac{1}{2}}$

$$\begin{cases} \left(D^{\frac{1}{2}}\right)^2 u = f & \text{in } \Omega \\ u = (g_1, g_2) & \text{on } \partial\Omega \end{cases} \quad (22)$$

with

$$g_1 = \tau u|_{\partial\Omega} \quad \text{and} \quad g_2 = \tau D^{\frac{1}{2}} u|_{\partial\Omega}$$

has a solution  $u \in W^{1,2}(\Omega)$  given by

$$\begin{aligned} u(x) &= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \\ &\quad + \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(z) d\Omega_z d\Omega_y \end{aligned}$$

with the following properties

(i)

$$\begin{aligned}
u(x) &= \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \right) \\
&\quad \uplus \left( \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(z) d\Omega_z d\Omega_y \right) \\
&= [u]_{g_1, g_2} \uplus [u]_f.
\end{aligned}$$

(ii)

$$\begin{aligned}
\|u\|_{W^{1,2}(\Omega)}^2 &= \left\| \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \right) \right\|_{W^{1,2}(\Omega)}^2 \\
&+ \left\| \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(z-x) \Gamma_{\frac{1}{2}}(y-z) f(z) d\Omega_z d\Omega_y \right\|_{W^{1,2}(\Omega)}^2 \\
&= \| [u]_{g_1, g_2} \|_{W^{1,2}(\Omega)}^2 + \| [u]_f \|_{W^{1,2}(\Omega)}^2.
\end{aligned}$$

*Proof.* Clearly since the input function  $f \in \mathcal{L}^2(\Omega)$ , which is the weekly second order derivative of the solution  $u$ , we have  $u$  to be in  $W^{1,2}(\Omega)$ . This is because the Dirac operator  $D$  is a regularity exponent diminishing operator, between Sobolev spaces.

The repeated application of the integral representation given in ( 21 ) will be used. Let  $v(x) = D^{\frac{1}{2}}u(x)$ , then we have a first order BVP

$$\begin{cases} -D^{\frac{1}{2}}v = f & \text{in } \Omega \\ v = g_2 & \text{on } \partial\Omega \end{cases}$$

whose solution is given by

$$v(x) = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_2(y) d\partial\Omega_y + \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) f(y) d\Omega_y \right).$$

But  $D^{\frac{1}{2}}u = v$  and hence we have again a first order BVP

$$\begin{cases} -D^{\frac{1}{2}}u = f & \text{in } \Omega \\ u = g_1 & \text{on } \partial\Omega \end{cases}$$

with a solution

$$\begin{aligned}
u(x) &= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) u(y) d\partial\Omega_y - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) D^{\frac{1}{2}}u(y) d\Omega_y \\
&= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z + \left( - \int_{\Omega} \Gamma_{\frac{1}{2}}(z-y) f(z) d\Omega_z \right) \right) d\Omega_y \\
&= \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \\
&\quad + \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) f(z) d\Omega_z d\Omega_y
\end{aligned}$$

We need to show that this sum is again an orthogonal sum from the orthogonal decomposition

$$W^{1,2}(\Omega) = A^{2,2}(\Omega) \oplus \left( D^{\frac{1}{2}} \right)^2 \left( W_0^{2,2}(\Omega) \right)$$

proven in [1]. Clearly

$$\int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y$$

is annihilated by  $\left(D^{\frac{1}{2}}\right)^2$  since

$$\begin{aligned} & \left(D^{\frac{1}{2}}\right)^2 \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \right) \\ &= \left(D^{\frac{1}{2}}\right)^2 \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y \right) - \left(D^{\frac{1}{2}}\right)^2 \left( \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \right) \\ &= -D^{\frac{1}{2}} \left( \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \right) = 0. \end{aligned}$$

Thus

$$\int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y \in A^{2,2}(\Omega).$$

Next, we need to show that  $\exists \xi \in W_0^{2,2}(\Omega)$  such that

$$\int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) f(z) d\Omega_z d\Omega_y = \left(D^{\frac{1}{2}}\right)^2 \xi(x).$$

Clearly

$$\xi(x)|_{\partial\Omega} = \left(\xi(x)|_{\partial\Omega}, D^{\frac{1}{2}}\xi(x)|_{\partial\Omega}\right) = (0, 0).$$

By applying the result of integration by parts above twice and the fact that

$$\xi|_{\partial\Omega} = 0$$

we have

$$\xi(x) = \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) \Gamma_{\frac{1}{2}}(w-z) \Gamma_{\frac{1}{2}}(q-w) f(q) d\Omega_q d\Omega_w d\Omega_z d\Omega_y \in W_0^{2,2}(\Omega).$$

Setting

$$[u]_{g_1, g_2} = \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) v_{\frac{1}{2}}(y) g_1(y) d\partial\Omega_y - \int_{\Omega} \int_{\partial\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) v_{\frac{1}{2}}(z) g_2(z) d\partial\Omega_z d\Omega_y$$

and

$$[u]_f = \int_{\Omega} \int_{\Omega} \Gamma_{\frac{1}{2}}(y-x) \Gamma_{\frac{1}{2}}(z-y) f(z) d\Omega_z d\Omega_y$$

we have

$$u = [u]_{g_1, g_2} \uplus [u]_f$$

which proves (i).

(ii) follows from the fact that

$$u = [u]_{g_1, g_2} \uplus [u]_f.$$

## 4. Applications

We consider the application of fractional orthogonal decomposition. A common field of investigation of a physical phenomena to consider is a vector field  $F$  of electromagnetism [7, 8, 16] which is the orthogonal sum of the fractional

gradient of the scalar potential  $\psi := -(\nabla^2)^{\frac{-3}{4}}(\nabla \cdot F)$  and the fractional curl of the vector potential  $\phi := (\nabla^2)^{\frac{-3}{4}}(\nabla \times F)$ . That is

*Proposition 4.1.* For a vector field  $F$ , we have  $F = \underbrace{\nabla^{\frac{1}{2}}\psi}_{\text{irrotational/normal}} \oplus \underbrace{\text{Curl}^{\frac{1}{2}}\phi}_{\text{solenoidal/tangential}}$  with  $\phi$  and  $\psi$  defined above.

*Proof.* Orthogonality: We use Fourier transform to address and the Parseval's invariance of inner product of two vector fields and their Fourier transforms.

Thus,

$$\begin{aligned} \left\langle \nabla^{\frac{1}{2}}\psi, \text{Curl}^{\frac{1}{2}}\phi \right\rangle_{\Omega} &= \int_{\Omega} \nabla^{\frac{1}{2}}\psi(x) \cdot \text{Curl}^{\frac{1}{2}}\phi(x) d\Omega_x \\ &= \left\langle \mathcal{F}\left(\nabla^{\frac{1}{2}}\psi(x)\right), \mathcal{F}\left(\text{Curl}^{\frac{1}{2}}\phi(x)\right) \right\rangle_{\Omega} \\ &= \int_{\Omega} \mathcal{F}\left\{\left(\nabla^{\frac{1}{2}}\psi\right)\right\}(\xi) \cdot \overline{\mathcal{F}\left\{\left(\text{Curl}^{\frac{1}{2}}\phi\right)\right\}(\xi)} d\xi \end{aligned}$$

But

$$\mathcal{F}\left\{\left(\nabla^{\frac{1}{2}}\psi\right)\right\}(\xi) = i\xi |\xi|^{\frac{-1}{2}} \widehat{\psi}(\xi)$$

and

$$\begin{aligned} \mathcal{F}\left\{\left(\text{Curl}^{\frac{1}{2}}\phi\right)\right\}(\xi) &= i(\xi \times \widehat{\phi}(\xi)) |\xi|^{\frac{-1}{2}} \\ \implies \overline{\mathcal{F}\left\{\left(\text{Curl}^{\frac{1}{2}}\phi\right)\right\}(\xi)} &= -i\left(\xi \times \overline{\widehat{\phi}(\xi)}\right) |\xi|^{\frac{-1}{2}} \end{aligned}$$

Then

$$\begin{aligned} \left\langle \nabla^{\frac{1}{2}}\psi, \text{Curl}^{\frac{1}{2}}\phi \right\rangle_{\Omega} &= \int_{\Omega} \left[ i\xi |\xi|^{\frac{-1}{2}} \widehat{\psi}(\xi) \right] \left[ -i\left(\xi \times \overline{\widehat{\phi}(\xi)}\right) |\xi|^{\frac{-1}{2}} \right] d\xi \\ &= \int_{\Omega} |\xi|^{-1} \widehat{\psi}(\xi) \left[ \xi \bullet \left(\xi \times \overline{\widehat{\phi}(\xi)}\right) \right] d\xi \end{aligned}$$

From vector triple product property we have:

$$\xi \bullet \left(\xi \times \overline{\widehat{\phi}(\xi)}\right) = (\xi \times \xi) \cdot \overline{\widehat{\phi}(\xi)} = 0$$

Therefore,

$$\left\langle \nabla^{\frac{1}{2}}\psi, \text{Curl}^{\frac{1}{2}}\phi \right\rangle_{\Omega} = \int_{\Omega} |\xi|^{-1} \widehat{\psi}(\xi) [0] d\xi = 0$$

which proves orthogonality of the component vector parts.

Next, we show:  $\nabla^{\frac{1}{2}}\psi + \text{Curl}^{\frac{1}{2}}\phi = F$ .

Again, taking the Fourier transform of the left side of the equation and its linearity property: From the construction of  $\psi$  and  $\phi$ , we have

$$\mathcal{F}\{\psi(x)\}(\xi) = \widehat{\psi}(\xi) = -|\xi|^{\frac{-3}{2}} \left( i\xi \cdot \widehat{F}(\xi) \right) \text{ and } \mathcal{F}\{\phi(x)\}(\xi) = \widehat{\phi}(\xi) = |\xi|^{\frac{-3}{2}} \left( i\xi \times \widehat{F}(\xi) \right)$$

Therefore,

$$\begin{aligned} \mathcal{F}\left(\nabla^{\frac{1}{2}}\psi + \text{Curl}^{\frac{1}{2}}\phi\right) &= \mathcal{F}\left(\nabla^{\frac{1}{2}}\psi\right) + \mathcal{F}\left(\text{Curl}^{\frac{1}{2}}\phi\right) = |\xi|^{\frac{-1}{2}} (i\xi) \widehat{\psi}(\xi) + |\xi|^{\frac{-1}{2}} \left( i\xi \times \widehat{\phi}(\xi) \right) \\ &= |\xi|^{\frac{-1}{2}} (i\xi) \left[ -|\xi|^{\frac{-3}{2}} \left( i\xi \cdot \widehat{F}(\xi) \right) \right] + |\xi|^{\frac{-1}{2}} \left[ i\xi \times \left( |\xi|^{\frac{-3}{2}} \left( i\xi \times \widehat{F}(\xi) \right) \right) \right] \end{aligned}$$

$$= -|\xi|^{-2} \left[ i\xi \left( i\xi \cdot \widehat{F}(\xi) \right) \right] + |\xi|^{-2} \left[ i\xi \times \left( i\xi \times \widehat{F}(\xi) \right) \right]$$

Then from the property of triple vector product:

$$u \times (v \times w) = (u \cdot v)u - |u|^2 v$$

we have

$$\begin{aligned} & -|\xi|^{-2} \left[ i\xi \left( i\xi \cdot \widehat{F}(\xi) \right) \right] + |\xi|^{-2} \left[ i\xi \times \left( i\xi \times \widehat{F}(\xi) \right) \right] \\ &= -|\xi|^{-2} \left[ i\xi \left( i\xi \cdot \widehat{F}(\xi) \right) \right] + |\xi|^{-2} \left[ \left( i\xi \cdot \widehat{F}(\xi) \right) i\xi - |i\xi|^2 \widehat{F}(\xi) \right] \\ &= -|\xi|^{-2} |i\xi|^2 \widehat{F}(\xi) = \widehat{F}(\xi) \end{aligned}$$

That is

$$\mathcal{F} \left( \nabla^{\frac{1}{2}} \psi + \text{Curl}^{\frac{1}{2}} \phi \right) = \widehat{F}(\xi)$$

Then taking the inverse Fourier transform, we have

$$\nabla^{\frac{1}{2}} \psi + \text{Curl}^{\frac{1}{2}} \phi = F.$$

Therefore, we can state:

$$\text{Proposition 4.2. } \|F\|^2 = \|F_{\text{tangential}}\|^2 + \|F_{\text{normal}}\|^2 = \int_{\Omega} |\nabla^{\frac{1}{2}} \psi|^2 dv + \int_{\Omega} |\nabla^{\frac{1}{2}} \times \phi|^2 dv$$

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## Abbreviations

|     |                         |
|-----|-------------------------|
| BVP | Boundary Value Problems |
| CD  | Caputo Derivative       |
| RL  | Riemann - Liouville     |

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**Dejenie Alemayehu Lakew:** Conceptualization, Validation, Writing – original draft, Writing – review & editing

## Conflicts of Interest

The author declares that there are no known conflicts of interests that affect the research work presented.

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