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On a Certain Subclass of Multivalent Function Defined by Generalized Ruscheweyh Derivative

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Abstract: Fractional calculus is the prominent branch of applied mathematics, it deals with a lot of diverse possibility of finding differentiation as well as integration of function $f(z)$ when the order of differentiation operator 'D' and integration operator 'J' is a real number or a complex number. The combination of fractional calculus with geometric function theory is the dynamic field of the current research scenario. It has many applications not only in the field of mathematics but also in the different fields like modern mathematical physics, electrochemistry, viscoelasticity, fluid dynamics, electromagnetic, the theory of partial differential equations systems, Mathematical modeling. Various new subclasses of univalent and multivalent functions defined by using different operators. In this research paper, we work on the formation of new subclass of analytic and multivalent functions defined under the open unit disk. By using Generalized Ruscheweyh derivative operator we define a new subclass of analytic and multivalent functions. The main aim of this research article is to derive interesting characteristics of new subclass of multivalent functions, which mainly include coefficient bound, growth and distortion bounds for function and its first derivative, extreme point and obtain unidirectional results for the multivalent functions which are belonging to this new subclass.

Keywords: Fractional Derivative, Generalized Ruscheweyh Derivative, Multivalent Functions, Coefficient Bound

1. Introduction

In mathematics, geometric function theory is an important branch of complex analysis, it deals with the geometric properties of the analytic functions. regarding this in 1851, Riemann gave an important theorem that *"every simply connected domain D can be mapped conformally onto the unit disk"* [11]. The geometric function theory started to develop around the second decade of 20^{th} century and till now it is the dynamic field of the current research scenario. Apart from Riemann mapping theorem its root can be drawn in 1907 from Koebe [13] paper after that in 1914 Gronwall's [15] proved the area theorem. Inspired from this research, in 1916 Bieberbach [9] proposed a conjecture which is known as Bieberbach conjecture or coefficient problem. This conjecture existed as an extreme challenge to geometric function theorist and inspired the researcher to make new inventions in the field of geometric function theory. The combination of fractional calculus and geometric function theory is the dynamic field of the current research scenario. Fractional calculus is the prominent branch of applied mathematics, it deals with a lot of diverse possibility of finding differentiation as well as integration of function $f(x)$ when the order of differentiation operator 'D' and integration operator 'J' is a real number or a complex number (see, for instance; Miller and Ross [23]).

By making use of geometric function theory and complex analysis many researchers defined various new subclasses of univalent and multivalent functions. Many researcher like Ahuja [1], Agarwal and Paliwal [2], Aouf and Chen [3], Arif et al. [6], Atshan [7], Atshan and Buti [8], Deniz and Orhan [10], Elhaddad et al. [12], Khairnar and More [16], Khan et al. [17], Khan et al. [18], Lupas [19], Magesh et al. [20], Mahmood et al. [21], Mahzoon [22], Orhan et al. [24], Raina and Prajapat [25], Salim et al. [26], Shammaky and Seoudy [27], Parihar and Agarwal [28], Srivastava et al. [30] derived a new subclass of univalent and multivalent functions by using Ruscheweyh Derivative and also discussed the properties of new subclass.

In this paper we work on the new subclass of multivalent functions under the unit disk $U = \{z \ : \ |z| < 1\}$ which is defined with the help of generalized Ruscheweyh derivative. We have studied various problems of multivalent functions and obtained results for the new class $\mathcal{H}(p, \gamma, \lambda, \delta)$. For this new subclass we have derived various results which includes coefficient estimate, distortion theorem, extreme point.

Let $\mathcal{M}\ (p)$ denote the class of analytic and p-valent, function $f\ (z)$ in the open unit disk

$$U = \{ \ z \ : \ |z| < 1\}$$

which is in the form

$$f\ (z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k \tag{1}$$

p is some positive integer.

Following definitions we used in this article

Definition 1.1. Let $f\ (z) \in \ \mathcal{M}\ (p)$ and $g\ (z) \in \ \mathcal{M}\ (p)$ is defined as

$$f\ (z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k \quad and \quad g\ (z) = z^p - \sum_{k=n+p}^{\infty} \beta_k z^k$$

then their convolution product (Hadamard product) is denoted by $(f * g)(z)$ or $(g * f)(z)$ and defined as (see, for instance; Aouf [5])

$$(f * g)(z) = \ (g * f)(z) = \ z^p - \sum_{k=n+p}^{\infty} \alpha_k \beta_k z^k \tag{2}$$

Definition 1.2. The generalized fractional derivative operator of order δ is denoted by $J_{0,\ z}^{\delta,\mu,\upsilon}$ and defined as (see, for instance; Goyal and Goyal [14])

$$J_{0,\ z}^{\delta,\mu,\upsilon}\ f\ (z) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(1-\delta)} \frac{d}{dz}\, z^{\delta-\mu} \int_0^z (z-t)^{-\delta} \times & \\ {}_2\mathrm{F}_1\ \left(\mu-\delta, \upsilon; 1-\delta; 1-\frac{t}{z}\right) f\ (t)\, dt & 0 \leq \ \delta < 1 \\ \frac{d^n}{dz^n} J_{0,\ z}^{\delta-n,\mu,\upsilon}\ f\ (z) & n \leq \delta < n+1,\ n \in N \end{array} \right\} \tag{3}$$

Provided further that

$$f\ (z) = O\left(|z|^k\right), \ \ (z \to 0) \tag{4}$$

where $\mu, \upsilon \in R$ and the function $f\ (z)$ is an analytic function in a simply connected region of the z plane containing the origin and the multiplicity of $(z-t)^{-\delta}$ it is removed by assuming $log(z-t)$ be real when $z - t > 0$ (see, for instance; Goyal and Goyal [14]).

In terms of gamma function $J_{0,\ z}^{\delta,\mu,\upsilon}\ \ z^p$ is expressed as

$$J_{0,\ z}^{\delta,\mu,\upsilon}\ f\ (z) = \frac{\Gamma(\mathrm{p}+1)\Gamma(\mathrm{p}-\mu+\upsilon+2)}{\Gamma(\mathrm{p}-\mu+1)\Gamma(\mathrm{p}-\delta+\upsilon+2)} z^{p-\mu} \tag{5}$$

where $0 \leq \ \ \delta < 1$ and $p > max\ \{0, \mu - \upsilon - 1\}$

Definition 1.3. The fractional derivative operator of order δ of a function $f\ (z)$ is denoted by $D_z^\delta f\ (z)$ and it is defined as (see, for instance; Srivastava and Aouf [29])

$$D_z^\delta f\ (z) = \ \frac{1}{\Gamma\ (1-\delta)}\ \frac{d}{dz} \int_0^z \frac{f\ (t)\, dt}{(z-t)^\delta}\ \ , \ \ 0 \leq \ \ \delta < 1 \tag{6}$$

where the function $f\ (z)$ is same as we have taken in (3) and the multiplicity of $(z-t)^{-\delta}$ can be removed by taking $log(z-t)$ to be real when $z - t > 0$ (see, for instance; Srivastava and Aouf [29]).

$D_z^\delta\, z^p$ in terms of gamma function is expressed as (see, for instance; Aouf [4])

$$D_z^\delta\, z^p = \frac{\Gamma(\mathrm{p}+1)}{\Gamma(\mathrm{p}-\delta+1)} z^{p-\delta} \ ,\ \ 0 \leq \ \ \delta < 1 \tag{7}$$

on taking $\delta = \mu$ in (3), we get

$$J_{0,\,z}^{\delta,\delta,\upsilon}\ f\,(z) = \frac{1}{\Gamma\,(1-\delta)} \frac{d}{dz} \left\{ \int_0^z \frac{f(t)}{(z-t)^\delta}\, dt \right\} \tag{8}$$

on comparing (6) and (8) we get

$$J_{0,\,z}^{\delta,\delta,\upsilon}\ f\,(z) = D_z^\delta f\,(z) \ \ , \qquad\qquad 0 \leq \ \ \delta < 1 \tag{9}$$

The definition of new class of multivalent function defined by using Generalized Ruscheweyh Derivative Operator is as follow

Definition 1.4. Let $f\,(z) \in \mathcal{M}\,(p)$ is said to be in the class $\mathcal{H}\,(p, \gamma, \lambda, \delta)$ if $f\,(z)$ satisfies the following condition

$$\Re \left\{ \frac{z^2 \left(J_p^{\delta,\mu}\ f\,(z)\right)^{''} + \ z(1-\gamma)\left(J_p^{\delta,\mu}\ f\,(z)\right)^{'}}{(1-\gamma)\left(J_p^{\delta,\mu}\ f\,(z)\right) + \gamma z^2 \left(J_p^{\delta,\mu}\ f\,(z)\right)^{''}} \right\} > \lambda \tag{10}$$

where $z \in U,\ 0 \leq \gamma < 1, \ \ 0 \leq \lambda < \beta\ ,\ \delta > -1$

$J_p^{\delta,\mu}$ is generalized Ruscheweyh derivative defined by Goyal and Goyal as [14]

$$J_p^{\delta,\mu}\ f\,(z) = \frac{\Gamma\mu - \delta + \upsilon + 2}{\Gamma\mu + 1)\Gamma\upsilon + 2} z^p\ J_{0,\,z}^{\delta,\mu,\upsilon}\ (z^{\mu-p} f\,(z)) \tag{11}$$

$$J_p^{\delta,\mu}\ f\,(z) = z^p - \sum_{k=n+p}^{\infty} B_p^{\delta,\mu}(k) \alpha_k z^k \tag{12}$$

where

$$B_p^{\delta,\mu}\,(k) = \frac{\Gamma\,(\mu-\delta+\upsilon+2)\,\Gamma\,(\mu-p+k+1)\,\Gamma(k+\upsilon+2-p)}{\Gamma\,(\upsilon+2)\,\Gamma\,(\mu+1)\,\Gamma\,(k-p+1)\,\Gamma(\mu-p+k-\delta+\upsilon+2)} \tag{13}$$

for $\delta = \mu$ and $\upsilon = 1$ the Generalized Ruscheweyh Derivative reduces into ordinary Ruscheweyh Derivative of order δ Goyal and Goyal [14]

$$D^\delta f\,(z) = \frac{z^p}{\Gamma(\delta+1)} \frac{d^\delta}{dz^\delta} \left(z^{\delta-p} f(z)\right) \tag{14}$$

on simplifying the above we get

$$D^\delta f\,(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma\,(\delta+k-p+1)}{\Gamma\,(k-p+1)\,\Gamma\,(\delta+1)} \alpha_k z^k \tag{15}$$

$$D^\delta f\,(z) = z^p - \sum_{k=p+1}^{\infty} B_k\,(\delta) \alpha_k z^k \tag{16}$$

where

$$B_k\,(\delta) = \ \frac{\Gamma\,(\delta+k-p+1)}{\Gamma\,(k-p+1)\,\Gamma\,(\delta+1)}$$

2. Main Results for New Subclass $\mathcal{H}\,(p, \gamma, \lambda, \delta)$

2.1. Coefficient Estimate

In this section we find the coefficient estimate of function $f\,(z)$ where $f\,(z)$ belongs to new subclass $\mathcal{H}\,(p, \gamma, \lambda, \delta)$ of multivalent function.

Theorem 2.1. Let $f\,(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k$, $f\,(z) \in \ \mathcal{M}\,(p)$, then $f\,(z) \in \ \mathcal{H}\,(p, \gamma, \lambda, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\left\{\frac{k\left[(k-1)(1-\lambda\gamma)+1-\gamma\right]-\lambda\left(1-\gamma\right)}{p\left[(p-1)(1-\lambda\gamma)+1-\gamma\right]-\lambda\left(1-\gamma\right)}\right\}B_p^{\delta,\mu}\left(k\right)\alpha_k<1 \tag{17}$$

where $z\in U,\ 0\leq\gamma<1,\quad 0\leq\lambda<\beta\ ,\ \delta>-1$

Proof: Let us consider a function $f\left(z\right)$ and $f\left(z\right)\in\ \mathcal{H}\left(p,\gamma,\lambda,\delta\right)$ then we have

$$\Re\left\{\frac{z^2\left(J_p^{\delta,\mu}\ f\left(z\right)\right)^{''}+\ z(1-\gamma)\left(J_p^{\delta,\mu}\ f\left(z\right)\right)^{'}}{(1-\gamma)\left(J_p^{\delta,\mu}\ f\left(z\right)\right)+\gamma z^2\left(J_p^{\delta,\mu}\ f\left(z\right)\right)^{''}}\right\}>\lambda \tag{18}$$

since

$$f\left(z\right)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k \tag{19}$$

$$J_p^{\delta,\mu}\ f\left(z\right)=z^p-\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)\alpha_k z^k \tag{20}$$

we have

$$\left(J_p^{\delta,\mu}f\left(z\right)\right)^{'}=pz^{p-1}-\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)k\ \alpha_k z^{k-1} \tag{21}$$

$$z\left(J_p^{\delta,\mu}f\left(z\right)\right)^{'}=pz^{p}-\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)k\ \alpha_k z^{k} \tag{22}$$

$$z^2\left(J_p^{\delta,\mu}f\left(z\right)\right)^{''}=p(p-1)z^{p}-\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)k(k-1)\ \alpha_k z^{k} \tag{23}$$

Using (20), (21), (22) and (23) in (18) we get

$$\Re\left\{\frac{p(p-1)z^p\ -\ \sum_{k=n+p}^{\infty}\alpha_k B_p^{\delta,\mu}(k)k(k-1)z^k\ +(1-\gamma)\left[pz^p-\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)k\alpha_k z^k\right]}{(1-\gamma)\left[z^p\ -\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}\left(k\right)\alpha_k z^k\ \right]\ +\gamma\left[\ p\left(p-1\right)z^p\ -\sum_{k=n+p}^{\infty}B_p^{\delta,\mu}(k)k(k-1)\ \alpha_k z^k\right]}\right\}>\lambda$$

on simplifying the above inequality we get,

$$\Re\left\{\frac{\Big[p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]z^p\ -\ \sum_{k=n+p}^{\infty}\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}\left(k\right)\alpha_k z^k}{\left[(1-\gamma)+\gamma p(p-1)\right]z^p\ -\sum_{k=n+p}^{\infty}\Big[(1-\gamma)\,B_p^{\delta,\mu}\left(k\right)\alpha_k+\gamma k(k-1)\Big]\ z^k}\right\}>0$$

This inequality is correct for all $z\in U$and let $|z|\rightarrow 1$ yields

$$\Re\left\{\frac{\Big[p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]\ -\ \sum_{k=n+p}^{\infty}\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}\left(k\right)\alpha_k}{\left[(1-\gamma)+\gamma p(p-1)\right]\ -\sum_{k=n+p}^{\infty}\Big[(1-\gamma)\,B_p^{\delta,\mu}\left(k\right)\alpha_k+\gamma k(k-1)\Big]}\right\}>0 \tag{24}$$

by the mean value theorem we have

$$\Re\left\{\Big[p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]\ -\sum_{k=n+p}^{\infty}\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}\left(k\right)\alpha_k\right\}>0$$

which implies

$$\Big[p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big] > \sum_{k=n+p}^{\infty}\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(k)\,\alpha_k$$

on simplifying, we get

$$\sum_{k=n+p}^{\infty}\left\{\frac{k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}\right\}B_p^{\delta,\mu}(k)\,\alpha_k<1$$

Hence the theorem is proved.

Corollary 2.1. Let us consider a function $f(z)\in\mathcal{M}(p)$ defined as $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$ is a member of new class $\mathcal{H}(p,\gamma,\lambda,\delta)$ then

$$\alpha_k<\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(k)},\quad k\geq n+p\,,p\,\in\mathbb{N} \tag{25}$$

2.2. Growth and Distortion Bounds

We derive the Distortion Theorems in this section for those functions $f(z)$ which are the member of new subclass $\mathcal{H}(p,\gamma,\lambda,\delta)$ of the multivalent function.

Theorem 2.2. Let the function $f(z)$ be defined as $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$ be in the class $\mathcal{H}(p,\gamma,\lambda,\delta)$ then for $|z|=r$, we have

$$\begin{aligned} r^p\left[1-r^n\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)]\;B_p^{\delta,\mu}(n+p)}\right\}\right] &\leq \Big|f(z)\Big| \\ \leq r^p\left[1+r^n\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)]\;B_p^{\delta,\mu}(n+p)}\right\}\right] \end{aligned} \tag{26}$$

Proof: Let us consider a function $f(z)$, $f(z)\in\mathcal{H}(p,\gamma,\lambda,\delta)$ and $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$, we have

$$\sum_{k=n+p}^{\infty}\left\{\frac{k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}\right\}B_p^{\delta,\mu}(k)\,\alpha_k<1$$

since for $k\geq n+p$, from above inequality we can write

$$\begin{aligned} &\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)\sum_{k=n+p}^{\infty}\alpha_k \\ &\leq\sum_{k=n+p}^{\infty}\Big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(k)\,\alpha_k \\ &<\;p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma) \end{aligned}$$

Which can also be expressed as

$$\sum_{k=n+p}^{\infty}\alpha_k\;<\;\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)} \tag{27}$$

we know that $f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k z^k$, we get

$$|f(z)|\;\;\leq\;\;|z|^p+|z|^{n+p}\sum_{k=n+p}^{\infty}\alpha_k$$

$$= r^p + r^{n+p} \sum_{k=n+p}^{\infty} \alpha_k$$

$$\leq r^p \left[1 + r^n \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\}\right] \tag{28}$$

similarly

$$\left|f(z)\right| \geq r^p \left[1 - r^n \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\}\right] \tag{29}$$

from (28) and (29) we get

$$r^p \left[1 - r^n \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\}\right] \leq \left|f(z)\right| \leq$$

$$r^p \left[1 + r^n \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\}\right]$$

Hence, the theorem is proved.

Theorem 2.3. Let the function $f(z)$ be defined as $f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k$ be in the class $\mathcal{H}(p,\gamma,\lambda,\delta)$ then for $|z| = r$, we have

$$pr^{p-1} - (n+p)r^{n+p-1} \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\} \leq \left|f'(z)\right|$$

$$\leq pr^{p-1} + (n+p)r^{n+p-1} \left\{ \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \right\} \tag{30}$$

Proof: Let us consider a function $f(z), f(z) \in \mathcal{H}(p,\gamma,\lambda,\delta)$ and $f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k$, we have

$$\sum_{k=n+p}^{\infty} \left\{ \frac{k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right\} B_p^{\delta,\mu}(k)\,\alpha_k < 1$$

since for $k \geq n+p$, from above inequality we can write

$$\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p) \sum_{k=n+p}^{\infty} \alpha_k$$

$$\leq \sum_{k=n+p}^{\infty} \left[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(k)\,\alpha_k$$

$$< p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)$$

Which can also be expressed as

$$\sum_{k=n+p}^{\infty} \alpha_k < \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\left[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\right] B_p^{\delta,\mu}(n+p)} \tag{31}$$

we know that $f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k$ and $f^{'}(z) = pz^{p-1} - \sum_{k=n+p}^{\infty} k\alpha_k z^{k-1}$, we get

$$\begin{aligned} \left|f^{'}(z)\right| &\leq p\,|z|^{p-1} + \sum_{k=n+p}^{\infty} k\alpha_k\,|z|^{k-1} \\ &= p\,|z|^{p-1} + (n+p)|z|^{n+p-1} \sum_{k=n+p}^{\infty} \alpha_k \\ &= pr^{p-1} + (n+p)r^{n+p-1} \sum_{k=n+p}^{\infty} \alpha_k \end{aligned}$$

$$\leq\ pr^{p-1} +\ (n+p)r^{n+p-1}\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)}\right\} \tag{32}$$

similarly

$$f^{'}(z) \geq\ pr^{p-1} -\ (n+p)r^{n+p-1}\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)}\right\} \tag{33}$$

So from (32) and (38) we get

$$pr^{p-1} -\ (n+p)r^{n+p-1}\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)}\right\} \leq\ \left|f^{'}(z)\right|$$

$$\leq pr^{p-1} +\ (n+p)r^{n+p-1}\left\{\frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\Big[(n+p)[(n+p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\Big]B_p^{\delta,\mu}(n+p)}\right\}$$

Hence, the theorem is proved.

2.3. Extreme Point

In this section, we find extreme points of function $f(z)$ belonging to the new subclass $\mathcal{H}(p,\gamma,\lambda,\delta)$ of multivalent function.
Theorem 2.4. Let us consider a function $f_p(z) = z^p$ and $f_k(z)$ can be expressed as

$$f_k(z) = z^p\ -\ \frac{1}{B_p^{\delta,\mu}(k)}\left[\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}\right]z^k \tag{34}$$

where, $k\ \geq\ n+p,\ n\in\mathbb{N}$ then $f\ \in\ \mathcal{H}(p,\gamma,\lambda,\delta)$ if and only if the function $f(z)$ can be written in the form

$$f(z)\ =\ \sum_{k=p}^{\infty}\beta_k\, f_k(z)$$

where $\beta_k \geq 0$ and $\sum_{k=p}^{\infty}\beta_k\ =\ 1$
Proof: Let us consider a function $f(z)$ can be expressed in the form

$$f(z)\ =\ \sum_{k=p}^{\infty}\beta_k\, f_k(z)$$

$$f(z)\ =\ \beta_p\, f_p(z)\ +\ \sum_{k=n+p}^{\infty}\beta_k\, f_k(z)$$

$$= \beta_p\, f_p(z) + \sum_{k=n+p}^{\infty} \beta_k \left[z^p - \frac{1}{B_p^{\delta,\mu}(k)} \left(\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right) z^k \right]$$

$$= \beta_p\, z^p + \sum_{k=n+p}^{\infty} \beta_k z^p - \sum_{k=n+p}^{\infty} \beta_k \left[\frac{1}{B_p^{\delta,\mu}(k)} \left(\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right) \right] z^k$$

$$= \left(\sum_{k=p}^{\infty} \beta_k \right) z^p - \sum_{k=n+p}^{\infty} \beta_k \left[\frac{1}{B_p^{\delta,\mu}(k)} \left(\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right) \right] z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} m_k z^k$$

where,

$$m_k = \frac{\beta_k}{B_p^{\delta,\mu}(k)} \left(\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right)$$

This implies $f(z) \in \mathcal{H}(p,\gamma,\lambda,\delta)$ because,

$$\sum_{k=n+p}^{\infty} m_k \times \left\{ \frac{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right\} B_p^{\delta,\mu}(k) = \sum_{k=n+p}^{\infty} \beta_k = 1 - \beta_p < 1$$

Conversely : Let us consider a function $f(z) \in \mathcal{H}(p,\gamma,\lambda,\delta)$
since,

$$\alpha_k < \frac{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{\big[k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)\big] B_p^{\delta,\mu}(k)}$$

now we consider β_k such that

$$\beta_k = \sum_{k=n+p}^{\infty} \left\{ \frac{k[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{p[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right\} B_p^{\delta,\mu}(k)\, \alpha_k$$

and

$$\beta_p = 1 - \sum_{k=n+p}^{\infty} \beta_k, \;\; k \geq n+p$$

since,

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{1}{B_p^{\delta,\mu}(k)} \left(\frac{p\,[(p-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)}{k\,[(k-1)(1-\lambda\gamma)+(1-\gamma)]-\lambda(1-\gamma)} \right) z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \beta_k \left[z^p - f_k(z) \right]$$

$$= z^p - \sum_{k=n+p}^{\infty} \beta_k z^p + \sum_{k=n+p}^{\infty} \beta_k f_k(z)$$

$$= z^p \left(1 - \sum_{k=n+p}^{\infty} \beta_k \right) + \sum_{k=n+p}^{\infty} \beta_k f_k(z)$$

$$= z^p \beta_p + \sum_{k=n+p}^{\infty} \beta_k f_k(z)$$

$$= f_p(z)\beta_p + \sum_{k=n+p}^{\infty} \beta_k f_k(z)$$

$$= \sum_{k=n}^{\infty} \beta_k f_k(z)$$

Hence, the theorem is proved.

3. Conclusion

In this research article we derived new subclass of multivalent function by using generalized Ruscheweyh derivative operator. We have also derived various properties for the function belonging to this new subclass. These properties include coefficient estimate, growth and distortion bounds, and extreme point. In future properties like convolution property, integral representation and partial sum property and many more can also be develop for this new subclass.

Conflicts of Interest

The authors declare no conflicts of interest.

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