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Methodology Article

Maximizing Efficiency in the Computation of Generalized Harmonic Numbers Through Recursive Binary Splitting

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Abstract: In this article, an efficient algorithm is implemented in Mathematica for the exact calculation of Generalized Harmonic Numbers (GHN). This is achieved through the combination of two methods. The first method is binary division, where terms formed by the powers of the reciprocals of odd and even numbers are summed separately. The second method is a recursive function that iterates the same sequence of operations until all calculations are completed. Within each cycle, the algorithm processes half of the remaining terms, a feature that significantly improves its efficiency. The computer code is notably concise, consisting of only 11 lines, depending on how they are counted. A remarkable event occurs when the argument is a power of two, as the code condenses into a single line. The most distinctive feature of this algorithm lies in the fact that to calculate the GHN for an argument 'n', it requires only the terms formed by the reciprocals of odd numbers. This provides a clear advantage over algorithms that use the complete numerical sequence of the reciprocals of all numbers from 1 to n. An intriguing aspect of this algorithm, is the unexpected discontinuity in the powers of two within the denominators of the common factors across each layer. Contrary to expected, these do not form a continuous sequence from 0 to $numberoflayers - 1$.

Keywords: Generalized Harmonic Numbers, Recursive Function, Binary Splitting

1. Introduction

When teaching recursive functions in elementary mathematics, educators frequently rely on examples such as factorials and Fibonacci sequences. However, in this work, a slightly more intricate yet equally captivating and elegant approach is delved: the combination of binary splitting and recursion for computing Generalized Harmonic Numbers (GHN). The GHN for a positive integer power p is defined as follows:

$$H(n,p) = \sum_{k=1}^{n} \frac{1}{k^p} \tag{1}$$

When $p = 1$, the celebrated harmonic numbers are obtained. However, as the power p becomes a complex number and n tends to infinity, these harmonic numbers transform into the Riemann Zeta function. To underscore the significance of this function, it suffices to mention the German mathematician Bernhard Riemann (1826 – 1866). Riemann postulated that the real part of all the complex zeros of this function equals $1/2$. While this conjecture has been numerically verified, an exact mathematical proof remains elusive. This intriguing proposition constitutes the famous *Riemann Hypothesis*, widely regarded by many mathematicians as the most renowned unsolved problem in mathematics.

The first demonstration that the harmonic series diverges is attributed to Nicole Oresme (circa 1320-1325 to 1382) [1]. A similar proof was later obtained by Johann Bernoulli (1654-1748)

The harmonic series is an essential component of the Euler constant, also known as the Euler–Mascheroni constant [2–3]. It is defined as follows:

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) \tag{2}$$

The harmonic series manifests in various ways: in cases of both convergent and divergent series; spanning all natural numbers or only the odd numbers; with all positive terms or alternating signs; involving even or odd positive powers, or even negative powers. The literature on both the classical and generalized harmonic series and the Euler constant is extensive [4–9]. Below are some of the most notable examples.

1. In 1670, the Scottish mathematician James Gregory (1638–1675) discovered the following series for $\pi/4$ [10]:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \tag{3}$$

In 1674, the German mathematician Gottfried Wilhelm Leibniz (1646–1716) independently discovered the same series. Although this series is commonly known as the Gregory-Leibniz series, it was originally found by the Indian mathematician Nilakantha (1445–1545) [10]. Legend has it that Leibniz once offered to share his formula for calculating $\pi/4$ with anyone who could demonstrate how to compute harmonic numbers in return. This anecdote underscores Leibniz's interest in harmonic series and his willingness to share mathematical knowledge.

2. Arguably one of the most celebrated problems in the entire history of mathematics is the calculation of the sum of reciprocals of the squares of natural numbers. This problem had been under study by Leonhard Euler (1707–1783). In 1735, Euler wrote, "quite unexpectedly, I have found an elegant formula involving the quadrature of the circle", by which he meant π. Euler demonstrated that:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \tag{4}$$

In 1759, Euler published the general solution for denominators raised to even powers [4]:

$$\varsigma(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \tag{5}$$

Where B_{2n} represents the Bernoulli numbers. Surprisingly, a similar formula for $\varsigma(n)$ when n is an odd number greater than one has not been discovered. This chapter in the history of mathematics remains an ongoing and fascinating tale [1, 4, 10].

3. An especially intriguing case involves negative integer powers (where $p < 0$ in (1)). Recall that a negative power in the denominator is equivalent to a positive power in the numerator (and vice versa). Series such as $1^k + 2^k + 3^k + \cdots + n^k$ certainly do not converge. However it is possible to compute the sum of the first n powers using the formula of the German mathematician Johann Faulhaber (1580–1635) [12]:

$$1^{k-1} + 2^{k-1} + \dots\ \mathrm{n}^{k-1} = \frac{\text{“}(n+B)^k - B^k\text{”}}{k} \tag{6}$$

B represents the Bernoulli numbers, a well-studied sequence. Here are some examples:

$$\begin{aligned}
B^0 &= 1\\
B^1 &= 1/2\\
B^2 &= 1/6\\
B^3 &= B^5 = B^7 \ldots \text{all odd numbers} = 0\\
B^4 &= B^8 = -1/30\\
B^6 &= \frac{1}{42}\\
B^{10} &= 5/66\\
&\dots
\end{aligned}$$

The quotation marks in the numerator expression "$(n+B)^k - B^(k)$" should be interpreted as follows: $(n+B)^k$ represents a binomial raised to the k^{th} power, and B^k denotes the k^{th} Bernoulli number [12]. Bernoulli numbers can be calculated based on preceding numbers using the following method:

$$\begin{aligned}
B^2 - 2B^1 + 1 = B^2, \text{ where } B^1 &= \frac{1}{2}\\
B^3 - 3B^2 + 3B^1 - 1 = B^3, \text{ where } B^2 &= \frac{1}{6}\\
B^4 - 4B^3 + 6B^2 - 4B^1 + 1 = B^4, \text{ where } B^3 &= 0
\end{aligned}$$

Or by the recursive formula shown in (7).

$$B_k = 1 - \frac{1}{k+1} \sum_{j=0}^{k-1} \binom{k+1}{j} B_j \tag{7}$$

These numbers play a significant role in various mathematical contexts, including number theory, combinatorics, and special functions.

More recently, other methods have been proposed for computing the sum of powers of natural numbers. For example the following formula [11]:

$$\sum_{k=0}^{n-1} k^p = \frac{B[p+1,n] - B[p+1]}{p+1} \tag{8}$$

Where $B(p+1,n)$ represents the $n-th$ Bernoulli polynomial of power $p+1$, while $B(p+1)$ is the $(p+1)$ Bernoulli number. The Bernoulli polynomials can be computed using the following formula:

$$B[p,x] = \sum_{k=0}^{p} \binom{p}{k} B_k x^{p-k} \tag{9}$$

4. As a final example of variations of the harmonic series, there are two more formulas [4]:

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32} \tag{10}$$

$$\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \ldots = \frac{5\pi^5}{1536} \tag{11}$$

These formulas were derived by Euler, but they differ significantly from the GHN sequence in two aspects: the sum involves reciprocals of odd powers of numbers, and they also exhibit alternating signs.

It's fascinating how certain mathematical topics garner more attention than others. The Riemann zeta function, with its deep connections to number theory and complex analysis, has indeed captivated the interest of many mathematicians due to its profound implications in understanding prime numbers and the distribution of their multiples [13].

The Generalized Harmonic Number (GHN) might not receive as much attention, but its significance lies in various applications, particularly in fields like physics, computer science, and even in some areas of economics. Its calculation for finite values of n is crucial in various computational algorithms and models. Sometimes, the popularity of certain mathematical problems can overshadow others, but each contributes its own unique insights and applications to the broader landscape of mathematics. The pursuit of understanding these less explored areas often leads to valuable discoveries and advancements, enriching the field as a whole.

There are many algorithms for the exact computation of harmonic numbers using binary splitting [14]. In a wider approach this work will focus on the computation of GHN by means of a recursive function that relies solely on the sum of the odd terms of the type $1/(2k+1)^p$ and the geometric series.

The exact computation of the GHN through direct application of (1) is so straightforward and evident that, apparently, no one has thought to explore the possibility of reducing the number of terms to be considered. To achieve this goal, two approaches were combined in this work: the use of binary splitting and a recursive function, as explained below.

The results will be presented in two forms: a) A (short) computer program written in Wolfram Research Mathematica 12.0, for the general case, and b) A single line formula for the special case when the first argument of $H[n,p]$ is of the form $n = 2^m$. This second case is interesting particularly when n grows rapidly. For the benefit of those readers interested in this problem, but with limited experience writing computer code, the implementation of the program will be explained step by step.

2. Program Deduction

1. The first step is to separate the terms whose denominator is a power of an odd number, from those terms whose denominators are a power of even numbers, as shown in (12).

$$\sum_{k=1}^{n} \frac{1}{k^p} == \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{(2k+1)^p} + \frac{1}{2^p} \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k^p}; \text{ for n odd} \\ \sum_{k=0}^{\frac{n}{2}-1} \frac{1}{(2k+1)^p} + \frac{1}{2^p} \sum_{k=1}^{\frac{n}{2}} \frac{1}{k^p}; \text{ for n even} \end{cases} \tag{12}$$

2. Observe that the two sums in the right hand side (RHS) of (12):

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k^p} \text{ and } \sum_{k=1}^{\frac{n}{2}} \frac{1}{k^p} \tag{13}$$

are similar to the expression on the LHS, except for the upper bounds of the sums and the coefficients $(1/2^p)$ that can be extracted from the sums as constant coefficients. This situation is the necessary condition to apply recursive functions.

3. Pay close attention to the definition of the upper bounds of the sums in the right-hand side of (12). Since this part of the algorithm might not be evident, let us explain in detail how the upper bounds of the sums in the right-hand side of (12) are defined as a function of n. (Later, it will be explained how the lower limits are defined).
   a) If n is an odd number, then the upper bounds of both expressions in the first row of the RHS of (12) should be set equal to $(n-1)/2$,
   b) If n is an even number, then the upper limit of the first of (12) should be set equal to $(n/2-1)$, while the upper limit of the second of (12) should be set equal to $(n/2)$.

By proceeding in the described way, it is ensured that all bounds are integers, and that index k will never go outside the range of $k = 1, 2, 3, \ldots n$.

4. For the development of the central part of the program the use of specific numerical values is of great help. So consider, for example. $n = 69$ and $p = 3$. Now the problem will be to calculate in a more efficient way the sum:

$$\sum_{k=1}^{69} \frac{1}{k^p} \tag{14}$$

5. After successively applying (12) the following result is obtained:

$$\begin{aligned}\sum_{k=1}^{69} \frac{1}{k^p} &= \sum_{k=0}^{34} \frac{1}{(2k+1)^p} + \\ &\frac{1}{2^p}\left(\sum_{k=0}^{16} \frac{1}{(2k+1)^p} + \frac{1}{2^p}\left(\sum_{k=0}^{8} \frac{1}{(2k+1)^p} + \frac{1}{2^p}\left(\sum_{k=0}^{3} \frac{1}{(2k+1)^p} + \frac{1}{2^p}\left(\sum_{k=0}^{1} \frac{1}{(2k+1)^p} + \frac{1}{2^p}\sum_{k=1}^{2}\frac{1}{k^p}\right)\right)\right)\right)\end{aligned} \tag{15}$$

6. After removing the parentheses and performing all the required operations, the following result is obtained:

$$\begin{aligned}\sum_{k=1}^{69} \frac{1}{k^p} = \sum_{k=0}^{34} \frac{1}{(2k+1)^p} + \left(\frac{1}{2^p}\right)\sum_{k=0}^{16} \frac{1}{(2k+1)^p} + \left(\frac{1}{2^{2p}}\right)\sum_{k=0}^{8} \frac{1}{(2k+1)^p} + \\ \left(\frac{1}{2^{3p}}\right)\sum_{k=0}^{3} \frac{1}{(2k+1)^p} + \left(\frac{1}{2^{4p}}\right)\sum_{k=0}^{1} \frac{1}{(2k+1)^p} + \left(\frac{1}{2^{5p}}\right)\sum_{k=1}^{2} \frac{1}{k^p}\end{aligned} \tag{16}$$

7. Now it is time to calculate the values of the lower bounds in the sums of the RHS equations. The lower bounds should be set equal to the upper limit of the following sum, plus unity. For instance, the upper bound in the second sum is 16. This means that the lower bound of the previous sum should start at $17 = 16 + 1$. But, what happens with the first 16 values of the index k, that were transferred from the first to the second sum. They are incorporated into the second sum. The coefficients are updated accordingly. (See (17)).

$$\begin{aligned}\sum_{k=1}^{69} \frac{1}{k^p} = \sum_{k=17}^{34} \frac{1}{(2k+1)^p} &+ \left(1 + \frac{1}{2^p}\right)\sum_{k=9}^{16} \frac{1}{(2k+1)^p} + \\ &\left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}}\right)\sum_{k=4}^{8} \frac{1}{(2k+1)^p} + \\ &\left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \frac{1}{2^{3p}}\right)\sum_{k=2}^{3} \frac{1}{(2k+1)^p} + \\ &\left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \frac{1}{2^{3p}} + \frac{1}{2^{4p}}\right)\sum_{k=0}^{1} \frac{1}{(2k+1)^p} + \\ &\left(\frac{1}{2^{5p}}\right)\sum_{k=1}^{2} \frac{1}{k^p}\end{aligned} \tag{17}$$

8. The sequences within brackets are geometric series that can be calculated by means of any one of the following expressions:

$$\left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \frac{1}{2^{3p}} + \ldots + \frac{1}{2^{n\,p}}\right) = \frac{2^{(n+1)p} - 1}{2^{(n+1)p} - 2^{n\,p}} = \frac{1}{2^p - 1}\left(2^p - \frac{1}{2^{n\,p}}\right) \tag{18}$$

9. Substituting (18) in (17) the following expression is obtained:

$$\begin{aligned}\sum_{k=1}^{69}\frac{1}{k^p} =&\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{0\,p}}\right)\sum_{k=17}^{34}\frac{1}{(2k+1)^p}+\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{1\,p}}\right)\sum_{k=9}^{16}\frac{1}{(2k+1)^p}+\\ &\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{2\,p}}\right)\sum_{k=4}^{8}\frac{1}{(2k+1)^p}+\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{3\,p}}\right)\sum_{k=2}^{3}\frac{1}{(2k+1)^p}+\\ &\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{4\,p}}\right)\sum_{k=0}^{1}\frac{1}{(2k+1)^p}+\left(\frac{1}{2^{5p}}\right)\sum_{k=1}^{2}\frac{1}{k^p}\end{aligned} \tag{19}$$

10. Equation (19) is correct, but its last sum seems to be out of context, since it would be better to express all the terms of (19) as functions only of odd numbers. After some elaborate algebraic manipulations, a somewhat unexpected equality is discovered, which seems to fit perfectly:

$$\begin{aligned}&\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{(j\text{-}1)\,p}}\right)\sum_{k=0}^{1}\frac{1}{(2k+1)^p}+\left(\frac{1}{2^{j\,p}}\right)\sum_{k=1}^{lsp}\frac{1}{k^p}=\\ &\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{(j\text{-}1)\,p}}\right)\sum_{k=1}^{1}\frac{1}{(2k+1)^p}+\frac{1}{2^p-1}\left(2^p-\frac{1}{2^{(j+lsp\text{-}1)\,p}}\right)\end{aligned} \tag{20}$$

This particular equality applies when $n = 69$, number of loop $j = 5$ and $lsp = 2$.

11. Finally, after proper substitution, the following remarkable expression is obtained:

$$\begin{aligned}\sum_{k=1}^{69}\frac{1}{k^p} = \left(\frac{1}{2^p-1}\right)\Bigg(&\left(2^p-\frac{1}{2^{0\,p}}\right)\sum_{k=17}^{34}\frac{1}{(2k+1)^p}+\left(2^p-\frac{1}{2^{1\,p}}\right)\sum_{k=9}^{16}\frac{1}{(2k+1)^p}+\\ &\left(2^p-\frac{1}{2^{2\,p}}\right)\sum_{k=4}^{8}\frac{1}{(2k+1)^p}+\left(2^p-\frac{1}{2^{3\,p}}\right)\sum_{k=2}^{3}\frac{1}{(2k+1)^p}+\\ &\left(2^p-\frac{1}{2^{4\,p}}\right)\sum_{k=1}^{1}\frac{1}{(2k+1)^p}+\left(2^p-\frac{1}{2^{6\,p}}\right)\Bigg)\end{aligned} \tag{21}$$

Equation (21) applies when $n = 69$, but similar expressions can be obtained for different values of n. Equation (21) can be expanded for a better appreciation of its structure, as is shown in (22).

$$\begin{aligned}\sum_{k=1}^{69}\frac{1}{k^p} = \left(\frac{1}{2^p-1}\right)\Bigg(&\left(2^p-\frac{1}{2^{6\,p}}\right)\left(\frac{1}{(1)^p}\right)+\\ &\left(2^p-\frac{1}{2^{4\,p}}\right)\left(\frac{1}{(3)^p}\right)+\\ &\left(2^p-\frac{1}{2^{3\,p}}\right)\left(\frac{1}{(5)^p}+\frac{1}{(7)^p}\right)+\\ &\left(2^p-\frac{1}{2^{2\,p}}\right)\left(\frac{1}{(9)^p}+\frac{1}{(11)^p}+\frac{1}{(13)^p}+\frac{1}{(15)^p}+\frac{1}{(17)^p}\right)+\\ &\left(2^p-\frac{1}{2^{1\,p}}\right)\left(\frac{1}{(19)^p}+\frac{1}{(21)^p}+\frac{1}{(23)^p}+\frac{1}{(25)^p}+\frac{1}{(27)^p}+\frac{1}{(29)^p}+\frac{1}{(31)^p}+\frac{1}{(33)^p}\right)+\\ &\left(2^p-\frac{1}{2^{0\,p}}\right)\Bigg(\frac{1}{(35)^p}+\frac{1}{(37)^p}+\frac{1}{(39)^p}+\frac{1}{(41)^p}+\frac{1}{(43)^p}+\frac{1}{(45)^p}+\frac{1}{(47)^p}+\frac{1}{(49)^p}+\\ &\quad\frac{1}{(51)^p}+\frac{1}{(53)^p}+\frac{1}{(55)^p}+\frac{1}{(57)^p}+\frac{1}{(59)^p}+\frac{1}{(61)^p}+\frac{1}{(63)^p}+\frac{1}{(65)^p}+\\ &\quad\frac{1}{(67)^p}+\frac{1}{(69)^p}\Bigg)\Bigg)\end{aligned} \tag{22}$$

An analysis, even superficial, from the image of (21) reveals the following:

a) There are no missing or repeating odd numbers in the range of $k = 1, 2, 3 \ldots 69$,

b) In (21) the number of terms (of the form $1/(2k+1)^p$) is roughly half the numbers of terms calculated by (1), 35 vs 69.

c) It is important to highlight the presence of the common factor: $1/(2^p - 1)$. The same factor is also observed in the study of the Riemann Zeta function, denoted as $\zeta(s)$ [15].

3. Mathematica Computer Code

The computations involved when applying (21) can be automated by a computer program for $n > 2$. To develop such program, Wolfram Mathematica 12.0 was utilized due to its availability, although alternative platforms could have been equally suitable. It is only important that the selected language could handle large integers, because the results of calculation often are fractions with long (even very long) numerators and denominators. Figure 1 shows the listing of the code.

```
(* MATHEMATICA CODE FOR THE COMPUTATION OF GENERALIZED HARMONIC NUMBERS *)
hg[n_Integer, p_Integer, j_Integer] := Module[{lsn, lin, lsp, lsn2},
    If[OddQ [n], lsn = (n - 1)/2; lsp = (n - 1)/2, lsn = n/2 - 1; lsp = n/2];
    N If[OddQ[lsp], lsn2 = (lsp - 1)/2, lsn2 = lsp/2 - 1];
    If[lsp ≤ 2, lin = 1, lin = lsn2 + 1];
    k = lin; kf = lsn;  num = 1; den = (2 k + 1)^p; k = k + 1;
    While[k ≤ kf, num = num (2 k + 1)^p + den; den = den (2 k + 1)^p ; k++];
    suma = 1/(2^p - 1) (2^p - 1/2^(p (j-1))) num/den;
    If[lsp ≤ 2, sumafinal = 1/(2^p - 1) (2^p - 1/2^(p (j+lsp-1)))];
    If[lin == 1, suma = suma + sumafinal, suma = suma + hg[lsp, p, j + 1]]];
hgen[n_Integer, p_Integer] := hg[n, p, 1];
```

Figure 1. *Mathematica code.*

4. Special Case: The Argument Is of the Form $n = 2^m$

When the argument n is of the form $n = 2^m$ (m is a positive integer) (21) and its expansions to greater numbers can be significantly compacted, remaining as follows:

$$\sum_{k=1}^{2^n} \frac{1}{k^p} = \left(\frac{1}{2^p - 1}\right)\left(\left(2^p - \frac{1}{2^{n\,p}}\right) + \sum_{i=0}^{n-2}\left(2^p - \frac{1}{2^{i\,p}}\right) \sum_{k=2^{(n\text{-}i\text{-}2)}}^{2^{(n\text{-}i\text{-}1)}-1} \frac{1}{(2k+1)^p}\right) \tag{23}$$

5. Discussion

Many mathematicians, both professionals and amateurs, have long been unsuccessfully attempting to find a closed formula to compute the Generalized Harmonic Numbers (GHN) solely based on n (the number of terms to be summed), similar to, for example, the formula used to calculate the sum of the cubes of the first n natural numbers: $((n(n+1)/2)^2$. Equation (23) constitutes a step in the desired direction as it computes the GHN solely based on the reciprocals of odd numbers which constitute half of n.

Probably the most amazing feature of the model is the fact that it is possible to calculate the GHN using only the odd numbers. This confers a notable advantage to the approach over (1), which utilizes all numbers, even and odd. However, this advantage comes at the expense of calculating specific coefficients, one per layer, as explicitly demonstrated in (22). The coefficients are of the form $(2^p - 1/2^{p\,i})$, where p is the power of the GHN and i is the number of layer, varying from zero for the first layer, (which is the largest layer and the corresponding coefficient is 1), up to the total number of layers. In the given example, it would be expected that i to vary this way: 0, 1, 2, 3, 4, 5. However from 4 the Index jumps to 6, missing past the 5. This is not an error; the same behavior was observed in all numerical examples carried up. Another unexpected feature of this model, is the appearance of the common factor of the form: $1/(2^p - 1)$.

In order to get an idea of the efficiency of the proposed algorithm in comparison with the algorithm represented by (1), observe that the computational complexity of the latter algorithm is proportional to n, while with the proposed algorithm such amount is roughly $n/2 + log_2(n)$. As an example of computation time for $p = 3$ and $n = 4096$ the following results were obtained:

Table 1. *Processing times.*

Formula	Time (seconds)
(1)	0.0349916
Program hgen[4096, 3]	0.0264583
(10), with $n = 12$, so that $2^{12} = 4096$	0.0220535

6. Conclusion

A computer program and a compact formula for the exact calculation of the GHN have been developed. The computer program is based on two methods: 1) Binary splitting that separates the odd and the even terms, and 2) a recursive function that operates on half of the remaining terms. The number of calls of the recursive function is roughly $log_2(n)$). This feature gives an advantage over the direct computation with (1), which utilizes all numbers, even and odd. A disadvantage of the proposed algorithm, is the necessity to compute certain coefficients for each one of the layers into which n has been subdivided (roughly $log_2(n)$)), as shown in (22).

Several questions remain unanswered that will require additional research. First of all, the possibility to calculate the GHN taking into account the even terms only (instead of the odd terms). A second question is to investigate the meaning of the common coefficient $1/(2^p - 1)$ that appears also in the Riemann Zeta function. Finally, why the penultimate layer is bypassed in the analysis.

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Conflicts of Interest

The authors declare no conflicts of interest.

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