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On the Polya Permanent Problem over Finite Commutative Rings

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Abstract: In this paper we address the Polya permanent problem that was first raised in the second decade of the last century. Despite this, it continues to be treated in several surveys, of which we highlight the studies that point out Polya's permanent problem over finite fields. Unlike previous papers, we focus on finite commutative rings, and to this end, we start by considering a commutative ring with identity R and its decomposition into a direct sum of finite local rings. Next we suppose that the characteristic of each residue field $\mathbb{F}_i$ is different from two, and we proof that if n is greater than or equal to 3, then no bijective map Φ from $M_n(R)$ to $M_n(R)$ transforms the permanent into a determinant. We developed this technique to estimate the order of the general linear group of degree n over a finite commutative ring with identity. The paper begins with the introduction where we present the title, the preliminaries that help the understanding of the following subject, then we talk about the unit permanent and unit determinant in $M_n(R)$, we demonstrate the main result and conclusions. Regarding the methodology, we use the previous results on finite fields and the structure of finite commutative rings and also radical theory of rings.

Keywords: Determinant, Permanent, Nilradical, Finite Ring

1. Introduction

The study of Pólya's permanent problem began in 1913 [1]. Currently, it has been generalized in many directions [3, 4, 6, 8, 13].

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Two basic parameters of A are its *determinant* and its *permanent* defined by

$$det\ A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

$$\wedge\ per\ A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1}$$

respectively. In both cases, S_n is the symmetric group of degree n and $sgn(\sigma) \in \{-1, 1\}$ is the sign of the permutation σ.

Recently, in [8] the authors proved that the number of zeros of the permanent function is strictly less than the number of zeros of the determinant function for square matrices of arbitrary size $n \geq 3$ over an arbitrary finite field $\mathbb{F}$ of characteristic different from 2. As a consequence they obtain the answer to the Pólya problem over $\mathbb{F}$, by showing that, for $n \geq 3$ there are no bijective transformations which converts the permanent to the determinant for square matrices of an arbitrary size $n \geq 3$ over any finite field of characteristic not 2.

The aim of this paper is to extend the remarkable results obtained in [8], for finite fields, to a general finite commutative ring with identity. It should be noted that the determinant and the permanent may be defined, using the formulas (1), for matrices with entries in an arbitrary commutative ring with identity.

Our main result can be formulated as follows:

Theorem 1.1. Let R be a finite commutative ring with identity and $R = \bigoplus_{i=1}^{k} R_i$ its decomposition in a finite direct sum of local rings. Suppose that $n \geq 3$. Then, if each residue field $\mathbb{F}_i = R_i/\mathcal{N}(R_i)$ verifies $char\mathbb{F}_i \geq 3$, for every $i = 1, \ldots, k$, no bijective map $\Phi : M_n(R) \to M_n(R)$ satisfies

$$per A = det\Phi(A). \tag{2}$$

The proof has three stages. First we make use of the fact that a finite commutative ring R with identity is a direct sum of local rings. This allows us to reduce the problem to the case where the ring is local. Secondly, for a local ring, we reduce the problem to its residue field. Finally, for fields we use the results obtained in [8].

2. Preliminaries

In this section we establish some facts about rings that are used throughout the argument. All rings considered have an identity element denoted by 1. The set of all units in the ring R is denoted by $U(R)$ and the set of all zero divisors by $Z(R)$. Following [15] we include the zero element of R also as a zero divisor.

The following fact about finite rings is well known. However, for the sake of completeness, we present a proof.

Lemma 2.1. Let R be a finite ring. Then, every element of R is either a unit or a zero divisor.

Proof Assume that $x \in R$ is not a zero divisor and consider the map $\phi : R \to R$ given by $\phi(r) = xr$. Note that, if $\phi(r) = \phi(s)$, then $x(r - s) = 0$. Since x is not a zero divisor it follows that $r = s$. Hence, ϕ is injective. Since R is finite, by the pigeonhole principle ϕ is surjective. Thus, there exists $y \in R$ such that $xy = 1$. Similarly, we can prove that there exists $z \in R$ such that $zx = 1$. Now we have $z = z(xy) = (zx)y = y$. Hence, $xy = yx = 1$. That is, x is a unit. ■

We denote the nilradical of a ring R by $\mathcal{N}(R)$. If the ring R is commutative, then the nilradical is the set of all nilpotent elements.

It is well-known that for a commutative ring if $x \in \mathcal{N}(R)$, then $1 + x$ (and also $1 - x$) is a unit. In fact, if $x^n = 0$, then

$$1 = (1 + x)(1 - x + x^2 + \ldots + (-1)^{n-1}x^{n-1}).$$

The next lemma, while a simple generalization of this fact, is very useful for our purposes.

Lemma 2.2. An element u in a commutative ring R is a unit if and only if $u + x$ is a unit, for every $x \in \mathcal{N}(R)$.

Proof Assume that u is a unit and let x be an element of $\mathcal{N}(R)$. Since the radical $\mathcal{N}(R)$ is an ideal of R, it follows that $u^{-1}x \in \mathcal{N}(R)$. Therefore, $u^{-1}x$ is a nilpotent element and thus $1 + u^{-1}x$ is a unit. Since the set $U(R)$ of all units in R forms a group it follows that $u(1 + u^{-1}x) = u + x$ is a unit.

Conversely, assume that $u + x$ is a unit and denote by $\widehat{x}$ the inverse of $u + x$. We have that

$$(u + x)\widehat{x} = 1 \Leftrightarrow u\widehat{x} = 1 - x\widehat{x}.$$

Since $x \in \mathcal{N}(R)$, the element $x\widehat{x}$ also belongs to $\mathcal{N}(R)$. So, $x\widehat{x}$ is nilpotent which implies that $1 - x\widehat{x}$ is a unit. Thus, $u\widehat{x}$ is a unit and then u is a unit. ■

The following lemma deals with polynomials. Roughly speaking, it asserts that in order to decide when the evaluation of a polynomial $p(x_1, \ldots, x_k) \in R[x_1, \ldots, x_k]$ at a point $(a_1, \ldots, a_k) \in R^k$ is a unit, we can avoid the radical.

Lemma 2.3. Let R be a commutative ring, $(a_1, \ldots, a_k)$ an element in R^k and $p(x_1, \ldots, x_k)$ a polynomial in the polynomial ring $R[x_1, \ldots, x_k]$. Then, $p(a_1, \ldots, a_k)$ is a unit in R if and only if $p(a_1 + n_1, \ldots, a_k + n_k)$ is a unit in R, for all $n_1, \ldots, n_k \in \mathcal{N}(R)$.

Proof Let ψ denote the canonical epimorphism from R onto $R/\mathcal{N}(R)$. Suppose that $a_1, \ldots, a_k \in R$ and $n_1, \ldots, n_k \in \mathcal{N}(R)$. Since ψ preserves addition and multiplication, we have

$$\psi\left[p(a_1 + n_1, \ldots, a_k + n_k)\right] = p(\psi(a_1) + \psi(n_1), \ldots, \psi(a_k) + \psi(n_k)).$$

Since, for $i = 1, \ldots, k$, each n_i belongs to $Ker(\psi)$, it follows that

$$\psi[p(a_1 + n_1, \ldots, a_k + n_k)] = p[\psi(a_1), \ldots, \psi(a_k)] = \psi[p(a_1, \ldots, a_k)].$$

Therefore, $p(a_1 + n_1, \ldots, a_k + n_k) = p(a_1, \ldots, a_k) + n$, for some $n \in \mathcal{N}(R)$. The claim follows by Lemma 2.2. ■

The above lemma is more useful when the ring R is finite. As usual, we write $|S|$ to denote the number of elements in a finite set S. If R is a finite commutative ring and $p(x_1, \ldots, x_k)$ is a polynomial in $R[x_1, \ldots, x_k]$, we define the set

$$U_p(R) = \left\{(a_1, \ldots, a_k) \in R^k \quad : \quad p(a_1, \ldots, a_k) \in U(R)\right\}.$$

That is, $U_p(R)$ is the set of all points $(a_1, \ldots, a_k)$ in R^k such that the evaluation of the polynomial $p(x_1, \ldots, x_k)$ at $(a_1, \ldots, a_k)$ is a unit. It follows by Lemma 2.3 that

$$|U_p(R)| = |\mathcal{N}(R)|^k |U_p(R/\mathcal{N}(R))|. \tag{3}$$

We end this section by recalling a well known fact in the structure theory of finite commutative rings. A commutative ring R is called local if it has exactly one maximal ideal M. The field $\mathbb{F} = R/M$ is called the residue field of the local ring R. Every finite commutative ring R can be uniquely expressed as a direct sum of finite local rings, [16]. That is,

$$R = R_1 \oplus \ldots \oplus R_l,$$

where R_i, for $i = 1, \ldots, l$, are uniquely determined local rings.

3. Unit Permanent and Unit Determinant in $M_n(R)$

In this section we derive results concerning the number of matrices with unit permanent and the number of matrices with unit determinant in $M_n(R)$.

Let R be a finite commutative ring; we define the sets

$$L_n(R) = \{A \in M_n(R) \ : \ per\ A \in U(R)\},$$

and

$$S_n(R) = \{A \in M_n(R) \ : \ det\ A \in U(R)\}$$

of all matrices with unit permanent and unit determinant, respectively. We will investigate the cardinality of these sets. Let us also define the sets

$$P_n(R) = \{A \in M_n(R) \ : \ per\ A \in Z(R)\} \ and \ D_n(R) = \{A \in M_n(R) \ : \ det\ A \in Z(R)\}$$

of all matrices with zero divisor permanent and zero divisor determinant, respectively. By Lemma 2.1 we conclude that

$$|S_n(R)| + |D_n(R)| = |P_n(R)| + |L_n(R)| = |M_n(R)| = |R|^{n^2}. \tag{4}$$

We note that any bijective map $\Phi : M_n(R) \to M_n(R)$ satisfying (2) would induce a bijection from the set $S_n(R)$ onto the set $L_n(R)$. Therefore, to prove the Theorem 1.1 it is enough to show that the cardinality of $S_n(R)$ does not equal the cardinality of $L_n(R)$.

In the following theorem we reduce the question of the cardinality of $S_n(R)$ and $L_n(R)$, where R a finite commutative ring, to the case where R is a local ring.

Theorem 3.1. Let R be a finite commutative ring and suppose that R decomposes as $R = R_1 \oplus \ldots \oplus R_k$, where each R_i is a local ring, for $i = 1, \ldots, k$. Then

$$|S_n(R)| = \prod_{i=1}^{k} |S_n(R_i)| \quad and \quad |L_n(R)| = \prod_{i=1}^{k} |L_n(R_i)|.$$

Proof The decomposition $R = R_1 \oplus \ldots \oplus R_k$ induces a natural isomorphism,

$$M_n(R) \cong M_n(R_1) \oplus \ldots \oplus M_n(R_k).$$

So, each matrix $A \in M_n(R)$ decomposes, under this isomorphism, as $(A_1, \ldots, A_k)$, where each A_i is in $M_n(R_i)$, for $i = 1, \ldots k$.

Now we observe that each element of R can be identified with an k-tuple $(r_1, \ldots, r_k)$, where $r_i \in R_i$, for $i = 1, \ldots, k$, with componentwise addition and multiplication. Since the determinant and the permanent functions are defined by addition and multiplication, we have the decompositions

$$detA = (detA_1, \ldots, detA_k)$$

and

$$perA = (perA_1, \ldots, perA_k).$$

To complete the proof we observe that the set $U(R)$, of all units in R, decomposes as $U(R) = U(R_1) \oplus \ldots \oplus U(R_k)$, [16]. Hence, $detA$ is a unit if and only if each $detA_i$ is a unit and $perA$ is a unit if and only if each $perA_i$ is a unit and the result follows. ■

Finally, we reduce the local case to the finite field case.

Theorem 3.2. Let R be a finite local ring, M the maximal ideal of R and $\mathbb{F} = R/M$ the finite residue field of R. Then

$$|S_n(R)| = |\mathcal{N}(R)|^{n^2} |S_n(\mathbb{F})|$$

and

$$|L_n(R)| = |\mathcal{N}(R)|^{n^2} |L_n(\mathbb{F})|.$$

Proof Note that, since R is a finite local ring, the nilradical equals the maximal ideal M. On the other hand, the determinant and the permanent are polynomial maps, in n^2 variables, in the entries of the matrix. So, the claim follows from Eq. (3). ■

4. Proof of the Main Theorem

We now have all the machinery needed to prove the main theorem.

Proof (of Theorem 1.1). As mentioned early it is a key observation that a bijective map $\Phi : M_n(R) \to M_n(R)$ satisfying $perA = det\Phi(A)$ for all matrices $A \in M_n(R)$ would induce a bijection from the set $S_n(R)$ onto the set $L_n(R)$. Therefore, to prove the theorem, it suffices to show that $|S_n(R)|$ does not equal $|L_n(R)|$.

Let R be a finite commutative ring. Suppose that R decomposes as $R = R_1 \oplus \ldots \oplus R_k$ in a direct sum of finite local rings. For $i = 1, \ldots, k$, let $\mathbb{F}_i = R_i/\mathcal{N}(R_i)$ be the residue field of each local ring R_i. From Theorem 3.1 and Theorem 3.2 we obtain

$$|S_n(R)| = \prod_{i=1}^{k} |S_n(R_i)| = \prod_{i=1}^{k} |\mathcal{N}(R_i)|^{n^2} |S_n(\mathbb{F}_i)|. \tag{5}$$

Note that an element r in the ring $R = R_1 \oplus \ldots \oplus R_k$ is nilpotent if and only if its projection in each local ring R_i is nilpotent. Since the nilradical of a commutative ring is the set of all nilpotent elements it follows that

$$\mathcal{N}(R) = \mathcal{N}(R_1) \oplus \ldots \oplus \mathcal{N}(R_k),$$

and thus $|\mathcal{N}(R)| = \prod_{i=1}^{k} |\mathcal{N}(R_i)|$. From this identity and Eq. (5) we conclude

$$|S_n(R)| = |\mathcal{N}(R)|^{n^2} \prod_{i=1}^{k} |S_n(\mathbb{F}_i)|. \quad (6)$$

Similarly, we also have

$$|L_n(R)| = |\mathcal{N}(R)|^{n^2} \prod_{i=1}^{k} |L_n(\mathbb{F}_i)|. \quad (7)$$

Now we make use of the results obtained in [8], where it was proven that for $n \geq 3$ and any finite field $\mathbb{F}$, with characteristic different from two, the number of zeros of the permanent function is strictly less than the number of zeros of the determinant function, that is, $|P_n(\mathbb{F})| < |D_n(\mathbb{F})|$.

From Eq. (4) we conclude that $|P_n(\mathbb{F})| < |D_n(\mathbb{F})|$ is equivalent to

$$|S_n(\mathbb{F})| < |L_n(\mathbb{F})|. \quad (8)$$

Finally, combining Eq. (8) with Eqs. (6) and (7), we obtain

$$|S_n(R)| < |L_n(R)|.$$

Hence, $|S_n(R)|$ does not equal $|L_n(R)|$ and the claim follows. ■

Remark. As it is well-known, see for example [16], if $\mathbb{F}$ is a finite field of order q, the general linear group $GL(n, \mathbb{F})$ of degree n over the field $\mathbb{F}$ has order

$$(q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{n-1}).$$

Our method leads us to generalize this fact to finite commutative rings. Over a commutative ring R, a matrix is invertible if and only if its determinant is a unit in R, [16]. So, let R be a finite commutative ring and assume that R decomposes as $R = R_1 \oplus \dots \oplus R_k$, in a direct sum of local rings. If $GL(n, R)$ denotes the general linear group of degree n over the finite ring R, we have from Eq. (6) that

$$|GL(n, R)| = |\mathcal{N}(R)|^{n^2} \prod_{i=1}^{k} |GL(n, \mathbb{F}_i)|,$$

where $\mathbb{F}_i$, for every $i = 1, \dots, k$, is the residue field of the local ring R_i.

5. Conclusion

For a finite commutative ring R with identity we proof that no bijective map

$$\Phi : M_n(R) \to M_n(R)$$

can transform the permanent into determinant. This extends to finite commutative rings the results obtained in [6] for finite fields.

For the proof, we have used the structure of finite commutative rings to reduce the problem into finite local rings. Another tool in the proof was radical theory, namely the nilradical.

The method we introduced allowed us to estimate the order of the general linear group of degree n over a finite commutative ring, which generalizes the known result to finite fields.

Abbreviations

M_n	Matrice of order n
det	Determinant
per	Permanent
S_n	Symmetric group of degree n
$Sgn(\sigma)$	Sign of permutation σ
$Char$	Charateristic
$U(R)$	The set of all units in the ring R
$Z(R)$	The set of all zero in the ring R
$\mathcal{N}(R)$	Nilradical of a ring R
$\|S\|$	The number of elements in finite set S
$\mathbb{F} = R/M$	Residue field of the local ring R
$L_n(R)$	The set of all matrices with unit permanent of R
$S_n(R)$	The set of all matrices with unit determinant of R
$P_n(R)$	The set of all matrices with zero divisor permanent of R
$D_n(R)$	The set of all matrices with zero divisor determinant of R
$Eq.$	Equation
$Eqs.$	Equations
$GL(n, \mathbb{F})$	The general linear group of degree n over the field $\mathbb{F}$
$M_n(R)$	Matrices of order n in R

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Conflicts of Interest

The author declares no conflicts of interest.

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Biography

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Research Field

Abrantes Malaquias Belo Caiúve: Graph Theory, Analytic Number Theory, Abstract Algebra.