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# Recursive Kernel Density Estimators Under Censoring Verifying an $\beta$-mixing Dependence Structure

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**Abstract :** In this paper, we consider the nonparametric recursive kernel density estimator on a compact ensemble when observations are censored and $\beta$-mixing. In this type of model, it is widely recognized that the traditional empirical distribution does not allow the densities F and G to be efficiently evaluated. Thus, Kaplan and Meier suggested a consistent estimator of $G_n$ to properly estimate $G$. Let $\{T_k, k \geq 1\}$ be a strictly stationary sequence of random variables distributed as $T$. We aims to establish a strong uniform consistency on a compact set with a rate of recursive kernel estimator of the underlying density function $f$ when the random variable of interest $T$ is right censored by another $C$ variable. In censoring, the observation is only partially known, which means that there are only the n pairs $(Y_i, \delta_i)$, $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$, where $\mathbb{I}_A$, where the indicator function for event $A$. Firstly, we propose the uniform convergence of this recursive estimator towards the density $f$. Then, we showed the veracity of our results by establishing all the necessary proofs. In other words we will prove our main result by establishing three lemmas. And finally we validated our theoretical results with a simulation study.

**Keywords :** Censored Data, Kernel Estimator, Density Function, Sure Convergence, $\beta$-mixing

## 1. Introduction

Let be $\{T_n, n \geq 1\}$ a sequence of positive and continuous random variables (lifetime) defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. It refers to the time spent until the occurrence of a specific event, commonly referred to as "death", which corresponds to a statutory change (usually a change from "alive" to "deceased")). In probability theory, most of the conclusions we have about random variables are generally only applicable to autonomous random variables. However, numerous concrete cases demonstrate that this postulate of independence would not be realistic. For example, if we look at air pollution in the city of Saint Louis, the pollution rate observed during the month will be significantly lower.

This information therefore induces a certain form of dependence, illustrated in mathematics by dependent data. The study of weak dependence, which is modeled by various notions, notably includes the notion of $\beta$ mixture. Volkonskii and Rozanov [19] introduced it as a dependence structure, mainly for pragmatic motivations.

For any two $\sigma$-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$ define the following measure of dependence :

$$\beta(\mathcal{A}, \mathcal{B}) = \sup \left\{ \frac{1}{2} \sum_{i=1}^{I} \sum_{i=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \right\} \tag{1}$$

see [1]
where the latter supremum is taken over all pairs of finite partitions $(A_1, A_2, ..., A_I)$ and $(B_1, B_2, ..., B_J)$ of ? such that $A_i \in \mathcal{A}$ for each $i$ and $B_j \in \mathcal{B}$ for each $j$.

Now suppose $T := (T_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on $(\Omega, \mathcal{F}, \mathcal{P})$. For the given random sequence $T$, for any positive integer $n$, define the dependence coefficient, define the dependence coefficient.

$$\beta(n) = \beta(T, n) = \sup_{j\in\mathbb{Z}} \beta(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \tag{2}$$

We say that the sequence $(T_k)$ is $\beta$ mixing if the mixing coefficient $\beta(n) \to 0$ when $n \to \infty$.

Currently, attention has become more focused on the specification of nonparametric estimates, models where the predictor does not come in a pre-established form, but is developed from information gleaned from the data. Parzen [15] and Rosenblatt [14] introduced kernel density estimators (KDE), which are frequently found in various scientific applications, especially in the medical field Reviews, epidemiology, decision-making theory and forecasting, genetics as an instrument of exploration.

In many cases, the full $T_1, \ldots, T_n$ data are not available because we do not have all the information. Suppose that the censoring instants $C_1 \ldots C_n$ are independent identically distributed (i.i.d.) and independent of $T_i; i = 1; 2; \ldots; n$ of distribution function $G$ unknown. Among the various forms of data where there is partial information, mainly on censorship and truncation. Right censoring is the most commonly observed case in survival analysis and has appeared extensively in the literature ([4, 7]), [6].

In censoring, the observation is only partially known, which means that there are only the n pairs $(Y_i, \delta_i)$, $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{1}_{\{T_i \leq C_i\}}$, where $\mathbb{1}_A$, where the indicator function for event $A$. For a long time, statisticians were wondering how to identify non-parametric estimation and inference techniques for f and its mode $\theta$ within the n actually observed pairs $(Y_i, \delta_i)$ ? We know that in this type of model, the traditional empirical distribution does not make it possible to effectively evaluate the densitie $G$. Therefore, Kaplan and Meier[9] proposed consistent estimator $G_n$ for $G$ (see (4)).

Their work presents important results regarding the estimation of the nonparametric survival function for right-censored random variables.

This paper looks at the almost sure uniform convergence of a non-parametric density function estimator based on a recursive kernel estimator with censored data and $\beta$-mixing. Thus, the estimator can be updated with each additional additional observation. These recursive characteristics present multiple storage benefits : their application, their interpretation, their calculation do not require a considerable amount of data. In certain specific circumstances, they also appear to perform better than traditional methods.

We design a stochastic algorithm that approximates the function $f$ at a specific point $t$, while determining the zero of the function $g : y \to f(t) - y$. According to the Robbins-Monro model (see [13]), we set $\hat{f}_0(t) \in \mathbb{R}$ and for all $n \geq 1$, $\hat{f}_n(t) = \hat{f}_{n-1}(t) + \gamma_n W_n(t)$ where $\gamma_n$ is not random positive sequence tending towards zero when $n$ tends towards infinity.

To determine $W_n$ at a point $t$, we follow [16–18]. Going in this direction we refer to the reader [2, 3, 5, 8, 10, 11]. We integrate the kernel $K$ (i.e. a function which satisfies $\int_{\mathbb{R}} K(x)dx = 1$) and the bandwidth $h_n$ (i.e. say a sequence of positive real numbers which lie towards zero). Based on [18], we substitute $\hat{\pi}_n^{-1}$ with $\bar{G}_n^{-1}$ and we obtain.

$$W_n = h_n^{-1} \delta_n \bar{G}_n^{-1} K(h_h^{-1}(t - Y_n)) - f_{n-1}(t) \tag{3}$$

Where $\bar{G}_n = 1 - G_n$ and $G_n$ is the Kaplan-Meier estimator associated with it has the expression :

$$G_n(t) = \begin{cases} \prod_{i=1}^{n} \left[1 - \dfrac{1 - \delta_{(i)}}{n - i + 1}\right]^{\mathbb{I}\{Y_{(i)} \leq t\}}, & \text{if} \quad t < Y_{(n)} \\ 0 & \text{if} \quad t \geq Y_{(n)} \end{cases} \tag{4}$$

Subsequently, we consider in this article the recursive estimator of the function $f$ at the point $t$ indicated by the following relation

$$\hat{f}_n(t) = (1 - \gamma_n) \hat{f}_{n-1}(t) + \gamma_n \delta_n \bar{G}_n^{-1} K\left(h_n^{-1}(t - Y_n)\right). \tag{5}$$

We assume that $\tilde{f}_0(x) = 0$ and $Q_n = \prod_j^n (1 - \gamma_j)$. We suggest examining the following estimator of $f$ at the point.(see[18])

$$\hat{f}_n(t) = Q_n \sum_{k=1}^{n} Q_k^{-1} \gamma_k \delta_k h_k^{-1} \bar{G}_n^{-1}(Y_k) K\left(\frac{t - Y_k}{h_k}\right). \tag{6}$$

When $G$ is known, the recursive pseudo-estimator $\tilde{f}_n$ of $f$ estimates the common density well lifespans.

$$\tilde{f}_n(t) = Q_n \sum_{k=1}^{n} Q_k^{-1} \gamma_k \delta_k h_k^{-1} \bar{G}^{-1}(Y_k) K\left(\frac{t - Y_k}{h_k}\right). \tag{7}$$

## 2. Hypotheses and Main Results

To establish our results we will need these classic hypotheses :

$\mathbf{M_1}$. $\{T_i, i \geq 1\}$ is a strictly stationary sequence of random variable $\beta$- mixing with a common distribution function $F$ which has a probability density function $f$, admitting finite second moments.

$\mathbf{M_2}$. The censoring times $\{C_i, 1 \leq i \leq n\}$ are i.i.d with distribution function $G$, and are independent of $\{T_i, i \geq 1\}$.

$\mathbf{K}$. $K$ is a Lipshitz density with true compact support satisfying $\int uK(u)du = 0$.

$\mathbf{D_1}$. The density $f(.)$ is twice continuously differentiable on $[0, \tau]$, where $\tau < \tau_F = \inf\{t, F(t) = 1\}$

$\mathbf{D_2}$. The coefficient of $\beta$-mixture of $T_i$ verifies $\beta(n) = \mathcal{O}(n^{-\nu})$ for all $\nu > 4$ :

$\mathbf{H}$. The smoothing parameter $h = h_n$ is such that $h \to 0$, $nh_n \to \infty$, $n^{3/2-\nu}h^{-3/2} \to 0$ and $n^{1/2-\nu}h^{-3/2} \to 0$

$i)\gamma_n \in \mathcal{G}S(-\alpha)$ with $\alpha \in (1/2, 1]$

$ii)h_n \in \mathcal{G}S(-a)$ with $a \in (0, 1)$

we recall that a sequence $(v)_n$ is said to have regular variation and we note $(v_n) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n\to\infty} n\left[1 - \frac{v_{n-1}}{v_n}\right] = \gamma$$

To facilitate monitoring of the main results presented in this document, we highlight that under hypotheses i) and ii), we have

$$Q_n \sum_{k=1}^{n} Q_k^{-1}\gamma_k = 1 + o(1), \qquad Q_n \sum_{k=1}^{n} Q_k^{-1}\gamma_k h_k^2 = \mathcal{O}\left(h_n^2\right) \quad et \quad Q_n^2 \sum_{k=1}^{n} Q_k^{-2}\gamma_k^2 h_k^{-1} = \mathcal{O}\left(\gamma_n h_n^{-1}\right)$$

The uniform convergence p.s from $\hat{f}_n$ to $f$ is given by the following theorem.

*Theorem* 2.1. Under slighty restrictive hypothesis on the mixing coefficient, the kernel $K$ and the density $f$ $M_1, M_2, K, D_1$ and $D_2$ and $H$

$$\sup_{t\in[0,\tau]} |\hat{f}_n(t) - f(t)| = \mathcal{O}\left\{\max\left(\sqrt{\frac{\log n}{nh_n}}, h_n^2\right)\right\} \quad a.s \quad \text{as} \quad n \to \infty \tag{8}$$

The proof of Theorem 2.1 is based on the following lemmas, before enoucing those lemma, let's give this proposition, This propisition is an adaptation of the Noureddine Rhomari [12] theorem in the case where $Y_1....Y_n$ are in $\mathbb{R}$ and $p = 1$, we will use it for established the follow lemma 2.1.

*Proposition* 2.1. Let $Y_t, 1 \leq t \leq n$ be a sequence of real centered random variables, with coefficient $\beta - mixing$, and for $n \geq 2$, $1 \leq i \leq 2r$, $|Y_i| < M$ and $\mathbb{E}(|U_i|)^2) \leq \sigma_i^2 = Q_i^{-2}\gamma_i^2\mathcal{O}(h_i^{-1})$ we have

$$\mathbb{P}\left(\sum_{t=1}^{n} |Y_t| > \varepsilon\right) \leq 4exp\left(-\frac{\varepsilon^2}{4[2\tilde{\sigma}_r^2 + \varepsilon M/3]}\right) + (n+2)\beta(1) \tag{9}$$

where

$\tilde{\sigma}_r^2 = \max\left(\sum_{i=1}^{r} \sigma_{2i}^2, \sum_{i=1}^{r} \sigma_{2i-1}^2\right)$ and $r = r_n$ is a sequence of positive numbers, defined by

$$r = \left[\frac{n}{2}\right] + 1$$

[.] being the whole part.

*Lemma* 2.1. Under Assumptions $K$, $D_1$ et $H$

$$\sup_{t\in[0,\tau]} \left|\mathbb{E}\tilde{f}_n(t) - f(t)\right| = \mathcal{O}\left(h_n^2\right) \tag{10}$$

*Lemma* 2.2. Under Assumptions $K$,$D_1$ and $H$

$$\sup_{t\in[0,\tau]} |\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh_n}}\right) \quad a.s \quad \text{when} n \to \infty \tag{11}$$

*Lemma* 2.3. Under Assumptions *D1*, *D2*, *K* et *H*

$$\sup_{t\in[0,\tau]} |\hat{f}_n(t) - \tilde{f}_n(t)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}} h_n^2\right) \quad a.s \tag{12}$$

# 3. Proofs

### *3.1. Proof of Lemma 2.1*

$$\begin{aligned}
\mathbb{E}\tilde{f}_n(t) &= \mathbb{E}\left[Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k\delta_k h_k^{-1}\bar{G}^{-1}(Y_k)K\left(\frac{t-Y_k}{h_k}\right)\right]\\
&= Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k\mathbb{E}\left[\delta_k\bar{G}^{-1}(Y_k)h_k^{-1}K\left(\frac{t-T_k}{h_k}\right)\right]
\end{aligned}$$

with

$$\begin{aligned}
\mathbb{E}\left[\bar{G}^{-1}(Y_k)\delta_k h_k^{-1}K\left(\frac{t-Y_k}{h_k}\right)\right] &= h_k^{-1}\mathbb{E}\left[\mathbb{E}\left(\bar{G}^{-1}(Y_k)\delta_k K\left(\frac{t-Y_1}{h_k}\right)|T_k\right)\right]\\
&= h_k^{-1}\mathbb{E}\left[\mathbb{E}\left(\mathbb{1}_{\{T_k\le C_k\}}\bar{G}^{-1}(Y_k)K\left(\frac{t-Y_1}{h_k}\right)\right)|T_k\right]\\
&= h_k^{-1}\mathbb{E}\left[\bar{G}^{-1}(T_k)K\left(\frac{t-T_1}{h_k}\right)\mathbb{E}\left(\mathbb{1}_{\{T_k\le C_k\}}|T_1\right)\right]\\
&= h_k^{-1}\mathbb{E}\left[\bar{G}^{-1}(T_k)K\left(\frac{t-T_1}{h_k}\right)\mathbb{P}\left(T_k\le C_k\right)\right]\\
&= h_k^{-1}\mathbb{E}\left[\bar{G}^{-1}(T_k)K\left(\frac{t-T_1}{h_k}\right)\bar{G}(T_k)\right]\\
&= h_k^{-1}\int K\left(\frac{t-t_1}{h_k}\right)f(t_1)dt_1\\
&= \int K(z)f(t-h_kz)dz
\end{aligned}$$

By setting $z=\dfrac{t-t_1}{h}$. We use a Taylor expansion to order 2 so *Taylor* to order 2 so.
If $\mathbf{D1}$ is verified and ($t^*$ being between $t$ and $t-hu$) then,

$$f(t-h_kz) = f(t)-h_kzf'(t)+\frac{h_k^2}{2}z^2f''(t^*)$$

therefore

$$\begin{aligned}
\mathbb{E}\left[\bar{G}^{-1}(Y_k)\delta_k h_k^{-1}K\left(\frac{t-Y_k}{h_k}\right)\right] &= \int K(z)\left[f(t)-h_kzf'(t)+\frac{h^2z^2}{2}f''(t^*)\right]dz\\
&= \int K(z)f(t)dz-h_kf'(t)\int zK(z)dz+\frac{h_k^2f''(t^*)}{2}\int z^2K(z)dz\\
&\le f(t)+\frac{h_k^2F(\tau)}{2}\sup_{t\in[0,\tau]}|f''(t)|\int z^2K(z)dz\\
&\le f(t)+\frac{h_k^2}{2}\sup_{t\in[0,\tau]}|f''(t)|\int z^2K(z)dz\\
&\le f(t)+Ch_k^2 \quad with \quad C=\frac{F(\tau)}{2}\sup_{t\in[0,\tau]}|f''(t)|\int z^2K(z)dz
\end{aligned}$$

Thus

$$\mathbb{E}\left[\bar{G}^{-1}(Y_k)\delta_k h_k^{-1}K\left(\frac{t-Y_k}{h_k}\right)\right]\le f(t)+Ch_k^2$$

Under the hypotheses $H$, we have

$$\begin{aligned}\mathbb{E}\tilde{f}_n(t) &= Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k\mathbb{E}\left[\bar{G}^{-1}(Y_k)\delta_k h_k^{-1}K\left(\frac{x-Y_k}{h_k}\right)\right]\\ &\leq Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k\left(f(t)+Ch_k^2\right)\\ &\leq Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k f(t)+Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k Ch_k^2\\ &\leq (1+o(1))f(t)+\mathcal{O}\left(h_k^2\right)\\ &\leq f(t)+\mathcal{O}\left(h_k^2\right)\end{aligned}$$

Then

$$\mathbb{E}\tilde{f}_n(t)-f(t)=\mathcal{O}\left(h_k^2\right)$$

### *3.2. Proof Lemma 2.2*

To establishe the convergence of $\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t)$ let's calculate first

$$\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t)=Q_n\sum_{k=1}^{n}Q_k^{-1}\gamma_k\left(Z_k(x)-\mathbb{E}\left[Z_k(x)\right]\right)$$

where $Z_k(x)=h_k^{-1}\bar{G}^{-1}(Y_n)\delta_k K\left(\dfrac{x-Y_n}{h_k}\right)$

Let $[0,\tau]$ be compact, it can be covered by a finite number $q_n$ of intervals $I_j$ of center $t_j^*$ ;$1\leq j\leq q_n$, and half length $a_n=\sqrt{\dfrac{h^3}{n}}$, soit $I_j=\left[t_j^*-\sqrt{\dfrac{h^3}{n}};t_j^*+\sqrt{\dfrac{h^3}{n}}\right]$.

$[0,\tau]$ being bounded, there exists a constant $c_1>0$ such that $q_n\leq c_1\sqrt{\dfrac{n}{h^3}}$ (Indeed,

$l([0,\tau])=2q_na_n=2q_n\sqrt{\dfrac{h^3}{n}}\leq 2c_1\sqrt{\dfrac{n}{h^3}}\sqrt{\dfrac{h^3}{n}}=2c_1)$

by posing

$$M_k(t)=Q_k^{-1}\gamma_k\left(Z_k(x)-\mathbb{E}\left[Z_k(x)\right]\right)$$

we have

$$\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t)=Q_n\sum_{k=1}^{n}M_k(t),$$

which we break down as follows

$$\begin{aligned}Q_n\sum_{k=1}^{n}M_k(t) &= \left\{\left[\tilde{f}_n(t)-\tilde{f}_n(t_j^*)\right]-\left[\mathbb{E}\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t_j^*)\right]\right\}+\left[\tilde{f}_n(t_j^*)-\mathbb{E}\tilde{f}_n(t_j^*)\right]\\ &= Q_n\sum_{k=1}^{n}\tilde{M}_k(t)+Q_n\sum_{k=1}^{n}\tilde{M}_k(t_j^*)\end{aligned}$$

Then

$$\sup_{t\in[0,\tau]}\left|Q_n\sum_{k=1}^{n}M_k(t)\right|\leq\max_{1\leq j\leq q_n}\sup_{t\in I_j}\left|Q_n\sum_{k=1}^{n}\tilde{M}_k(t)\right|+\max_{1\leq j\leq q_n}\left|Q_n\sum_{k=1}^{n}\tilde{M}_k(t_j^*)\right|=A_1+A_2$$

On the other hand

$$\begin{aligned}
\left|Q_n\sum_{k=1}^{n}\tilde{M}_k(t)\right| &= \left|\tilde{f}_n(t)-\tilde{f}_n(t_j^*)-\mathbb{E}\left[\tilde{f}_n(t)-\tilde{f}_n(t_j^*)\right]\right|\\
&\leq Q_n\sum_{k=1}^{n}Q_k^{-1}\delta_k\bar{G}^{-1}(Y_k)\gamma_k h_k^{-1}\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\\
&\quad+\mathbb{E}\left[Q_n\sum_{k=1}^{n}Q_k^{-1}\delta_k\bar{G}^{-1}(Y_k)\gamma_k h_k^{-1}\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\right]\\
&\leq Q_n\sum_{k=1}^{n}Q_k^{-1}\delta_k\bar{G}^{-1}(Y_k)\gamma_k h_k^{-1}\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\\
&\quad+Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-1}\gamma_k\mathbb{E}\left[\delta_k\bar{G}^{-1}(Y_k)\left|K\left(\frac{t-Y_k}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\right]\\
&\leq B_1(t)+B_2(t)
\end{aligned}$$

with

$$\begin{aligned}
\sup_{t\in I_j}B_1(t) &= \sup_{t\in I_j}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-1}\bar{G}^{-1}(Y_n)\gamma_k\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\\
&= Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-1}\sup_{t\in I_j}\delta_k\gamma_k\bar{G}^{-1}(Y_n)\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\\
&\leq \frac{1}{\bar{G}(\tau)}Q_n\sum_{i=1}^{n}Q_k^{-1}h_k^{-1}\gamma_k\sup_{t\in I_j}\left|K\left(\frac{t-Y_i}{h_k}\right)-K\left(\frac{t_j^*-Y_i}{h_k}\right)\right|\\
&\leq \frac{1}{\bar{G}(\tau)}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-1}\gamma_k\frac{\lambda|t-t_j^*|}{h_k}\quad K\ being\ lipschitzian\ .Moreover\ \ t\in I_j\Rightarrow|t-t_j^*|\leq 2a_n\\
&\leq \frac{2\lambda a_n}{\bar{G}(\tau)}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-2}\gamma_k\\
&= \frac{2\lambda}{\bar{G}(\tau)}\sqrt{\frac{h_n^3}{n}}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-2}\gamma_k\\
&= \frac{2\lambda}{\bar{G}(\tau)\sqrt{nh_n^{-3}}}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{-2}\gamma_k\\
&\leq \frac{2\lambda}{\bar{G}(\tau)\sqrt{nh_n^{-3}}}Q_n\sum_{k=1}^{n}Q_k^{-1}h_k^{2}\gamma_k\\
&\leq \frac{2\lambda}{\bar{G}(\tau)\sqrt{nh_n^{-3}}}\mathcal{O}\left(h_n^2\right)\\
&\leq C\left(\frac{1}{\sqrt{nh^{-7}}}\right)\qquad with\ \ C=\frac{2\lambda c}{\bar{G}(\tau)}\ \ and\ \ \lambda\ \ being\ \ the\ \ Lipschitz\ \ constant\\
&= \mathcal{O}\left(\frac{1}{\sqrt{nh^{-7}}}\right)
\end{aligned}$$

Then

$$\sup_{t\in I_j}B_1(t)=\mathcal{O}\left(\frac{1}{\sqrt{nh^{-7}}}\right)$$

In the same way arguments similar to the above give :

$$\sup_{t\in I_j} B_2(t) = \mathcal{O}\left(\frac{1}{\sqrt{nh^{-7}}}\right)$$

Which leads to

$$A_1(t) = \max_{1\leq j\leq q_n} \sup_{t\in I_j} \left| Q_n \sum_{k=1}^{n} \tilde{M}_k(t) \right| = \mathcal{O}\left(\frac{1}{\sqrt{nh^{-7}}}\right)$$

For the study of $A_2$, we will use the technique developed in [6].

For $t \in [0, \tau]$, we apply proposition 2.1 to this sequence of random variables $\{U_i, i \in \mathbb{N}\}$, with coefficient $\beta$-mixing, checking for everything $n \in \mathbb{N}$, $|U_i| < M, 1 \leq i \leq n$.

Let's put

$$U_i(t_k^*) = Q_i^{-1}\gamma_i \left(Z_i(t_k^*) - \mathbb{E}[Z_i(t_k^*)]\right) \quad and \quad Z_i(t_k^*) = \delta_i \bar{G}^{-1}(Y_i) h_i^{-1} K\left(\frac{t_k^* - Y_i}{h_i}\right)$$

$$\begin{aligned}
\mathbb{E}(|U_k|)^2) &= \mathbb{E}\left(Q_k^{-1}\gamma_k \left(Z_k(t_k^*) - \mathbb{E}[Z_k(t_k^*)]\right)\right)^2 \\
&= Q_k^{-2}\gamma_k^2 \mathbb{E}\left(\left(Z_k(t_k^*) - \mathbb{E}[Z_k(t_k^*)]\right)\right)^2 \\
&= Q_k^{-2}\gamma_k^2 \mathbb{E}\left(Z_k(t_k^*)^2 + \mathbb{E}[Z_k(t_k^*)]^2 - 2Z_k(t_k^*)\mathbb{E}[Z_k(t_k^*)]\right) \\
&= Q_k^{-2}\gamma_k^2 \left(\mathbb{E}(Z_k(t_k^*)^2) + \mathbb{E}[Z_k(t_k^*)]^2 - 2\mathbb{E}(Z_k(t_k^*)\mathbb{E}[Z_k(t_k^*)])\right) \\
&= Q_k^{-2}\gamma_k^2 \left(\mathbb{E}(Z_k(t_k^*)^2) - \mathbb{E}[Z_k(t_k^*)]^2\right) \\
&= Q_k^{-2}\gamma_k^2 Var(Z_k(t_k^*))
\end{aligned}$$

$$\begin{aligned}
Var(Z_k(t_k^*)) &= \mathbb{E}[Z_k(t_k^*)]^2 - \mathbb{E}^2[Z_k(t_k^*)] \\
&\leq \mathbb{E}[Z_k(t_k^*)]^2 \\
&\leq \mathbb{E}\left[\delta_k \bar{G}^{-1}(Y_k) h_k^{-1} K\left(\frac{t_k^* - Y_k}{h_k}\right)\right]^2 \\
&\leq h_k^{-2}\mathbb{E}\left[\mathbb{E}\left[\left(K\left(\frac{t_k^* - Y_k}{h_k}\right)\frac{\nVdash_{\{T_k\leq C_k\}}}{\bar{G}^{-1}(Y_k)}\right)^2 |T_k\right]\right] \\
&\leq h_k^{-2}\mathbb{E}\left[\left(K\left(\frac{t_k^* - T_k}{h_k}\right)\frac{1}{\bar{G}(T_k)}\right)^2 \mathbb{E}\left[\nVdash_{\{T_k\leq C_k\}}|T_k\right]\right] \\
&\leq h_k^{-2}\mathbb{E}\left[\left(K\left(\frac{t_k^* - T_k}{h_k}\right)\frac{1}{\bar{G}(T_k)}\right)^2 \mathbb{P}(T_k \leq C_k)\right] \\
&\leq h_k^{-2}\mathbb{E}\left[\left(K\left(\frac{t_k^* - T_k}{h_k}\right)\frac{1}{\bar{G}(T_k)}\right)^2 \mathbb{P}(T_k \leq C_k)\right] \\
&\leq h_k^{-2}\mathbb{E}\left[\left(K\left(\frac{t_k^* - T_k}{h_k}\right)\right)^2 \frac{1}{\bar{G}(T_k)}\right] \\
&\leq h_k^{-2}G(\tau)^{-1}\mathbb{E}\left[K^2\left(\frac{t_k^* - T_k}{h_k}\right)\right] \\
&\leq h_k^{-1}\bar{G}(\tau)^{-1}\int K^2(z)f(t - h_k z)dz \\
&= \mathcal{O}(h_k^{-1})
\end{aligned}$$

thus

$$\begin{aligned}
\mathbb{E}(|U_k|)^2) &= Q_k^{-2}\gamma_k^2\mathcal{O}(h_k^{-1}) \\
&\leq CQ_k^{-2}\gamma_k^2 h_k^{-1} = \sigma_k^2
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}\left(|\tilde{f}_n(t_k^*)-\mathbb{E}\tilde{f}_n(t_k^*)|>\varepsilon\right) &= \mathbb{P}\left(\left|Q_n\sum_{t=1}^{n}Q_t^{-1}\gamma_t\left(Z_t(t_k^*)-\mathbb{E}[Z_t(t_k^*)]\right|>\varepsilon\right)\\
&= \mathbb{P}\left(\left|Q_n\sum_{t=1}^{n}U_t\right|>\varepsilon\right)\\
&\le \mathbb{P}\left(\sum_{t=1}^{n}|U_t|>\varepsilon Q_n^{-1}\right)\\
&\le 4exp\left(-\frac{\varepsilon^2Q_n^{-2}}{4[2\tilde{\sigma}_r^2+\varepsilon Q_n^{-1}M/3]}\right)+(n+2)\beta(1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le q_n}\left|\tilde{f}_n(t_k^*)-\mathbb{E}\tilde{f}_n(t_k^*)\right|>\varepsilon\right) &\le \sum_{k=1}^{q_n}\mathbb{P}\left(\left|\sum_{t=1}^{n}U_t(t_k^*)\right|>\varepsilon Q_n^{-1}\right)\\
&\le 4q_nexp\left(-\frac{\varepsilon^2Q_n^{-2}}{4[2\tilde{\sigma}_r^2+\varepsilon Q_n^{-1}M/3]}\right)+q_n(n+2)\beta(1)
\end{aligned}
$$

Let $\varepsilon=\varepsilon_0\sqrt{\dfrac{\log n}{nh}}$ then $4q_nexp\left(-\dfrac{\varepsilon_0^2\frac{\log n}{nh}Q_n^{-2}}{4[2\tilde{\sigma}_r^2+\varepsilon_0\sqrt{\frac{\log n}{nh}}Q_n^{-1}M/3]}\right)$

$$
\begin{aligned}
q_nexp\left(-\frac{\varepsilon_0^2\frac{\log n}{nh}Q_n^{-2}}{4[2\tilde{\sigma}_r^2+\varepsilon_0\sqrt{\frac{\log n}{nh}}Q_n^{-1}M/3]}\right) &= q_nexp\left(-\frac{\varepsilon_0^2\log nQ_n^{-2}}{8\tilde{\sigma}_r^2nh+4\varepsilon_0(\log n)^{1/2}(nh)^{1/2}Q_n^{-1}M/3]}\right)\\
&= q_nexp\left(-\frac{\varepsilon_0^2\log nQ_n^{-2}}{8\tilde{\sigma}_r^2nh+4\varepsilon_0(n\log n)^{1/2}h^{1/2}Q_n^{-1}M/3]}\right)\\
&= q_nexp\left(-\frac{\varepsilon_0^2Q_n^{-2}}{8\tilde{\sigma}_r^2(\log n)^{-1}nh+4\varepsilon_0(n\log n)^{-1/2}h^{1/2}Q_n^{-1}M/3]}\right)\\
&= q_nexp\left(-\frac{\varepsilon_0^2Q_n^{-2}}{8\tilde{\sigma}_r^2(\frac{\log n}{n})^{-1}h+4\varepsilon_0(n\log n)^{-1/2}h^{1/2}Q_n^{-1}M/3]}\right)\\
&\to 0
\end{aligned}
$$

So

$$
4q_nexp\left(-\frac{\varepsilon_0^2\frac{\log n}{nh}Q_n^2}{4[2\tilde{\sigma}_r^2+\varepsilon_0\sqrt{\frac{\log n}{nh}}Q_nM/3]}\right)\to 0 \qquad when \qquad n\longrightarrow\infty \tag{13}
$$

We will look at the case where $\sigma_{2i}^2\ge\sigma_{2i-1}^2$, by framing $\sum_{i=1}^{r}\sigma_{2i}^2\ge\sum_{i=1}^{r}\sigma_{2i-1}^2$ and

$$
\begin{aligned}
\tilde{\sigma}_r^2=\max\left(\sum_{i=1}^{r}\sigma_{2i}^2,\sum_{i=1}^{r}\sigma_{2i-1}^2\right)=\sum_{i=1}^{r}\sigma_{2i}^2 &= \sum_{i=1}^{r}Q_{2i}^{-2}\gamma_{2i}^2\mathcal{O}(h_{2i}^{-1})\\
&= \sum_{j=2}^{r}Q_j^{-2}\gamma_j^2\mathcal{O}(h_j^{-1})=Q_r^{-2}Q_r^2\sum_{j=2}^{r}Q_j^{-2}\gamma_j^2\mathcal{O}(h_j^{-1})\\
&\le cQ_r^{-2}Q_r^2\sum_{j=1}^{n}Q_j^{-2}\gamma_j^2h_j^{-1}=cQ_r^{-2}\mathcal{O}(\gamma_rh_r^{-1})\\
&= \mathcal{O}(\gamma_rh_r^{-1})
\end{aligned}
$$

If $\sigma_{2i}^2 \le \sigma_{2i-1}^2$, by framing $\sum_{i=1}^r \sigma_{2i}^2 \le \sum_{i=1}^r \sigma_{2i-1}^2$ and

$$\begin{aligned}\tilde{\sigma}_r^2 = \max\left(\sum_{i=1}^r \sigma_{2i}^2, \sum_{i=1}^r \sigma_{2i-1}^2\right) = \sum_{i=1}^r \sigma_{2i-1}^2 &= \sum_{i=1}^r Q_{2i-1}^{-2}\gamma_{2i-1}^2 \mathcal{O}(h_{2i-1}^{-1}) \\ &= \sum_{j=1}^r Q_j^{-2}\gamma_j^1 \mathcal{O}(h_j^{-1}) = Q_r^{-2}Q_r^2 \sum_{j=1}^r Q_j^{-2}\gamma_j^2 \mathcal{O}(h_j^{-1}) \\ &\le cQ_r^{-2}Q_r^2 \sum_{j=1}^r Q_j^{-2}\gamma_j^2 h_j^{-1} = cQ_r^{-2}\mathcal{O}(\gamma_r h_r^{-1}) \\ &= \mathcal{O}(\gamma_r h_r^{-1})\end{aligned}$$

In conclusion

$$\tilde{\sigma}_r^2 = \mathcal{O}(\gamma_n h_r^{-1})$$

The nonincreasing of $\beta$ allows us

$$\begin{aligned} q_n(n+2)\beta(1) &\le q_n(n+2)\beta(n) \le c_1\sqrt{\frac{n}{h^3}}(n+2)\beta(n) \\ &\le c_1 c_2 \frac{n^{1/2}}{h^{3/2}}(n+2)n^{-\nu} \le c_1 c_2 (n^{3/2-\nu}h^{-3/2} + 2n^{1/2-\nu}h^{-3/2}) \to 0 \end{aligned} \tag{14}$$

Considering the fact 13 et 14, we obtain

$$A_2 = \max_{1\le j\le q_n}\left|Q_n \sum_{k=1}^n \tilde{M}_k(t_j^*)\right| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh_n}}\right).$$

In conclusion

$$\sup_{t\in[0,\tau]} |\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh_n}}\right)$$

### 3.3. *Proof of Lemma 2.3*

$$\begin{aligned} |\hat{f}_n(t) - \tilde{f}_n(t)| &= Q_n \sum_{k=1}^n Q_k^{-1}\delta_k \gamma_k h_k^{-1} K\left(\frac{t-Y_k}{h_k}\right)\left(\frac{1}{\bar{G}_k(Y_k)} - \frac{1}{\bar{G}(Y_k)}\right) \\ &\le Q_n \sum_{k=1}^n Q_k^{-1}\gamma_k h_k^{-1} K\left(\frac{t-Y_k}{h_k}\right)\frac{|\bar{G}_k(Y_i) - \bar{G}(Y_i)|}{\bar{G}_k(Y_i)\bar{G}(Y_i)} \\ &\le \frac{\sup_{y\in[0,\tau]}|\bar{G}_n(y) - \bar{G}(y)|}{\bar{G}_k(\tau)\bar{G}(\tau)} Q_n \sum_{k=1}^n Q_k^{-1}\gamma_k h_k^{-1} K\left(\frac{t-Y_k}{h_k}\right) \\ &\le \frac{\sup_{y\in[0,\tau]}|\bar{G}_n(y) - \bar{G}(y)|}{\bar{G}_k(\tau)\bar{G}(\tau)} Q_n \sum_{k=1}^n Q_k^{-1}\gamma_k h_k^2 K\left(\frac{t-Y_k}{h_k}\right) \\ &\le c_1 c_2 \sqrt{\frac{\log\log n}{n}} h_n^2 \\ &= \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}} h_n^2\right) \end{aligned}$$

Indeed

$$\sup_{y\in[0,\tau]} |\bar{G}_n(y)-\bar{G}(y)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}\right) \qquad p.s$$

According to the law of iterated logarithm (LIL) for censored data, in the case i.i.d. (see Deheuvels and Einmahl (2000)) and since K is a bounded function $\Rightarrow K\left(\frac{t-Y_k}{h_k}\right)\leq M$

$$\begin{aligned} Q_n\sum_{k=1}^{n} Q_k^{-1}\gamma_k h_k^2 K\left(\frac{t-Y_k}{h_k}\right) &\leq MQ_n\sum_{k=1}^{n} Q_k^{-1}\gamma_k h_k^2 \\ &\leq Mch_n^2 \\ &\leq c_2 h_n^2 \text{ with } c_2 = Mc \end{aligned}$$

### *3.4. Proof of Theorem 1*

The proof of the Theorem is based on the lemmas above and the triangular inequality allows us to write :

$$\begin{aligned} \sup_{t\in\Omega}\left|\hat{f}_n(t)-f(t)\right| &\leq \sup_{t\in\Omega}\left|\hat{f}_n(t)-\tilde{f}(t)\right| + \sup_{t\in\Omega}\left|\tilde{f}_n(t)-\mathbb{E}\left(\tilde{f}_n(t)\right)\right| + \sup_{t\in\Omega}\left|\mathbb{E}\left(\tilde{f}_n(t)\right)-f(t)\right| \\ &= S_1+S_2+S_3 \end{aligned}$$

where

$$\tilde{f}_n(t) = Q_n\sum_{k=1}^{n} Q_k^{-1}\gamma_k\delta_k h_k^{-1}\bar{G}^{-1}(Y_k)K\left(\frac{t-Y_k}{h_k}\right)$$

By applying these three lemmas, we obtain the result.

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# Conflicts of Interest

The authors declare no conflicts of interest.

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