

Stability of Quadratic Mappings in 2-Banach Spaces and Related Topics

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Abstract: Functional analysis is an important branch of mathematics widely used to study the stability of different functional equations. This includes the stability of various quadratic functional equations, which typically involve a specific number of variables to reach results more easily in this field. One of the methods used to study the stability of functional equations is the direct method, which is known for its simplicity in proving the stability of this type of functional equation. In this research paper, we have successfully proven the Hyers-Ulam-Rassias stability of the quadratic functional equation in 2-Banach spaces. Specifically, we have shown that the equation: $f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(x+z)$ holds true within this context. Our approach involved using either the usual or the direct method to establish this stability. Furthermore, we have also demonstrated the generalized Hyers-Ulam stability of the quadratic functional equation in 2-Banach spaces using the usual process by considering various conditions. This has led to many exciting results and revealed many related applications.

Keywords: Hyers-Ulam Stability, 2-Banach Spaces, Quadratic Mapping, Functional Equation, Usual Method

1. Introduction

Given a group (G, \circ) and a metric group (H, \star, d) with the metric $d(\cdot, \cdot)$, the question is whether there exists a $\delta = \delta(\epsilon) > 0$ such that if a mapping $F : G \rightarrow H$ satisfies the inequality:

$$d(f(x \circ y), f(x) \star f(y)) < \delta$$

for all $x, y \in G$, then a homomorphism $F : G \rightarrow H$ exists such that $d(f(x), F(x))$ for all $x \in G$?

In 1941, Hyers[1] provided the first (partial) affirmative answer to the question of Ulam for Banach spaces. This resulted in the introduction of the Hyers–Ulam stability.

The result of Hyers has been extended by Aoki [4] for approximate additive mappings and by Th. M. Rassias [5] for an approximate linear mapping following the difference

Cauchy equation

$$\|f(x+y) - f(x) - f(y)\| \text{ to be controlled by } \epsilon(\|x\|^p + \|y\|^p)$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) \quad (1)$$

Is related to a symmetric bi-additive function and is called a quadratic functional equation. Every solution of the quadratic (1) is referred to as a quadratic mapping. Skof [13] proved the Hyers-Ulam stability of the quadratic functional equation (1).

The concept of linear 2-normed spaces was initially formulated by Gahler [6] in the mid-1960s, while the study of 2-Banach spaces was subsequently undertaken by Gahler [7] and White [8].

The functional equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z) \quad (2)$$

was shown to be another quadratic mapping by Skof [13], who proved the Hyers-Ulam stability of the quadratic functional equation (2). Additionally, Jung [3] demonstrated the Hyers-Ulam stability of the same quadratic functional equation (2) in a Banach space, subject to certain conditions.

In 1978, Rassias [5] extended the Hyers-Ulam stability by considering variables. It was also generalized to the function case by Gavruta.

Hensel [14] introduced a normed space without the Archimedean property, and Rassias[5] proved the generalized Hyers-Ulam stability of the additive functional equation and the quadratic functional equation in non-Archimedean spaces.

Yun[15] proved the stability of the Cauchy functional equation, the Jensen functional equation, and another type of quadratic functional equation in 2-Banach spaces under certain conditions.

Recent positive developments have occurred in the study of the stability of quadratic-functional equations. For example, Zenada[16] utilized the difference method to establish the stability of such equations in A non-Archimedean 2-Banach spaces.

Using the fixed-point method, Pasupathi[17] contributed to proving the stability of some quadratic equations in Banach spaces.

In this paper, we present compelling evidence demonstrating the stability of the quadratic functional equation in (2) in 2-Banach spaces when subjected to approximately even or odd conditions. Furthermore, we will delve into the asymptotic behaviors of quadratic and additive mappings, ultimately extending the stability of the same functional equation in 2-Banach spaces.

2. Preliminaries

In the 1960's Gahler [2],[6] introduced the concept of linear 2-normed spaces.

Definition 2.1. [6] Let X be a real linear space with $\dim X > 1$ and let

$\|x, y\| : X \times X \rightarrow [0, \infty[$ be a function satisfying the following properties:

- (a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (b) $\|x, y\| = \|y, x\|$
- (c) $\|\alpha x, y\| = |\alpha| \|x, y\|$
- (d) $\|x, (y + z)\| \leq \|x, y\| + \|x, z\|$

Lemma 3.1. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and let $f : X \rightarrow Y$ satisfies the following inequality:

$$\begin{aligned} & \|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) \\ & - f(y + z) - f(z + x), u\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|u\| \end{aligned} \quad (3)$$

for all $x, y, z \in X$ and $u \in Y$. Then

$$\left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \leq \sum_{k=1}^n 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| \quad (4)$$

for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$. Then the function $(\| \cdot, \cdot \|)$ is called a 2- norm on X and the pair $(X, \| \cdot, \cdot \|)$ is called a linear 2-normed space. Sometimes the condition (d) called the triangle inequality.

We introduce a basic property of linear 2-normed spaces.

Lemma 2.1. [6] Let $(X, \| \cdot, \cdot \|)$ be a linear 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.

Definition 2.2. [7] A sequence x_n in a linear 2-normed space X is called a Cauchy sequence if

$$\lim_{n, m \rightarrow \infty} \|x_n - x_m, y\| = 0$$

for all $y \in X$

Definition 2.3. [8] A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there $x \in X$ such that

$$\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$$

for all $y \in X$. If $\{x_n\}$ converges to x , write $x_n \rightarrow x$ as $n \rightarrow \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n \rightarrow \infty} \|x_n, y\| = x$ for all $y \in X$

Triangle inequality implies the following Lemma:

Lemma 2.2. [9] For a convergent sequence $\{x_n\}$ in a linear 2-normed space X ,

$$\lim_{n \rightarrow \infty} \|x_n, y\| = \left\| \lim_{n \rightarrow \infty} x_n, y \right\|$$

for all $y \in X$.

In the 1960s, Gahler and White [7]-[8] introduced the concept of 2-Banach spaces. To define completeness, the concepts of Cauchy Sequences and convergence are required as follows:

Definition 2.4. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Definition 2.5. A 2-Banach space X is called a normed 2-Banach space if X is a normed space with the norm.

3. The Stability of Quadratic Mapping in 2-Banach Spaces

In this section, we prove the Hyers-Ulam stability of the quadratic equation (2) in 2- Banach spaces. Let X be a normed space and let Y be a normed 2-Banach space.

for all $x \in X$, $u \in Y$ and $n \in \mathbb{N}$.

Proof Suppose that (3), if we replace x, y and z by 0 in (3), we have, $\|f(0), u\| \leq 0$

This implies that $\|f(0), u\| = 0 \quad \forall u \in Y$ so by lemma (2.1) $f(0) = 0$.

Now put $x = y = -z$ in (3), we have,

$$\|3f(x) + f(-x) - f(2x), u\| \leq \theta \|x\|^{p+q+r} \|u\| \quad (5)$$

By substituting x by $-x$ in (5), we obtain

$$\|3f(-x) + f(x) - f(-2x), u\| \leq \theta \|x\|^{p+q+r} \|u\| \quad (6)$$

Now we will use the induction on n to prove our lemma,

For $n = 1$ on (4) we have for $x \in X$, $u \in Y$

$$\begin{aligned} & \left\| f(x) - \frac{3}{8}f(2x) + \frac{1}{8}f(-2x), u \right\| \\ &= \frac{1}{8} \|8f(x) - 3f(2x) + f(-2x), u\| \\ &\leq \frac{1}{8} (3 \|3f(x) + f(-x) - f(2x)\| + \|3f(-x) + f(x) - f(-2x)\|) \\ &= \frac{1}{2} \theta \|x\|^{p+q+r} \|u\| \end{aligned}$$

This means the inequality is true for $n=1$.

Assume that the inequality (4) is true for $n=m$, we want to show that it's true for $n=m+1$.

It clearly that, for all $x \in X$, $\forall u \in Y$ and $m \in \mathbb{N}$

$$\begin{aligned} & f(x) - \frac{2^{m+1}+1}{2^{2m+3}}f(2^{m+1}x) + \frac{2^{m+1}-1}{2^{2m+3}}f(-2^{m+1}x) \\ &= f(x) - \frac{2^m+1}{2^{2m+1}}f(2^mx) + \frac{2^m-1}{2^{2m+1}}f(-2^mx) + \frac{2^{m+1}+1}{2^{2m+3}}(3f(2^mx) + f(-2^mx) - f(2^{m+1}x)) \\ &\quad - \frac{2^{m+1}-1}{2^{2m+3}}(3f(-2^mx) + f(2^mx) - f(-2^{m+1}x)) \end{aligned}$$

By Triangle inequality, (4), (5), and (6) we have

$$\begin{aligned} & \left\| f(x) - \frac{2^{m+1}+1}{2^{2m+3}}f(2^{m+1}x) + \frac{2^{m+1}-1}{2^{2m+3}}f(-2^{m+1}x), u \right\| \\ &\leq \sum_{k=1}^m 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \left(\frac{1}{2^{m+1}}\right) \cdot 2^{(p+q+r)m} \theta \|x\|^{p+q+r} \|u\| \\ &= \sum_{k=1}^{m+1} 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| \end{aligned}$$

Therefore we verify the inequality (4) for $m+1$, hence we end prove.

In the following theorem, Hyers-Ulam's stability of equation (2) is proved in 2-Banach spaces under approximately even conditions.

Theorem 3.1. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ such that $p + q + r < 1$, and let $f : X \rightarrow Y$ Satisfies the following inequalities:

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|u\| \quad (7)$$

$$\|f(x) - f(-x), u\| \leq \delta \quad \text{for some } \delta \geq 0 \quad (8)$$

for all $x, y, z \in X$ and $\forall u \in Y$,

Then there exists a unique quadratic mapping which satisfies (2) and the inequality:

$$\|f(x) - Q(x), u\| \leq \frac{1}{2 - 2^{p+q+r}} \theta \|x\|^{p+q+r} \|u\| \quad (9)$$

$\forall x \in X$ and $\forall u \in Y$

Proof Suppose that $f : X \longrightarrow Y$ be a mapping satisfies the inequalities (7) and (8) Now for all $x \in X$, and $\forall u \in Y$, By (8) we have

$$\left\| \frac{2^n - 1}{2^{2n+1}} f(2^n x) - \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| = \frac{2^n - 1}{2^{2n+1}} \|f(2^n x) - f(-2^n x), u\| \leq \frac{2^n - 1}{2^{2n+1}} \delta \quad (10)$$

Now, for all $x \in X$, $n \in N$ and $\forall u \in Y$

$$\begin{aligned} & \left\| f(x) - \frac{2}{2^{2n+1}} f(2^n x), u \right\| \\ & \leq \left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) \right\| + \left\| -\frac{1}{2^{2n}} f(2^n x) + \frac{2^n + 1}{2^{2n+1}} f(2^n x) - \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \\ & \leq \sum_{k=1}^n 2^{(p+q+r)(k-1)k} \theta \|x\|^{p+q+r} \|u\| + \left\| \frac{2^n - 1}{2^{2n+1}} f(2^n x) - \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \\ & \leq \sum_{k=1}^n 2^{(p+q+r)(k-1)k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta \end{aligned}$$

for some $\delta \geq 0$.

This implies for some $\delta \geq 0$ and for all $n, m \in N$ we have

$$\left\| f(x) - \frac{1}{2^{2n}} f(2^n x), u \right\| \leq \sum_{k=1}^n 2^{(p+q+r)(k-1)k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta$$

By replacing n by $n-m$ in (11), we obtain,

$$\left\| f(x) - 2^{-2(n-m)} f(2^{n-m} x), u \right\| \leq \sum_{k=1}^{n-m} 2^{(p+q+r)(k-1)k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^{n-m} - 1}{2^{2(n-m)+1}} \delta \quad (11)$$

By replacing x by $2^m x$, we obtain for $n, m \in N$.

$$\left\| f(2^m x) - 2^{-2(n-m)} f(2^{n-m} 2^m x), u \right\| \leq 2^{m(p+q+r)} \sum_{k=1}^{n-m} 2^{(p+q+r)(k-1)k} \theta \|2^m x\|^{p+q+r} \|u\| + \frac{2^{n-m} - 1}{2^{2(n-m)+1}} \delta \quad (12)$$

Now for all $x \in X$, $n, m \in N$ $\forall u \in Y$,

$$\|2^{-2n} f(2^n x) - 2^{-2m} f(2^m x), u\| = 2^{-2m} \|f(2^m x) - 2^{-2(n-m)} f(2^{n-m} 2^m x), u\|$$

By (12) we have

$$\leq 2^{m(p+q+r-2)} \left(\sum_{k=1}^{n-m} 2^{(p+q+r)(k-1)k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^{n-m} - 1}{2^{2(n-m)+1}} \delta \right) \quad (13)$$

for $n \geq m$. Since the right side of the inequality (13) tend to 0 as m, n tends to ∞ , this implies that the sequence $2^{-2n} f(2^n x)$ is a Cauchy sequence in Y , by 2-Banach spaces of Y this sequence is convergent. So we can define the mapping $Q : X \rightarrow Y$ by that is $Q(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x) \forall x \in X$.

To show that Q is quadratic mapping, $\forall x \in X$ and $\forall u \in Y$

$$\begin{aligned} & \|Q(x+y+z) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y+z) - Q(z+x), u\| \\ & = \left\| \lim_{n \rightarrow \infty} 2^{-2n} f(2^n(x+y+z)) + \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x) + \lim_{n \rightarrow \infty} 2^{-2n} f(2^n y) + \lim_{n \rightarrow \infty} 2^{-2n} f(2^n z) \right. \\ & \quad \left. - \lim_{n \rightarrow \infty} 2^{-2n} f(2^n(x+y)) - \lim_{n \rightarrow \infty} 2^{-2n} f(2^n(y+z)) - \lim_{n \rightarrow \infty} 2^{-2n} f(2^n(z+x)), u \right\| \end{aligned}$$

By lemma (2.2) and (7)

$$= \lim_{n \rightarrow \infty} 2^{-2n} \|f(2^n(x+y+z)) + -2nf(2^n x) + f(2^n y) + f(2^n z) - f(2^n(x+y)) - f(2^n(y+z)) - f(2^n(z+x)), u\|$$

$$= \lim_{n \rightarrow \infty} 2^{n(p+q+r-2)} \theta \|x\|^p \|y\|^q \|z\|^r \|u\| = 0$$

as $n \rightarrow \infty$ (since $p+q+r < 1$).

So $\|Q(x+y+z) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y+z) - Q(z+x), u\| = 0$ for all $x \in X$ and $u \in Y$.

By lemma(2.1),

$$Q(x+y+z) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y+z) - Q(z+x) = 0$$

Hence, Q is a quadratic mapping.

Now for all $x \in X, \forall n \in N$ and $\forall u \in Y$

$$\|f(x) - Q(x)\| = \left\| f(x) - \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} f(2^n x), u \right\| = \lim_{n \rightarrow \infty} \left\| f(x) - \frac{1}{2^{2n}} f(2^n x), u \right\|$$

by lemma(2.2), we get

$$\leq \sum_{k=1}^{\infty} 2^{(p+1+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta \quad \text{by (11)}$$

$$= \frac{1}{2 - 2^{p+q+r}} \theta \|x\|^{p+q+r} \|u\|$$

This means that Q satisfies inequality (9).

Let $Q(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x)$ and $Q(-x) = \lim_{n \rightarrow \infty} 2^{-2n} f(-2^n x)$, So we have,

$$\|Q(x) - Q(-x), u\| = \left\| \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x) - \lim_{n \rightarrow \infty} 2^{-2n} f(-2^n x), u \right\|$$

$$= \lim_{n \rightarrow \infty} 2^{-2n} \|f(2^n x) - f(-2^n x), u\|$$

$$\leq \delta \lim_{n \rightarrow \infty} 2^{-2n} = 0 \quad \text{as } n \rightarrow \infty$$

$\|Q(x) - Q(-x), u\| = 0 \Rightarrow Q(x) = Q(-x)$, that is Q is even.

Now, let $T : X \rightarrow Y$ be another quadratic mapping, which satisfies Equation (2) and inequality (9), so we have for all $x \in X, u \in Y$ and $n \in N$,

$$\|Q(x) - T(x), u\| = \frac{1}{4^n} \|Q(2^n x) - T(2^n x), u\|$$

$$\leq \frac{1}{4^n} \|Q(2^n x) - f(2^n x), u\| + \frac{1}{4^n} \|T(2^n x) - f(2^n x), u\|$$

$$\leq 2 \cdot \left(\frac{2^{(p+q+r)n-2n}}{2 - 2^{p+q+r}} \right) \theta \|x\|^{p+q+r} \|u\|$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. By lemma (2.1), we conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Therefore, the theorem is proved.

Similarly as the proof of the theorem (3.1), the Hyers- Ulam stability of equation (2) in 2-Banach space under the approximately odd condition is proved.

Theorem 3.2. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ such that $p+q+r < 1$, and let $f : X \rightarrow Y$ Satisfies the following inequalities:

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|u\| \quad (14)$$

$$\|f(x) + f(-x), u\| \leq \delta \quad \text{for some } \delta \geq 0 \quad (15)$$

for all $x, y, z \in X$ and $\forall u \in Y$,

Then there exists a unique additive quadratic mapping

$F : X \rightarrow Y$ which satisfies (2) and the inequality:

$$\|f(x) - F(x), u\| \leq \frac{1}{2 - 2^{p+q+r}} \theta \|x\|^{p+q+r} \|u\| \quad (16)$$

$\forall x \in X$ and $\forall u \in Y$

Proof Suppose that $f : X \longrightarrow Y$ be a mapping satisfies the inequalities (14) and (15) Now for all $x \in X$, and $\forall u \in Y$, by (15) we have,

$$\left\| \frac{2^n - 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| = \frac{2^n - 1}{2^{2n+1}} \|f(2^n x) + f(-2^n x), u\| \leq \frac{2^n - 1}{2^{2n+1}} \delta \quad (17)$$

Now, for all $x \in X$, $n \in N$ and $\forall u \in Y$

$$\begin{aligned} & \left\| f(x) - \frac{1}{2^n} f(2^n x), u \right\| \\ & \leq \left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) \right\| + \left\| \frac{2^n - 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \end{aligned}$$

By (4) and (15)

$$\leq \sum_{k=1}^n 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta$$

for some $\delta \geq 0$

$$\Rightarrow \left\| f(x) - \frac{1}{2^n} f(2^n x), u \right\| \leq \sum_{k=1}^n 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta \quad (18)$$

Now, for all $x \in X$, $n, m \in N$ and $\forall u \in N$, we have by (18)

$$\begin{aligned} & \left\| 2^{-n} f(2^n x) - 2^{-m} f(2^m x), u \right\| \\ & = 2^{-m} \left\| f(2^m x) - 2^{-(n-m)} f(2^{n-m} 2^m x) \right\| \\ & \leq 2^{-m} \left(\sum_{k=1}^{n-m} 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^{n-m} - 1}{2^{2(n-m)+1}} \delta \right) \\ & = 0 \quad \text{as } m, n \rightarrow \infty \end{aligned}$$

This implies that the sequence $2^{-n} f(2^n x)$ is a Cauchy sequence in Y , by 2-Banach spaces of Y this sequence is convergent. Say that $F(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x) \forall x \in X$.

Now $\forall x \in X$ and $\forall u \in Y$, we obtain,

$$\begin{aligned} & \| F(x + y + z) + F(x) + F(y) + F(z) - F(x + y) - F(y + z) - F(z + x), u \| \\ & = \| \lim_{n \rightarrow \infty} 2^{-n} f(2^n(x + y + z)) + \lim_{n \rightarrow \infty} 2^{-n} f(2^n x) + \lim_{n \rightarrow \infty} 2^{-n} f(2^n y) + \lim_{n \rightarrow \infty} 2^{-n} f(2^n z) \\ & \quad - \lim_{n \rightarrow \infty} 2^{-n} f(2^n(x + y)) - \lim_{n \rightarrow \infty} 2^{-n} f(2^n(y + z)) - \lim_{n \rightarrow \infty} 2^{-n} f(2^n(z + x)), u \| \end{aligned}$$

By lemma (2.2) and (14)

$$\begin{aligned} & \leq \lim_{n \rightarrow \infty} 2^{n(p+q+r-1)} \theta \|x\|^p \|y\|^q \|z\|^r \|u\| \\ & = 0 \quad \text{as } n \rightarrow \infty \quad (\text{since } p + q + r < 1) \end{aligned}$$

By lemma(2.1), $F(x + y + z) + F(x) + F(y) + F(z) - F(x + y) - F(y + z) - F(z + x) = 0$

Thus, F is a quadratic mapping.

For all $n \in N$, $\forall x \in X$ and $u \in Y$,

$$\begin{aligned} \|f(x) - F(x), u\| &= \lim_{m \rightarrow \infty} \left\| f(x) - \frac{1}{2^m} f(2^m x), u \right\| \text{ by lemma (2.2)} \\ &\leq \lim_{m \rightarrow \infty} \left(\sum_{k=1}^n 2^{(p+q+r)(k-1)-k} \theta \|x\|^{p+q+r} \|u\| + \frac{2^n - 1}{2^{2n+1}} \delta \right) = \frac{1}{2 - 2^{p+q+r}} \|x\|^{p+q+r} \|u\| \end{aligned}$$

So F is satisfies (16).

The proof of the uniqueness is the same way as that of Theorem (3.1) by applying $F(2^n x) = 2^n F(x)$ and $T(2^n x) = 2^n T(x)$.

Similarly, as proof of theorem(3.1) due to (15), we see that mapping F is odd.

By putting $z = -x$ in (2) and considering the oddness of F and letting $u = x + y$, $v = x - y$, we get $2F(\frac{u+v}{2}) = F(u) + F(v)$, since $F(0) = 0$, the Mapping F is additive.

Hence, the proof is complete.

Remark 3.1. The approximately even condition in (8) guarantees the quadratic property of Q , whereas the approximately odd condition in (15) guarantees the additive behavior of F .

Theorem 3.3. Let $\theta \in [0, \infty)$, $p, q, r, k \in (0, \infty)$ with $k < 1$ and let $f : X \rightarrow Y$ be mapping satisfying

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z), u\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|u\|^k \quad (19)$$

for all $x, y, z, \in X$ and all $u \in Y$. Then $f : X \rightarrow Y$ is a quadratic mapping.

Proof Suppose that $f : X \rightarrow Y$ be mapping satisfying (19) for all $x, y, z, \in X$ and all $u \in Y$. Replacing u by vu in (19) for $v \in R \setminus \{0\}$, we get

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z), vu\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|vu\|^k$$

$$\Rightarrow \|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z), u\| \leq \theta \|x\|^p \|y\|^q \|z\|^r \|u\|^k \frac{|v|^k}{|v|} \quad (20)$$

for all $x, y, z, \in X$, all $u \in Y$ and for all $v \in R \setminus \{0\}$, by $k < 1$, the right side of (20) tends to zero as $v \rightarrow \infty$. Thus,

$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z), u\| = 0$ for all $x, y, z, \in X$ and all $u \in Y$. By lemma (2.1) we get,

$$f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z) = 0,$$

Hence f is quadratic mapping.

4. Stability of Generalized Quadratic Mapping in 2-Banach Spaces

P. Gavruta [11] proved a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias in normed space.

Also, G. Kim [12] studied the stability of generalized quadratic mapping under the spirit of Gavruta in one normed space as,

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) \quad (21)$$

In this section, we discuss the stability of quadratic mapping (21) as the spirit of Gavruta but in 2- normed spaces.

Throughout this section, let X be a normed space and let Y be a normed 2-Banach spaces. Also R and N stand for the set of all real numbers and natural numbers, respectively.

Lemma 4.1. Let $f : X \rightarrow Y$ be mapping satisfies the following inequality:

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u\| \leq \varphi(x, y, z, u) \quad (22)$$

for all $x, y, z \in X$ and $u \in Y$. where $\varphi : X^3 \times Y \rightarrow [0, \infty)$ is arbitrary mapping. Then,

$$\begin{aligned} & \left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \\ & \leq \sum_{k=1}^n \frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}x, 2^{k-1}x, u) + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) \end{aligned} \quad (23)$$

Proof Let $f : X \rightarrow Y$ be mapping satisfies the inequality (22), if we replace x, y and z by 0 in (22), we have,

$$\|f(0), u\| \leq \varphi(0, 0, 0, u) \quad \text{for all } u \in Y \quad (24)$$

Now by putting $x = y = -z$ in (22), we have,

$$\|3f(x) + f(-x) - f(2x) - 2f(0), u\| \leq \varphi(x, x, -x, u)$$

So by (24) we obtain

$$\| f(x) + f(x) + f(x) + f(-x) - f(2x), u \| \leq \varphi(x, x, -x, u) + 2\varphi(0, 0, 0, u) \quad (25)$$

By substituting x by $-x$ in (25), we obtain

$$\| 3f(-x) + f(x) - f(-2x), u \| \leq \varphi(-x, -x, x, u) + 2\varphi(0, 0, 0, u) \quad (26)$$

Now we will use the induction on n to prove our lemma,

For $n = 1$ on (23) we have for all $x \in X$ and $u \in Y$,

$$\begin{aligned} & \| f(x) - \frac{3}{8}f(2x) + \frac{1}{8}f(-2x), u \| \\ &= \frac{1}{8} \| 9f(x) - f(x) - 3f(2x) + f(-2x) + 3f(-x) - 3f(-x), u \| \\ &\leq \frac{1}{8} (3 \| 3f(x) + f(-x) - f(2x) \| + \| 3f(-x) + f(x) - f(-2x), u \|) \end{aligned}$$

By (25) and (26)

$$\leq \frac{3}{8}\varphi(x, x, -x, u) + \frac{1}{8}\varphi(-x, -x, x, u) + \varphi(0, 0, 0, u) \quad (27)$$

This means the inequality is true for $n=1$.

Assume that the inequality (23) is true for $n=m$, we want to show that it's true for $n=m+1$, but Previous, we used the inequality, for all $x \in X$, $\forall u \in Y$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \left\| f(x) - \frac{2^{m+1}+1}{2^{2m+3}}f(2^{m+1}x) + \frac{2^{m+1}-1}{2^{2m+3}}f(-2^{m+1}x), u \right\| \\ &\leq \left\| f(x) - \frac{2^m+1}{2^{2m+1}}f(2^mx) + \frac{2^m-1}{2^{2m+1}}f(-2^mx), u \right\| + \left\| \frac{2^{m+1}+1}{2^{2m+3}}(3f(2^mx) + f(-2^mx) - f(2^{m+1}x)), u \right\| \\ &+ \left\| \frac{2^{m+1}-1}{2^{2m+3}}(3f(-2^mx) + f(2^mx) - f(-2^{m+1}x)), u \right\| \end{aligned}$$

By (23), (25), (26)

$$\begin{aligned} &\leq \sum_{k=1}^m \frac{1}{2^{k-1}}\varphi(0, 0, 0, u) + \frac{2^k+1}{2^{2k+1}}\varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) + \frac{2^k-1}{2^{2k+1}}\varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) \\ &+ \frac{2^{m+1}+1}{2^{2m+3}}(\varphi(2^mx, 2^mx, -2^mx, u) + 2\varphi(0, 0, 0, u)) + \frac{2^{m+1}-1}{2^{2m+3}}(\varphi(-2^mx, -2^mx, 2^mx, u) + 2\varphi(0, 0, 0, u)) \\ &\leq \sum_{k=1}^{m+1} \frac{1}{2^{k-1}}\varphi(0, 0, 0, u) + \frac{2^k+1}{2^{2k+1}}\varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) + \frac{2^k-1}{2^{2k+1}}\varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) \end{aligned}$$

Therefore we verify the inequality (23) for $m+1$, thus the induction is ended. Hence the proof is complete.

Theorem 4.1. Assume that a mapping $f : X \rightarrow Y$ satisfies the following inequalities:

$$\| f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u \| \leq \varphi(x, y, z, u) \quad (28)$$

for all $x, y, z \in X$ and $\forall u \in Y$, where $\varphi : X^3 \times Y \rightarrow [0, \infty)$. Let the mappings $\psi : X \rightarrow [0, \infty)$ such that, for all $x, y, z \in X$, and $\forall u \in Y$

$$\| f(x) - f(-x), u \| \leq \psi(x) \quad (29)$$

and the mappings $\phi : X^3 \times Y \rightarrow [0, \infty)$, satisfies the inequalities:

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y, 2^n z, u)}{2^{2n}} = 0 \quad (30)$$

for all $x, y, z \in X$, $\forall u \in Y$ and $n \in N$,

$$\lim_{n \rightarrow \infty} \frac{\psi(2^n x)}{2^{2n}} = 0 \quad (31)$$

$$\lim_{n \rightarrow \infty} \frac{2^n - 1}{2^{2n+1}} \psi(2^n x) = 0 \quad (32)$$

for all $x \in X$, and

$$\begin{aligned} \phi(x, y, z, u) &= \sum_{k=1}^{\infty} \left(\frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}y, -2^{k-1}z, u) \right. \\ &\quad \left. + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}y, 2^{k-1}z, u) \right) < \infty \end{aligned} \quad (33)$$

for all $x, y, z \in X$, $u \in Y$ and $n \in N$,

Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ which satisfies (21) and the inequality:

$$\|f(x) - Q(x), u\| \leq \phi(x, x, x, u) \quad \forall x \in X, \text{ and } \forall u \in Y \quad (34)$$

Proof Suppose that $f : X \rightarrow Y$ be mapping satisfies inequalities (28) and (29).

So we have, for all $x \in X$, $u \in Y$ and $n \in N$,

$$\|f(2^n x) - f(-2^n x)\| \leq \psi(2^n x) \quad (35)$$

$$\begin{aligned} \text{but } &\left\| 2^{-2(n+1)} f(2^{n+1}x) - 2^{2n} f(2^n x), u \right\| \\ &\leq 2^{-2(n+1)} \|3f(2^n x) + f(-2^n x) - f(2 \cdot 2^n x), u\| + \|f(2^n x) - f(-2^n x), u\| \end{aligned}$$

By (25) and (35) we obtain

$$\left\| 2^{-2(n+1)} f(2^{n+1}x) - 2^{2n} f(2^n x), u \right\| \leq 2^{-2(n+1)} (2\varphi(0, 0, 0) + \varphi(2^n x, 2^n x, -2^n x) + \psi(2^n x)) \quad (36)$$

Now define $g_n(x) = 2^{-2n} f(2^n x)$, for all $x \in X$, and $\forall u \in Y$ we have $n \in N$,

$$\begin{aligned} \|g_n(x) - g_m(x), u\| &\leq \sum_{j=m}^{n-1} \|g_{j+1}(x) - g_j(x), u\| \\ &= \sum_{j=m}^{n-1} \left\| 2^{-2(j+1)} f(2^{j+1}x) - 2^{-2j} f(2^j x), u \right\| \\ &\leq \sum_{j=m}^{n-1} 2^{-2(j+1)} (2\varphi(0, 0, 0) + \varphi(2^j x, 2^j x, -2^j x) + \psi(2^j x)) \quad \text{by (36)} \\ &= \frac{1}{4} \left(\sum_{j=m}^{n-1} 2^{-2j} 2\varphi(0, 0, 0) + \sum_{j=m}^{n-1} 2^{-2j} \varphi(2^j x, 2^j x, -2^j x) + \sum_{j=m}^{n-1} 2^{-2j} \psi(2^j x) \right) \end{aligned} \quad (37)$$

By (30) and (31) the right-hand side in (37) tends to zero as m tends to infinity. Thus $\{g_n(x)\}$ is Cauchy sequence for all $x \in X$, and hence $\{2^{-2n} f(2^n x)\}$ is Cauchy sequence for all $x \in X$, by completeness of Y every Cauchy sequence is convergent.

Define $Q : X \rightarrow Y$ by

$$Q(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x) \quad (38)$$

$\forall x \in X$, $\forall u \in Y$. By lemma (2.2),

$$\|Q(x + y + z) + Q(x) + Q(y) + Q(z) - Q(x + y) - Q(y + z) - Q(z + x), u\|$$

$$= \lim_{n \rightarrow \infty} 2^{-2n} \|f(2^n(x+y+z)) + -2nf(2^n x) + f(2^n y) + f(2^n z) - f(2^n(x+y)) - f(2^n(y+z)) - f(2^n(z+x)), u\|$$

by (28) and (30)

$$\leq \lim_{n \rightarrow \infty} 2^{-2n} \varphi(2^n x, 2^n x, 2^n z) = 0$$

for all $x, y, z \in X$, and $\forall u \in Y$.

By lemma(2.1),

$$Q(x+y+z) + Q(x) + Q(y) + Q(z) - Q(x+y) - Q(y+z) - Q(z+x) = 0$$

Hence, Q is a quadratic mapping.

Let

$$Q(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x),$$

$$Q(-x) = \lim_{n \rightarrow \infty} 2^{-2n} f(-2^n x)$$

, then by(31)

$$\begin{aligned} \|Q(x) - Q(-x), u\| &= \lim_{n \rightarrow \infty} 2^{-2n} \|f(2^n x) - f(-2^n x), u\| \\ &\leq \lim_{n \rightarrow \infty} 2^{-2n} \psi(2^n x) \\ &= 0 \end{aligned}$$

By lemma (2.1) $Q(x) = Q(-x)$, that is Q is even.

Now, for $x \in X, n \in N$ and $\forall u \in Y$

$$\begin{aligned} &\|f(x) - \frac{1}{2^{2n}} f(2^n x), u\| \\ &\leq \left\| f(x) - \frac{2^n + 1}{2^{2n+1}} f(2^n x) + \frac{2^n - 1}{2^{2n+1}} f(-2^n x) \right\| + \left\| -\frac{1}{2^{2n}} f(2^n x) + \frac{2^n + 1}{2^{2n+1}} f(2^n x) - \frac{2^n - 1}{2^{2n+1}} f(-2^n x), u \right\| \end{aligned}$$

By (23) and (35)

$$\begin{aligned} &\leq \sum_{k=1}^n \left(\frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) \right. \\ &\quad \left. + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) + \frac{2^n - 1}{2^{2n+1}} \psi(2^n x) \right) \end{aligned} \quad (39)$$

for all $x \in X, u \in Y$ and $n \in N$.

$$\begin{aligned} \text{But } \|f(x) - Q(x), u\| &= \lim_{n \rightarrow \infty} \left\| f(x) - \frac{1}{2^{2n}} f(2^n x), u \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \left(\frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) \right. \right. \\ &\quad \left. \left. + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) + \frac{2^n - 1}{2^{2n+1}} \psi(2^n x) \right) \right) \end{aligned} \quad (40)$$

by (32) and (33)

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) \right)$$

Hence $\|f(x) - Q(x)\| \leq \phi(x, x, x, u)$. for all $x \in X$ and $u \in Y$, such that $\phi(x, y, z, u)$ as defined in (33) with $x = y = z$, thus Q is satisfies (34).

Let $T : X \longrightarrow Y$ be another quadratic mapping, which satisfies Equation (21) and inequality (34),

By [10], we have

$$Q(2^n x) = 4^n Q(x) \text{ and } T(2^n x) = 4^n T(x) \quad \text{for all } x \in X \text{ and } n \in N$$

So for all $x \in X, u \in Y$ and $n \in N$,

$$\begin{aligned}
\|Q(x) - T(x), u\| &= \frac{1}{4^n} \|Q(2^n x) - T(2^n x), u\| \\
&\leq \frac{1}{4^n} \|Q(2^n x) - f(2^n x), u\| + \frac{1}{4^n} \|T(2^n x) - f(2^n x), u\| \\
&\leq \frac{2}{4^n} \phi(x, x, x) \quad \text{by (34)}
\end{aligned}$$

which tends to zero as $n \rightarrow \infty$, by lemma (2.1), we obtain $Q(x) = T(x)$,

Therefore, Q is unique. Hence the proof is complete.

Similarly as the proof of the theorem (4.1), the Hyers- Ulam stability of generalized functional equation (21) under the approximately odd condition is proved.

Theorem 4.2. Assume that a mapping $f : X \rightarrow Y$ satisfies the following inequalities:

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u\| \leq \varphi(x, y, z, u) \quad (41)$$

for all $x, y, z \in X$ and $\forall u \in Y$, where $\varphi : X^3 \times Y \rightarrow [0, \infty)$. Let the mappings $\psi : X \rightarrow [0, \infty)$, such that for all $x, y, z \in X$, and $u \in Y$

$$\|f(x) + f(-x), u\| \leq \psi(x) \quad (42)$$

and the mappings $\phi : X^3 \times Y \rightarrow [0, \infty)$, satisfies the inequalities:

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y, 2^n z, u)}{2^n} = 0 \quad (43)$$

for all $x, y, z \in X$, $\forall u \in Y$ and $n \in N$,

$$\lim_{n \rightarrow \infty} \frac{\psi(2^n x)}{2^n} = 0 \quad (44)$$

$$\lim_{n \rightarrow \infty} \frac{2^n - 1}{2^{2n+1}} \psi(2^n x) = 0 \quad (45)$$

for all $x \in X$, and

$$\begin{aligned}
\phi(x, y, z, u) &= \sum_{k=1}^{\infty} \left(\frac{1}{2^{k-1}} \varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}} \varphi(2^{k-1}x, 2^{k-1}y, -2^{k-1}z, u) \right. \\
&\quad \left. + \frac{2^k - 1}{2^{2k+1}} \varphi(-2^{k-1}x, -2^{k-1}y, 2^{k-1}z, u) \right) < \infty
\end{aligned} \quad (46)$$

for all $x, y, z \in X$, $u \in Y$ and $n \in N$.

Then there exists a unique quadratic mapping $F : X \rightarrow Y$ which satisfies (21) and the inequality:

$$\|f(x) - F(x), u\| \leq \phi(x, x, x, u) \quad (47)$$

for all $x \in X$, and $u \in Y$

Proof Suppose that $f : X \rightarrow Y$ be a mapping satisfies inequalities (41) and (42)

Now for all $x \in X$, $u \in Y$, $n \in N$ and by (42), we have

$$\|f(2^n x) + f(-2^n x)\| \leq \psi(2^n x) \quad (48)$$

But, we have

$$\|2^{-(n+1)} f(2^{n+1}x) - 2^{-n} f(2^n x), u\| \leq 2^{-(n+1)} (\|3f(2^n x) + f(-2^n x) - f(2 \cdot 2^n x), u\| + \| -f(2^n x) - f(-2^n x), u\|)$$

By(25) and (48) we obtain,

$$\|2^{-(n+1)} f(2^{n+1}x) - 2^{-n} f(2^n x), u\| \leq 2^{-(n+1)} (2\varphi(0, 0, 0) + \varphi(2^n x, 2^n x, -2^n x) + \psi(2^n x)) \quad (49)$$

Now define $g_n(x) = 2^{-n}f(2^n x)$. For all $x \in X$, $\forall u \in Y$ and $\forall n \in N$,

$$\begin{aligned} \|g_n(x) - g_m(x), u\| &\leq \sum_{j=m}^{n-1} \|g_{j+1}(x) - g_j(x), u\| = \sum_{j=m}^{n-1} \|2^{-(j+1)}f(2^{j+1}x) - 2^{-j}f(2^j x), u\| \\ \text{by(49)} & \\ &\leq \sum_{j=m}^{n-1} 2^{-(j+1)}(2\varphi(0, 0, 0) + \varphi(2^j x, 2^j x, -2^j x) + \psi(2^j x)) \end{aligned} \quad (50)$$

By (43) and (44) the right-hand side in (50) tends to zero as m tends to infinity. Thus $\{g_n(x)\}$ is Cauchy sequence for all $x \in X$, and hence $\{2^{-n}f(2^n x)\}$ is Cauchy sequence for all $x \in X$, by completeness of Y every Cauchy sequence is convergent.

Define $F : X \rightarrow Y$ by $F(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x)$.

It clearly F is quadratic mapping as in the proof of theorem (4.1) with $F(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x)$. Now for $x \in X$, $n \in N$ and $\forall u \in Y$

$$\left\| f(x) - \frac{1}{2^n}f(2^n x), u \right\| \leq \left\| f(x) - \frac{2^n + 1}{2^{2n+1}}f(2^n x) + \frac{2^n - 1}{2^{2n+1}}f(-2^n x), u \right\| + \left\| \frac{2^n + 1}{2^{2n+1}}f(2^n x) + \frac{2^n - 1}{2^{2n+1}}f(-2^n x), u \right\|$$

By (23) and (48)

$$\leq \sum_{k=1}^n \left(\frac{1}{2^{k-1}}\varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}}\varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) + \frac{2^k - 1}{2^{2k+1}}\varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) + \frac{2^n - 1}{2^{2n+1}}\psi(2^n x) \right)$$

for all $x \in X$, $u \in Y$ and $n \in N$.

$$\begin{aligned} \text{But } \|f(x) - F(x), u\| &= \lim_{n \rightarrow \infty} \left\| f(x) - \frac{1}{2^n}f(2^n x), u \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \left(\frac{1}{2^{k-1}}\varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}}\varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) \right. \right. \\ &\quad \left. \left. + \frac{2^k - 1}{2^{2k+1}}\varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) + \frac{2^n - 1}{2^{2n+1}}\psi(2^n x) \right) \right) \end{aligned}$$

by (45) and (46)

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2^{k-1}}\varphi(0, 0, 0, u) + \frac{2^k + 1}{2^{2k+1}}\varphi(2^{k-1}x, 2^{k-1}x, -2^{k-1}x, u) + \frac{2^k - 1}{2^{2k+1}}\varphi(-2^{k-1}x, -2^{k-1}x, 2^{k-1}x, u) \right) = \phi(x, x, x, u)$$

Hence $\|f(x) - F(x)\| \leq \phi(x, x, x, u)$. for all $x \in X$ and $u \in Y$. This means that F satisfies (47).

The proof of the uniqueness is same way as that of Theorem (4.1) by applying $F(2^{-n}x) = 2^{-n}F(x)$ and $T(2^{-n}x) = 2^{-n}T(x)$.

Similarly, as proof of theorem (4.1) due to (42), we see that mapping F is odd.

By putting $z=x$ in (21) and considering the oddness of F and letting $u=x+y$, $v=x-y$, we get $2F(\frac{u+v}{2}) = F(u) + F(v)$, since $F(0)=0$, the F is additive.

Hence the proof of this theorem is complete.

In the next corollaries there is many application of the Hyers Ulam stability of generalized quadratic functional equation (21) in 2-Banach spaces under the approximately even or odd condition which is the result of theorems (4.1) and (4.2) with $\phi(x, y, z, u) = \delta$ in corollaries (4.2.1) and (4.2.2), and with $\phi(x, y, z, u) = \theta \|x\|^p \|y\|^q \|z\|^r \|u\|$ in Remark(4.1).

Corollary 4.2.1. Assume that a mapping $f : X \rightarrow Y$ satisfies the following inequalities:

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(z + x), u\| \leq \delta$$

$$\|f(x) - f(-x)\| \leq \theta \quad (51)$$

for some $\delta, \theta \geq 0$, for all $x, y, z \in X$, and $\forall u \in Y$,

Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ which satisfies (21) and the inequality:

$$\|f(x) - Q(x), u\| \leq \frac{3}{2}\delta \quad \text{foe all } x \in X, \text{ and } u \in Y \quad (52)$$

Proof Assume that (51) and (51), it clear the condition (30), (31) and (32) satisfies, then by theorem (4.1) Then there exist a unique quadratic mapping $Q : X \rightarrow Y$ which satisfies (21) and (52).

Corollary 4.2.2. Assume that a mapping $f : X \rightarrow Y$ satisfies the following inequalities:

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), u\| \leq \delta \quad (53)$$

$$\|f(x) + f(-x)\| \leq \theta \quad (54)$$

for some $\delta, \theta \geq 0$, for all $x, y, z \in X$, and $u \in Y$,

Then there exists a unique quadratic mapping $F : X \rightarrow Y$ which satisfies (21) and the inequality:

$$\|f(x) - F(x), u\| \leq \frac{3}{2}\delta \quad \text{foe all } x \in X, \text{ and } u \in Y \quad (55)$$

Proof Similarly as proof of corollary

Remark 4.1. By applying Theorem (4.1) and Theorem (4.2) with $\varphi(x, y, z, u) = \theta \|x\|^p \|y\|^q \|z\|^r \|u\|$ and $\psi(x) = \delta$, we obtain matching result with Theorem (3.1) and Theorem (3.2).

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Author Contributions

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Conflicts of Interest

The author declares no conflicts of interest.

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