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# Pompeiu-Hausdorff Fuzzy $b$ -metric Spaces Are Associated with a Common Fixed Point and Multivalued Mappings

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**Abstract:** The notion of fuzzy logic was introduced by Zadeh. Unlike traditional logic theory, where an element either belongs to the set or does not, in fuzzy logic, the affiliation of the element to the set is expressed as a number from the interval  $[0, 1]$ . The study of the theory of fuzzy sets was prompted by the presence of uncertainty as an essential part of real-world problems, leading Zadeh to address the problem of indeterminacy. The theory of a fixed point in fuzzy metric spaces can be viewed in different ways, one of which involves the use of fuzzy logic. Fuzzy metric spaces, which are specific types of topological spaces with pleasing “geometric” characteristics, possess a number of appealing properties and are commonly used in both pure and applied sciences. Metric spaces and their various generalizations frequently occur in computer science applications. For this reason, a new space called a Pompeiu-Hausdorff fuzzy  $b$ -metric space is constructed in this paper. In this space, some new fixed point results are also formulated and proven. Additionally, a general common fixed point theorem for a pair of multi-valued mappings in Pompeiu-Hausdorff fuzzy  $b$ -metric spaces is investigated. The findings obtained in fuzzy metric spaces, such as those discussed in Remark 3.1, are generalized by the results in this paper, and additional specific findings are produced and supported by examples. The study of denotational semantics and their applications in control theory using fuzzy  $b$ -metric spaces and Pompeiu-Hausdorff fuzzy  $b$ -metric spaces will be an important next step.

**Keywords:** Fuzzy Metric Space, Fuzzy  $b$ -metric Space,  $t$ -norm, Fixed Point, Implicit Relation

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## 1. Introduction

The study of fixed point theory in metric spaces has a variety of applications in mathematics, particularly in solving differential equations. Many authors have studied the new class of generalized metric space, known as  $b$ -metric space, introduced by Bakhtin [4] in 1989. For example, see [1–3, 5–8, 10, 16–19]. In 1975, Kramosil and Michalek [14] proposed the idea of a fuzzy distance between two elements of a nonempty set, using the concepts of a fuzzy set and a  $t$ -norm.

A binary operation  $\star : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if it satisfies the following conditions:  $\star$  is continuous, associative, and commutative,  $\alpha \star 1 = \alpha$  for all  $\alpha \in [0, 1]$  and for all  $\alpha, \beta, \gamma, \delta \in [0, 1]$  if  $\alpha \leq \gamma$  and  $\beta \leq \delta$  then  $\alpha \star \beta \leq \gamma \star \delta$ .

George and Veeramani [11] generalized the concept of fuzzy metric spaces introduced by Kramosil and Michalek [14]. Given a non empty set  $\xi$ , and  $\star$  is a continuous  $t$ -norm, the

3-tuple  $(\xi, \Lambda, \star)$  is said to be a fuzzy metric space [11, 12] if  $\Lambda$  is a fuzzy set on  $\xi \times \xi \times (0, \infty)$  satisfying the following conditions for all  $x, y, z \in \xi, \iota, \mu > 0$ :

- $$\left\{ \begin{array}{l} 1) \quad \Lambda(x, y, \iota) > 0, \\ 2) \quad \Lambda(x, y, \iota) = \Lambda(y, x, \iota) = 1 \quad \text{iff} \quad x = y, \\ 3) \quad \Lambda(x, z, \iota + \mu) \geq \Lambda(x, y, \iota) \star \Lambda(y, z, \mu), \\ 4) \quad \Lambda(x, y, \cdot) \text{ is left continuous function from } (0, \infty) \rightarrow [0, 1]. \end{array} \right.$$

The relationship between  $b$ -metric and fuzzy metric spaces is considered in [13]. Conversely, [20] introduced the concept of a fuzzy  $b$ -metric space, substituting the triangle inequality with a weaker one.

In this paper, a general common fixed point theorem for a pair of multi-valued mappings in Pompeiu-Hausdorff fuzzy  $b$ -metric spaces is of interest to be proven.

## 2. Preliminary

**Definition 2.1.** [20] A 3-tuples  $(\xi, \Lambda, \star)$  is called a fuzzy  $b$ -metric space if  $\xi$  is an arbitrary nonempty set,  $\star$  is a continuous  $t$ -norm, and  $\Lambda$  is a fuzzy set on  $\xi \times \xi \times (0, \infty)$  satisfying the conditions for all  $x, y, z \in \xi$ ,  $\iota, \mu > 0$  and a given real number  $s \geq 1$ :

- (b<sub>1</sub>)  $\Lambda(x, y, \iota) > 0$ ,
- (b<sub>2</sub>)  $\Lambda(x, y, \iota) = 1$  if and only if  $x = y$ ,
- (b<sub>3</sub>)  $\Lambda(x, y, \iota) = \Lambda(y, x, \iota)$ ,
- (b<sub>4</sub>)  $\Lambda(x, z, s(\iota + \mu)) \geq \Lambda(x, y, \iota) \star \Lambda(y, z, \mu)$ ,
- (b<sub>5</sub>)  $\Lambda(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.

**Remark 2.1.** In this paper, a fuzzy  $b$ -metric space with the additional condition  $\lim_{\iota \rightarrow \infty} \Lambda(x, y, \iota) = 1$  will further be used.

**Example 2.1.** [9] Let  $\Lambda(x, y, \iota) = \exp^{-\frac{|x-y|^p}{\iota}}$ , where  $p > 1$  is a real number, and  $\alpha \star \beta = \alpha \cdot \beta$  for all  $\alpha, \beta \in [0, 1]$ . So  $(\xi, \Lambda, \star)$  is a fuzzy  $b$ -metric space with  $s = 2^{p-1}$ .

**Definition 2.2.** [20, 21] Let  $(\xi, \Lambda, \star)$  be a fuzzy  $b$ -metric space.

- (i) A sequence  $(x_n)$  is said to converge to  $x$  if  $\Lambda(x_n, x, \iota) \rightarrow 1$  as  $n \rightarrow \infty$  for each  $\iota > 0$ . In this case,  $\lim_{n \rightarrow \infty} x_n = x$  is written.
- (ii) A sequence  $(x_n)$  is called a Cauchy sequence if,  $\forall \varepsilon \in (0, 1)$  and  $\iota > 0$ ,  $\exists n_0 \in \mathbb{N} \forall n, m \geq n_0$ ,  $\Lambda(x_n, x_m, \iota) > 1 - \varepsilon$ .
- (iii) The fuzzy  $b$ -metric space  $(\xi, \Lambda, \star)$  is said to be complete if every Cauchy sequence is convergent.
- (iv) A subset  $C \subset \xi$  is said to be closed if, for every sequence  $x_n \in C$  such that  $x_n \rightarrow x$ ,  $x$  is contained in  $C$ .
- (v) A subset  $C \subset \xi$  is said to be compact if every sequence  $x_n \in C$  has a convergent subsequence.

## 3. Main Results

In this section, a new space called a Pompeiu-Hausdorff fuzzy  $b$ -metric space is constructed. Several examples are presented in this section. In this space, some new fixed point results are also formulated and proven. The following is presented as the starting point:

Throughout this paper, the family of nonempty compact subsets of  $\xi$  will be denoted by  $C(\xi)$ , and the family of nonempty closed subsets of  $\xi$  will be denoted by  $CL(\xi)$ . For all  $K, L \in C(\xi)$  and for all  $\iota > 0$ , a function on  $C(\xi) \times C(\xi) \times (0, \infty)$  is defined by

$$\Pi_\Lambda(K, L, \iota) = \min \left\{ \inf_{k \in K} \Lambda(k, L, \iota), \inf_{l \in L} \Lambda(K, l, \iota) \right\},$$

$$\text{where } \Lambda(M, y, \iota) = \sup_{z \in M} \Lambda(z, y, \iota).$$

Then, the Pompeiu-Hausdorff fuzzy  $b$ -metric induced by the fuzzy  $b$ -metric  $\Lambda$  is called  $\Pi_\Lambda$ . The triplet  $(C(\xi), \Pi_\Lambda, \star)$  is referred to as the Pompeiu-Hausdorff fuzzy  $b$ -metric space.

Additionally,  $\delta_\Lambda(K, L, \iota)$  is defined by

$$\delta_\Lambda(K, L, \iota) = \inf \{ \Lambda(k, l, \iota), \quad k \in K \quad l \in L \}, \quad \iota > 0.$$

It is immediately followed from the definition of  $\delta_\Lambda$  that

$$\delta_\Lambda(K, L, \iota) = 1 \iff K = L = \{.\} \text{ and}$$

$$\Lambda(k, l, \iota) \geq \delta_\Lambda(K, L, \iota) \quad \forall k \in K \quad \forall l \in L, \quad \iota > 0.$$

Indeed:

If  $K = L = \{.\}$ , then

$$\delta_\Lambda(K, L, \iota) = \delta_\Lambda(\{k\}, \{k\}, \iota) = \Lambda(k, k, \iota) = 1.$$

Now, if  $\delta_\Lambda(K, L, \iota) = 1$ ,  $\forall \iota > 0$ .

Then  $\forall (k, l) \in K \times L$ ,  $\Lambda(k, l, \iota) \geq \delta_\Lambda(K, L, \iota) = 1$ .

So,  $K = L = \{k\} = \{l\} = \{.\}$ .

**Proposition 3.1.** Let  $(\xi, \Lambda, \star)$  be fuzzy  $b$ -metric space with constant  $s$ .

Then  $\Pi_\Lambda$  is a fuzzy set on  $C(\xi) \times C(\xi) \times (0, \infty)$  satisfying the conditions for all  $\iota, \mu > 0$  and  $K, L, M \in C(\xi)$ :

- ( $\Pi_1$ )  $\Pi_\Lambda(K, L, \iota) > 0$ ,
- ( $\Pi_2$ )  $\Pi_\Lambda(K, L, \iota) = 1 \iff K = L$ ,
- ( $\Pi_3$ )  $\Pi_\Lambda(K, M, s(\iota + \mu)) \geq \Pi_\Lambda(K, L, \iota) \star \Pi_\Lambda(L, M, \mu)$ ,
- ( $\Pi_4$ )  $\Pi_\Lambda(K, L, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.
- ( $\Pi_5$ )  $\lim_{\iota \rightarrow \infty} \Pi_\Lambda(K, L, \iota) = 1 \iff \lim_{\iota \rightarrow \infty} \Lambda(k, l, \iota) = 1$ .

**Proof** ( $\Pi_1$ ) Let  $K, L \in C(\xi)$ , let us show that

$$\Pi_\Lambda(K, L, \iota) > 0, \forall \iota > 0,$$

we have

$$\begin{aligned} \Pi_\Lambda(K, L, \iota) &= \min \left\{ \inf_{k \in K} \Lambda(k, L, \iota), \inf_{l \in L} \Lambda(K, l, \iota) \right\}, \\ &= \min \left\{ \inf_{k \in K} \sup_{l \in L} \Lambda(k, l, \iota), \inf_{l \in L} \sup_{k \in K} \Lambda(k, l, \iota) \right\}. \end{aligned}$$

Suppose that  $\Pi_\Lambda(K, L, \iota) = \inf_{k \in K} \sup_{l \in L} \Lambda(k, l, \iota)$ , by the characterization of inf and sup, there exist  $(k_n, l_n) \in K \times L$  such that  $\lim_{n \rightarrow \infty} \Lambda(k_n, l_n, \iota) = \Pi_\Lambda(K, L, \iota)$ ,  $\forall \iota > 0$ .

Since  $K$  and  $L$  are two compacts, then  $(k_n)$  and  $(l_n)$  respectively admit two subsequences, as also noted by  $(k_n)$  and  $(l_n)$ , such that  $k_n \rightarrow k \in K$  and  $l_n \rightarrow l \in L$  ( $n \rightarrow \infty$ ).

By Definition 2.1 (b<sub>4</sub>), it is obtained that

$$\begin{aligned} \Lambda(k_n, l_n, \iota) &\geq \Lambda\left(k_n, k, \frac{\iota}{2s}\right) \star \Lambda\left(k, l_n, \frac{\iota}{2s}\right) \\ &\geq \Lambda\left(k_n, k, \frac{\iota}{2s}\right) \star \left( \frac{\Lambda\left(k, l, \frac{\iota}{4s^2}\right)}{\Lambda\left(l, l_n, \frac{\iota}{4s^2}\right)} \right), \end{aligned}$$

by taking  $n \rightarrow \infty$ , it is obtained that

$$\begin{aligned} \Pi_\Lambda(K, L, \iota) &\geq 1 \star \left( \frac{\Lambda\left(k, l, \frac{\iota}{4s^2}\right)}{\Lambda\left(l, l, \frac{\iota}{4s^2}\right)} \right) \\ &= \Lambda\left(k, l, \frac{\iota}{4s^2}\right) > 0, \quad \forall \iota > 0. \end{aligned}$$

Similarly, if  $\Pi_\Lambda(K, L, \iota) = \inf_{l \in L} \sup_{k \in K} \Lambda(k, l, \iota)$ .

Then  $\Pi_\Lambda(K, L, \iota) > 0, \quad \forall \iota > 0$ .

( $\Pi_2$ ) Let  $K, L \in C(\xi), \iota > 0$  and show that

$\Pi_\Lambda(K, L, \iota) = 1, \Leftrightarrow K = L$ .

If  $K = L$ , it is obtained that

$$\Pi_\Lambda(K, L, \iota) = \Pi_\Lambda(K, K, \iota) = \inf_{k \in K} \Lambda(k, K, \iota).$$

According to the lower bound characterization, there exists  $k_n \in K$ , such that  $\lim_{n \rightarrow \infty} \Lambda(k_n, K, \iota) = \Pi_\Lambda(K, K, \iota)$ . Since  $K$  is a compact, then  $(k_n)$  admits a subsequence also noted  $(k_n)$ , such that  $k_n \rightarrow k \in K$ . On the other hand, it is obtained that

$$\Lambda(k_n, K, \iota) = \sup_{x \in K} \Lambda(k_n, x, \iota) \geq \Lambda(k_n, k, \iota),$$

by taking  $n \rightarrow \infty$ , it is obtained that

$$\Pi_\Lambda(K, K, \iota) = \lim_{n \rightarrow \infty} \Lambda(k_n, K, \iota) \geq \lim_{n \rightarrow \infty} \Lambda(k_n, k, \iota) = 1.$$

So,  $\Pi_\Lambda(K, K, \iota) = 1$ .

Now, if  $\Pi_\Lambda(K, L, \iota) = 1$ , then

$$\begin{aligned} & \inf_{k \in K} \Lambda(k, L, \iota) = 1 \text{ and } \inf_{l \in L} \Lambda(K, l, \iota) = 1 \\ \Rightarrow & \forall k \in K, \Lambda(k, L, \iota) = 1 \text{ and } \forall l \in L, \Lambda(K, l, \iota) = 1, \\ \Rightarrow & \forall k \in K, \sup_{l \in L} \Lambda(k, l, \iota) = 1 \text{ and } \forall l \in L, \sup_{k \in K} \Lambda(k, l, \iota) = 1, \\ \Rightarrow & \forall k \in K, \exists l_n \in L : \lim_{n \rightarrow \infty} \Lambda(k, l_n, \iota) = 1 \text{ and } \forall l \in L, \exists k_n \in K : \lim_{n \rightarrow \infty} \Lambda(k_n, l, \iota) = 1, \\ \Rightarrow & \forall k \in K, \exists l_n \in L : l_n \rightarrow k \in L \text{ and } \forall l \in L, \exists k_n \in K : k_n \rightarrow l \in K, \\ \Rightarrow & K = L. \end{aligned}$$

( $\Pi_3$ ) Let  $K, L, M \in C(\xi), \iota, \mu > 0$  and  $s \geq 1$ , let us show that

$$\Pi_\Lambda(K, M, s(\iota + \mu)) \geq \Pi_\Lambda(K, L, \iota) \star \Pi_\Lambda(L, M, \mu).$$

Suppose that  $\Pi_\Lambda(K, M, s(\iota + \mu)) = \inf_{k \in K} \Lambda(k, M, s(\iota + \mu))$ . First, it is proven that, for all  $(k, l) \in K \times L$ :

$$\Lambda(k, M, s(\iota + \mu)) \geq \Lambda(k, l, \iota) \star \Lambda(l, M, \mu).$$

Indeed, since  $\Lambda(l, M, \mu) = \sup_{m \in M} \Lambda(l, m, \mu)$ , then there exists  $m_n \in M$  such that:

$\lim_{n \rightarrow \infty} \Lambda(l, m_n, \mu) = \Lambda(l, M, \mu)$ . By definition 2.1 ( $b_4$ ), it is obtained that

$$\Lambda(k, m_n, s(\iota + \mu)) \geq \Lambda(k, l, \iota) \star \Lambda(l, m_n, \mu),$$

by taking  $n \rightarrow \infty$ , it is obtained that

$$\begin{aligned} \Lambda(k, M, s(\iota + \mu)) & \geq \lim_{n \rightarrow \infty} \Lambda(k, m_n, s(\iota + \mu)) \\ & \geq \lim_{n \rightarrow \infty} \Lambda(k, l, \iota) \star \Lambda(l, m_n, \mu), \\ & = \Lambda(k, l, \iota) \star \lim_{n \rightarrow \infty} \Lambda(l, m_n, \mu), \\ & = \Lambda(k, l, \iota) \star \Lambda(l, M, \mu). \end{aligned}$$

So,

$$\Lambda(k, M, s(\iota + \mu)) \geq \Lambda(k, l, \iota) \star \Lambda(l, M, \mu). \quad (1)$$

Now, it is obtained by (1) that

$$\begin{aligned} \Lambda(k, M, s(\iota + \mu)) & \geq \Lambda(k, l, \iota) \star \Lambda(l, M, \mu), \\ & \geq \Lambda(k, l, \iota) \star \inf_{l \in L} \Lambda(l, M, \mu), \quad \forall l \in L. \end{aligned}$$

Then

$$\begin{aligned}\Lambda(k, M, s(\iota + \mu)) &\geq \sup_{l \in L} \Lambda(k, l, \iota) \star \inf_{l \in L} \Lambda(l, M, \mu), \\ &= \Lambda(k, L, \iota) \star \inf_{l \in L} \Lambda(l, M, \mu), \quad \forall k \in K.\end{aligned}$$

This implies

$$\begin{aligned}\Pi_{\Lambda}(K, M, s(\iota + \mu)) &= \inf_{k \in K} \Lambda(k, M, s(\iota + \mu)) \\ &\geq \inf_{k \in K} \Lambda(k, L, \iota) \star \inf_{l \in L} \Lambda(l, M, \mu), \\ &\geq \Pi_{\Lambda}(K, L, \iota) \star \Pi_{\Lambda}(L, M, \mu).\end{aligned}$$

Because

$$\left( \begin{array}{l} \inf_{k \in K} \Lambda(k, L, \iota) \geq \Pi_{\Lambda}(K, L, \iota) \text{ and} \\ \inf_{l \in L} \Lambda(l, M, \mu) \geq \Pi_{\Lambda}(L, M, \mu) \end{array} \right).$$

Similarly, if  $\Pi_{\Lambda}(K, M, s(\iota + \mu)) = \inf_{m \in M} \Lambda(K, m, s(\iota + \mu))$ . Then

$$\Pi_{\Lambda}(K, M, s(\iota + \mu)) \geq \Pi_{\Lambda}(K, L, \iota) \star \Pi_{\Lambda}(L, M, \mu).$$

( $\Pi_4$ ) Let  $K, L \in C(\xi)$ , let us show that

$\Pi_{\Lambda}(K, L, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous, let  $(\iota_n)$  a sequence of elements in  $(0, \infty)$  which is converged to  $\iota$ , then it is proven that  $\lim_{n \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota_n) = \Pi_{\Lambda}(K, L, \iota)$ .

By Definition 2.1 ( $b_5$ ), for all  $(k, l) \in K \times L$ , it is obtained that

$$\lim_{n \rightarrow \infty} \Lambda(k, l, \iota_n) = \Lambda(k, l, \iota), \text{ i.e}$$

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n > n_0 :$$

$$\Lambda(k, l, \iota) - \varepsilon \leq \Lambda(k, l, \iota_n) \leq \Lambda(k, l, \iota) + \varepsilon.$$

Then

$$\sup_{l \in L} \Lambda(k, l, \iota) - \varepsilon \leq \sup_{l \in L} \Lambda(k, l, \iota_n) \leq \sup_{l \in L} \Lambda(k, l, \iota) + \varepsilon.$$

So,

$$\Lambda(k, L, \iota) - \varepsilon \leq \Lambda(k, L, \iota_n) \leq \Lambda(k, L, \iota) + \varepsilon.$$

This implies

$$\inf_{k \in K} \Lambda(k, L, \iota) - \varepsilon \leq \inf_{k \in K} \Lambda(k, L, \iota_n) \leq \inf_{k \in K} \Lambda(k, L, \iota) + \varepsilon. \quad (2)$$

Similarly, it is found that

$$\inf_{l \in L} \Lambda(K, l, \iota) - \varepsilon \leq \inf_{l \in L} \Lambda(K, l, \iota_n) \leq \inf_{l \in L} \Lambda(K, l, \iota) + \varepsilon. \quad (3)$$

Thus, it is deduced from (2) and (3) that

$$\Pi_{\Lambda}(K, L, \iota) - \varepsilon \leq \Pi_{\Lambda}(K, L, \iota_n) \leq \Pi_{\Lambda}(K, L, \iota) + \varepsilon.$$

So,  $\lim_{n \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota_n) = \Pi_{\Lambda}(K, L, \iota)$ .

( $\Pi_5$ ) Let  $K, L \in C(\xi)$ ,  $\iota > 0$ , and show that

$$\lim_{\iota \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota) = 1 \Leftrightarrow \lim_{\iota \rightarrow \infty} \Lambda(k, l, \iota) = 1.$$

If  $K = \{k\}$  and  $L = \{l\}$  it is obtained that

$$\begin{aligned} \lim_{\iota \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota) = 1 &\Rightarrow \lim_{\iota \rightarrow \infty} \Pi_{\Lambda}(\{k\}, \{l\}, \iota) = 1 \\ &\Rightarrow \lim_{\iota \rightarrow \infty} \Lambda(k, l, \iota) = 1. \end{aligned}$$

Now, if  $\lim_{\iota \rightarrow \infty} \Lambda(k, l, \iota) = 1$ . Suppose that

$$\Pi_{\Lambda}(K, L, \iota) = \inf_{k \in K} \sup_{l \in L} \Lambda(k, l, \iota),$$

according to the characterization of the lower bound and upper bound, there exist  $(k_n, l_n) \in K, \times L$  such that  $\lim_{n \rightarrow \infty} \Lambda(k_n, l_n, \iota) = \Pi_{\Lambda}(K, L, \iota)$ ,  $\forall \iota > 0$ . Thus

$$\begin{aligned} \lim_{\iota \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota) &= \lim_{\iota \rightarrow \infty} \left( \lim_{n \rightarrow \infty} \Lambda(k_n, l_n, \iota) \right) \\ &= \lim_{n \rightarrow \infty} \left( \lim_{\iota \rightarrow \infty} \Lambda(k_n, l_n, \iota) \right) = 1. \end{aligned}$$

Similarly, if

$$\Pi_{\Lambda}(K, L, \iota) = \inf_{l \in L} \sup_{k \in K} \Lambda(k, l, \iota).$$

Then  $\lim_{\iota \rightarrow \infty} \Pi_{\Lambda}(K, L, \iota) = 1$ .

### 3.1. Fixed Point and Multivalued Mappings Compact Subsets of $\xi$

**Definition 3.1.** Let  $\Theta$  be the set of all continuous functions  $\theta(t_1, t_2, t_3, t_4) : [0, 1]^4 \rightarrow \mathbb{R}$  such that:

( $\theta_0$ ) :  $\theta$  is nonincreasing in variable  $t_1$  and nondecreasing in variables  $t_2, t_3, t_4$ .

( $\theta_1$ ) :  $\forall u, v \in [0, 1]$

$$\theta(u, v, v, u) \leq 0 \text{ or } \theta(u, v, u, v) \leq 0 \implies v < u.$$

( $\theta_2$ ) :  $\forall u \in [0, 1] \quad \theta(u, u, 1, 1) \leq 0 \implies u = 1$ .

**Example 3.1.**  $\theta(t_1, t_2, t_3, t_4) = \frac{1}{p} \ln(t_2) - \ln(t_1)$ , with  $p > 1$ .

**Example 3.2.**

$$\theta(t_1, t_2, t_3, t_4) = \frac{1}{1+t_1} - \frac{t_1}{2} - \lambda \left( \frac{1}{1+t_2} - \frac{t_2}{2} \right), \text{ with } 0 < \lambda < 1$$

**Example 3.3.**  $\theta(t_1, t_2, t_3, t_4) = \beta(t_1) - \lambda\beta(t_2)$ , with  $\beta : (0, 1] \rightarrow [0, \infty)$  and  $\beta(1) = 0$  is a continuous function strictly decreasing,  $0 < \lambda < 1$ .

**Example 3.4.**  $\theta(t_1, t_2, t_3, t_4) = \beta(t_1) - \lambda\beta(\frac{t_2+t_3+t_4}{4})$ , with  $\beta : (0, 1] \rightarrow [0, \infty)$  and  $\beta(1) = 0$  is a continuous function strictly decreasing,  $0 < \lambda < 1$ .

**Example 3.5.**  $\theta(t_1, t_2, t_3, t_4) = \gamma(t_1) - k(t)\gamma(t_2)$ , with  $\gamma : (0, 1] \rightarrow [0, \infty)$ ,  $\gamma(1) = 0$  is a continuous function strictly decreasing and  $k$  be a function from  $(0, \infty)$  into  $(0, 1)$ .

**Definition 3.2.** A function  $\Psi : \xi \rightarrow CL(\xi)$ , where  $(\xi, \Lambda, \star)$  is a fuzzy  $b$ -metric space, is called closed if for all sequences  $(x_n)$  and  $(y_n)$  of elements from  $\xi$  and  $x, y \in \xi$  such that  $\lim_{n \rightarrow \infty} x_n = x$ ,  $\lim_{n \rightarrow \infty} y_n = y$  and  $y_n \in \Psi(x_n)$  for every  $n \in \mathbb{N}$ , it is obtained that  $y \in \Psi(x)$ .

**Theorem 3.1.** Let  $(\xi, \Lambda, \star)$  be a complete fuzzy  $b$ -metric space with constant  $s$ , let's suppose that  $\Lambda$  is continuous with respect to one of its variables. Let  $\Psi, \Phi : \xi \rightarrow C(\xi)$  be two closed maps and  $\theta \in \Theta$  such that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi(x), \Phi(y), \iota), \Lambda(x, y, \iota), \Lambda(\Psi(x), x, \iota), \\ \Lambda(\Phi(y), y, \iota) \end{array} \right) \leq 0. \quad (4)$$

Then  $\Psi$  and  $\Phi$  have a common fixed point  $x \in \xi$ .

Moreover, if  $x$  is absolutely fixed for  $\Psi$  and  $\Phi$  (which means that  $\Psi(x) = \{x\}$  and  $\Phi(x) = \{x\}$ ), then the fixed point is unique.

Two lemmas are needed for the proof of this theorem.

*Lemma 3.1.* In a fuzzy  $b$ -metric space  $(\xi, \Lambda, \star)$ , if the

function  $\Lambda$  is continuous with respect to one of its variable, then it is continuous with respect to the other.

*Proof* Suppose that  $\Lambda$  is continuous with respect to the first variable, and let  $(y_n)$  be a sequence of elements of  $\xi$  such that  $(y_n)$  is  $b$ -convergent to  $y \in \xi$ . Then since  $\Lambda$  is symmetric, for all  $x \in \xi$ , it is obtained that

$$\lim_{n \rightarrow \infty} \Lambda(x, y_n, \iota) = \lim_{n \rightarrow \infty} \Lambda(y_n, x, \iota) = \Lambda(y, x, \iota) = \Lambda(x, y, \iota), \quad \iota > 0.$$

*Lemma 3.2.* Let  $(\xi, \Lambda, \star)$  be a fuzzy  $b$ -metric space and let  $A \in C(\xi)$ , if  $\Lambda$  is continuous with respect to one of its variables, then for all  $x \in X$ , there exists  $y_0 \in A$  such that

$$\Lambda(x, A, \iota) = \sup_{y \in A} \Lambda(x, y, \iota) = \Lambda(x, y_0, \iota), \quad \iota > 0.$$

*Proof* Since  $\Lambda(x, A, \iota) = \sup_{y \in A} \Lambda(x, y, \iota)$ , then for every

$n \in \mathbb{N}^*$ ,  $\iota > 0$ , there exists  $x_n \in A$  such that

$$\Lambda(x, A, \iota) - \frac{1}{n} < \Lambda(x, x_n, \iota) \leq \Lambda(x, A, \iota) < \Lambda(x, A, \iota) + \frac{1}{n}.$$

Since  $A$  is compact,  $(x_n)$  has a subsequence, also noted  $(x_n)$ , which is  $b$ -convergent to  $y_0 \in A$ . So,

$$|\Lambda(x, x_n, \iota) - \Lambda(x, A, \iota)| < \frac{1}{n} \longrightarrow 0 \quad \text{when} \quad n \longrightarrow \infty.$$

from where  $\lim_{n \rightarrow \infty} \Lambda(x, x_n, \iota) = \Lambda(x, A, \iota)$ , so since  $\Lambda$  is continuous, it is deduced that

$$\lim_{n \rightarrow \infty} \Lambda(x, x_n, \iota) = \Lambda(x, y_0, \iota),$$

hence from the uniqueness of the limit in a fuzzy  $b$ -metric space, we have  $\Lambda(x, y_0, \iota) = \Lambda(x, A, \iota)$ ,  $\iota > 0$ .

*Proof of the Theorem.*

Let  $x_0 \in \xi$ , and  $x_1 \in \Psi x_0$ . Since  $x_1 \in \Psi x_0$ , it is obtained that

$$\begin{aligned} \Pi_{\Lambda}(\Psi x_0, \Phi x_1, \iota) &= \min \left\{ \inf_{x \in \Psi x_0} \Lambda(x, \Phi x_1, \iota), \inf_{y \in \Phi x_1} \Lambda(\Psi x_0, y, \iota) \right\} \\ &\leq \inf_{x \in \Psi x_0} \Lambda(x, \Phi x_1, \iota) \\ &\leq \Lambda(x_1, \Phi x_1, \iota) = \sup_{y \in \Phi x_1} \Lambda(x_1, y, \iota). \end{aligned}$$

And we have  $\Phi x_1$  is compact and  $\Lambda$  is continuous with respect to one of its variables, so according to Lemma 3.2 there exists  $x_2 \in \Phi x_1$  such that

$$\Pi_{\Lambda}(\Psi x_0, \Phi x_1, \iota) \leq \Lambda(x_1, x_2, \iota), \quad \iota > 0.$$

Similarly, since  $x_2 \in \Phi x_1$ , it is obtained that

$$\Pi_{\Lambda}(\Psi x_2, \Phi x_1, \iota) \leq \Lambda(\Psi x_2, x_2, \iota) = \sup_{x \in \Psi x_2} \Lambda(x, x_2, \iota).$$

And we have  $\Psi x_2$  is compact and  $\Lambda$  is continuous with respect to one of its variables, so according to Lemma 3.2 there exists  $x_3 \in \Psi x_2$  such that

$$\Pi_{\Lambda}(\Psi x_2, \Phi x_1, \iota) \leq \Lambda(x_2, x_3, \iota), \quad \iota > 0.$$

Similarly, there exist  $x_4 \in \Phi x_3$  such that

$$\Pi_{\Lambda}(\Psi x_2, \Phi x_3, \iota) \leq \Lambda(x_3, x_4, \iota), \quad \iota > 0.$$

A sequence  $(x_n)$  is constructed by recurrence such that  $x_{2n+1} \in \Psi x_{2n}$ , and  $x_{2n+2} \in \Phi x_{2n+1}$  which satisfies:

$$\Pi_{\Lambda}(\Psi x_{2n}, \Phi x_{2n-1}, \iota) \leq \Lambda(x_{2n+1}, x_{2n}, \iota), \quad (5)$$

and

$$\Pi_{\Lambda}(\Psi x_{2n}, \Phi x_{2n+1}, \iota) \leq \Lambda(x_{2n+1}, x_{2n+2}, \iota). \quad (6)$$

According to (4), with  $x = x_{2n}$  and  $y = x_{2n-1}$ , it is obtained that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x_{2n}, \Phi x_{2n-1}, \iota), \Lambda(x_{2n}, x_{2n-1}, \iota), \\ \Lambda(\Psi x_{2n}, x_{2n}, \iota), \Lambda(\Phi x_{2n-1}, x_{2n-1}, \iota) \end{array} \right) \leq 0.$$

Since  $x_{2n+1} \in \Psi x_{2n}$ , and  $x_{2n} \in \Phi x_{2n-1}$ , then

$$\Lambda(\Psi x_{2n}, x_{2n}, \iota) \geq \Lambda(x_{2n+1}, x_{2n}, \iota), \quad (7)$$

and

$$\Lambda(\Phi x_{2n-1}, x_{2n-1}, \iota) \geq \Lambda(x_{2n}, x_{2n-1}, \iota). \quad (8)$$

By (5),(7),(8) and  $(\theta_0)$ , it is obtained that

$$\theta \left( \begin{array}{c} \Lambda(x_{2n+1}, x_{2n}, \iota), \Lambda(x_{2n}, x_{2n-1}, \iota), \\ \Lambda(x_{2n+1}, x_{2n}, \iota), \Lambda(x_{2n}, x_{2n-1}, \iota) \end{array} \right) \leq 0.$$

By  $(\theta_1)$ , it is obtained that

$$\Lambda(x_{2n}, x_{2n+1}, \iota) < \Lambda(x_{2n-1}, x_{2n}, \iota), \quad n \in \mathbb{N}, \iota > 0. \quad (9)$$

Similarly, by (4) with  $x = x_{2n}$  and  $y = x_{2n+1}$ , it is obtained that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x_{2n}, \Phi x_{2n+1}, \iota), \Lambda(x_{2n}, x_{2n+1}, \iota), \\ \Lambda(\Psi x_{2n}, x_{2n}, \iota), \Lambda(\Phi x_{2n+1}, x_{2n+1}, \iota) \end{array} \right) \leq 0.$$

Since  $x_{2n+2} \in \Phi x_{2n+1}$ , then

$$\Lambda(\Phi x_{2n+1}, x_{2n+1}, \iota) \geq \Lambda(x_{2n+2}, x_{2n+1}, \iota). \quad (10)$$

By (6),(7),(10) and  $(\theta_0)$ , it is concluded that

$$\theta \left( \begin{array}{c} \Lambda(x_{2n+1}, x_{2n+2}, \iota), \Lambda(x_{2n}, x_{2n+1}, \iota), \\ \Lambda(x_{2n+1}, x_{2n}, \iota), \Lambda(x_{2n+2}, x_{2n+1}, \iota) \end{array} \right) \leq 0.$$

By  $(\theta_1)$ , it is obtained that

$$\Lambda(x_{2n}, x_{2n+1}, \iota) < \Lambda(x_{2n+1}, x_{2n+2}, \iota), \quad n \in \mathbb{N}, \iota > 0. \quad (11)$$

According to (9) and (11), it is determined that

$$\Lambda(x_{n-1}, x_n, \iota) < \Lambda(x_n, x_{n+1}, \iota), \quad n \in \mathbb{N}^*, \iota > 0. \quad (12)$$

So,  $(\Lambda(x_n, x_{n+1}, \iota))$  is a strictly increasing sequence of positive real, numbers in  $[0, 1]$ .

Consider  $\Lambda_n(\iota) = \Lambda(x_n, x_{n+1}, \iota)$ . Then  $(\Lambda_n(\iota))$  is a strictly increasing sequence.

So, there exists  $\Lambda(\iota)$  such that  $\lim_{n \rightarrow \infty} \Lambda_n(\iota) = \Lambda(\iota)$ ,  $\iota > 0$ .

Suppose that  $0 < \Lambda(\iota) < 1$ . According to (4), with  $x = x_{2n}$  and  $y = x_{2n+1}$ , it is obtained that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x_{2n}, \Phi x_{2n+1}, \iota), \Lambda(x_{2n}, x_{2n+1}, \iota), \\ \Lambda(\Psi x_{2n}, x_{2n}, \iota), \Lambda(\Phi x_{2n+1}, x_{2n+1}, \iota) \end{array} \right) \leq 0.$$

By (6),(7),(10) and  $(\theta_0)$ , it is obtained that

$$\begin{aligned} \theta \left( \begin{array}{c} \Lambda(x_{2n+1}, x_{2n+2}, \iota), \Lambda(x_{2n}, x_{2n+1}, \iota), \\ \Lambda(x_{2n+1}, x_{2n}, \iota), \Lambda(x_{2n+2}, x_{2n+1}, \iota) \end{array} \right) &\leq 0, \\ \Rightarrow \theta \left( \Lambda_{2n+1}(\iota), \Lambda_{2n}(\iota), \Lambda_{2n}(\iota), \Lambda_{2n+1}(\iota) \right) &\leq 0. \end{aligned}$$

Since  $\theta$  is continuous function, taking  $n \rightarrow \infty$ , it is obtained that

$$\theta \left( \Lambda(\iota), \Lambda(\iota), \Lambda(\iota), \Lambda(\iota) \right) \leq 0.$$

By  $(\theta_1)$ , it is obtained that  $\Lambda(\iota) < \Lambda(\iota)$ , which leads to a contradiction. Then  $\Lambda(\iota) = 1$ . Now, it will be proven that  $(x_n)$  is a Cauchy sequence. Suppose that  $(x_n)$  is not a Cauchy sequence, then for all  $0 < \varepsilon < 1$ , there exist two sub-sequences  $(x_{n(i)})$  and  $(x_{m(i)})$  such that for each  $i \in \mathbb{N}$ , let  $n(i), m(i) \in \mathbb{N}$  satisfying  $n(i) > m(i) \geq i$ , such that

$$\Lambda(x_{2n(i)}, x_{2m(i)+1}, \iota) \leq 1 - \varepsilon. \quad (13)$$

Now,  $2n(i)$  is chosen corresponding to  $2m(i) + 1$  such that it is the smallest even integer with  $n(i) > m(i)$  and satisfies Inequality (13). Hence,

$$\Lambda(x_{2n(i)-1}, x_{2m(i)+1}, \iota) > 1 - \varepsilon \quad (14)$$

By (13), it is obtained that

$$\begin{aligned} 1 - \varepsilon &\geq \Lambda(x_{2n(i)}, x_{2m(i)+1}, \iota) \\ &\geq \Lambda\left(x_{2n(i)}, x_{2n(i)-1}, \frac{\iota}{2s}\right) \star \Lambda\left(x_{2n(i)-1}, x_{2m(i)+1}, \frac{\iota}{2s}\right). \end{aligned}$$

By (14), it is obtained that

$$\begin{aligned} 1 - \varepsilon &\geq \Lambda(x_{2n(i)}, x_{2m(i)+1}, \iota) \\ &\geq \Lambda\left(x_{2n(i)}, x_{2n(i)-1}, \frac{\iota}{2s}\right) \star 1 - \varepsilon \\ &= \Lambda_{2n(i)-1}\left(\frac{\iota}{2s}\right) \star 1 - \varepsilon. \end{aligned}$$

Taking  $i \rightarrow \infty$ , it is obtained that

$$1 - \varepsilon \geq \lim_{i \rightarrow \infty} \Lambda(x_{2n(i)}, x_{2m(i)+1}, \iota) \geq 1 \star 1 - \varepsilon = 1 - \varepsilon.$$

So,

$$\lim_{i \rightarrow \infty} \Lambda(x_{2n(i)}, x_{2m(i)+1}, \iota) = 1 - \varepsilon.$$

By (4) with  $x = x_{2m(i)}$  and  $y = x_{2n(i)-1}$ , it is obtained that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x_{2m(i)}, \Phi x_{2n(i)-1}, \iota), \\ \Lambda(x_{2m(i)}, x_{2n(i)-1}, \iota), \\ \Lambda(\Psi x_{2m(i)}, x_{2m(i)}, \iota), \\ \Lambda(\Phi x_{2n(i)-1}, x_{2n(i)-1}, \iota) \end{array} \right) \leq 0.$$



Since  $x_{2m(i)+1} \in \Psi x_{2m(i)}$ , and  $x_{2n(i)} \in \Phi x_{2n(i)-1}$ , then

$$\Lambda(\Psi x_{2m(i)}, x_{2m(i)}, \iota) \geq \Lambda(x_{2m(i)+1}, x_{2m(i)}, \iota), \quad (15)$$

and

$$\Lambda(\Phi x_{2n(i)-1}, x_{2n(i)-1}, \iota) \geq \Lambda(x_{2n(i)}, x_{2n(i)-1}, \iota). \quad (16)$$

By (15),(16) and  $(\theta_0)$ , it is obtained that

$$\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x_{2m(i)}, \Phi x_{2n(i)-1}, \iota), \\ \Lambda(x_{2m(i)}, x_{2n(i)-1}, \iota), \\ \Lambda(x_{2m(i)+1}, x_{2m(i)}, \iota), \\ \Lambda(x_{2n(i)}, x_{2n(i)-1}, \iota) \end{array} \right) \leq 0.$$

Since  $x_{2m(i)+1} \in \Psi x_{2m(i)}$  and  $x_{2n(i)} \in \Phi x_{2n(i)-1}$ , then

$$\begin{aligned} \Pi_{\Lambda}(\Psi x_{2m(i)}, \Phi x_{2n(i)-1}, \iota) &\leq \Lambda(x_{2m(i)+1}, \Phi x_{2n(i)-1}, \iota) \\ &= \Lambda(x_{2m(i)+1}, \Phi x_{2n(i)-1}, \iota) \star 1 \\ &= \Lambda(x_{2m(i)+1}, \Phi x_{2n(i)-1}, \iota) \star \Lambda(x_{2n(i)}, \Phi x_{2n(i)-1}, \iota) \\ &\leq \Lambda(x_{2m(i)+1}, x_{2n(i)}, 2\iota s). \end{aligned}$$

So

$$\Pi_{\Lambda}(\Psi x_{2m(i)}, \Phi x_{2n(i)-1}, \iota) \leq \Lambda(x_{2m(i)+1}, x_{2n(i)}, 2\iota s). \quad (17)$$

By (17) and  $(\theta_0)$ , it is obtained that

$$\begin{aligned} &\theta \left( \begin{array}{c} \Lambda(x_{2n(i)}, x_{2m(i)+1}, 2s\iota), \\ \Lambda_{2m(i)}\left(\frac{\iota}{2s}\right) \star \Lambda(x_{2m(i)+1}, x_{2n(i)-1}, \frac{\iota}{2s}), \\ \Lambda_{2m(i)}(\iota), \Lambda_{2n(i)-1}(\iota) \end{array} \right) \leq 0 \\ \Rightarrow &\theta \left( \begin{array}{c} \Lambda(x_{2n(i)}, x_{2m(i)+1}, 2s\iota), \Lambda_{2m(i)}\left(\frac{\iota}{2s}\right) \star 1 - \varepsilon, \\ \Lambda_{2m(i)}(\iota), \Lambda_{2n(i)-1}(\iota) \end{array} \right) \leq 0. \end{aligned}$$

taking  $i \rightarrow \infty$ , it is obtained that

$$\begin{aligned} &\theta(1 - \varepsilon, 1 \star 1 - \varepsilon, 1, 1) \leq 0 \\ \Rightarrow &\theta(1 - \varepsilon, 1 - \varepsilon, 1, 1) \leq 0. \end{aligned}$$

By  $(\theta_2)$ , it is obtained that  $1 - \varepsilon = 1$ , which leads to a contradiction.

Hence  $(x_n)$  is a Cauchy sequence in a complete fuzzy  $b$ -metric space. So there exists  $x \in \xi$  such that  $\lim_{n \rightarrow \infty} \Lambda(x_n, x, \iota) = 1$ . Next, it is shown that  $x \in \Psi x \cap \Phi x$ , indeed, we have  $x_{2n} \in \Phi x_{2n-1}$  and  $x_{2n+1} \in \Psi x_{2n}$ ,  $n \in \mathbb{N}$ .

Since  $\lim_{n \rightarrow \infty} x_{2n-1} = \lim_{n \rightarrow \infty} x_{2n} = \lim_{n \rightarrow \infty} x_{2n+1} = x$ ,  $\Psi$  and  $\Phi$  are closed, then  $x \in \Psi x \cap \Phi x$ .

*Unicity.* Suppose that  $\Psi x = \{x\}$  and there exists  $y \in X$  is another common fixed point of  $\Psi$  and  $\Phi$ , such that  $\Phi y = \{y\}$ . Then by (4), it is obtained that

$$\begin{aligned} &\theta \left( \begin{array}{c} \Pi_{\Lambda}(\Psi x, \Phi y, \iota), \Lambda(x, y, \iota), \Lambda(\Psi x, x, \iota), \\ \Lambda(\Phi y, y, \iota) \end{array} \right) \leq 0 \\ \Rightarrow &\theta \left( \begin{array}{c} \Lambda(x, y, \iota), \Lambda(x, y, \iota), \Lambda(x, x, \iota), \\ \Lambda(y, y, \iota) \end{array} \right) \leq 0 \\ \Rightarrow &\theta(\Lambda(x, y, \iota), \Lambda(x, y, \iota), 1, 1) \leq 0. \end{aligned}$$

By  $(\theta_2)$ , it is obtained that  $\Lambda(x, y, \iota) = 1$ , then  $x = y$ .

### 3.2. Fixed Point and Multivalued Mappings Closed Subsets of $\xi$

**Theorem 3.2.** Let  $(\xi, \Lambda, \star)$  be a complete fuzzy  $b$ -metric space with constant  $s$ . Let  $\Psi : \xi \longrightarrow CL(\xi)$  be a closed maps and  $\theta \in \Theta$  such that

$$\theta \left( \begin{array}{c} \delta_{\Lambda}(\Psi(x), \Psi(y), \iota), \Lambda(x, y, \iota), \Lambda(\Psi(x), x, \iota), \\ \Lambda(\Psi(y), y, \iota) \end{array} \right) \leq 0. \quad (18)$$

Then  $\Psi$  has a unique fixed point  $x \in \xi$ .

*Proof* Let  $x_0 \in \xi$ , define the sequence  $(x_n)$  of elements from  $\xi$  such that:  $x_{n+1} \in \Psi x_n$  for every  $n \in \mathbb{N}$ .

According to (18), with  $x = x_{n-1}$  and  $y = x_n$ , it is obtained that

$$\theta \left( \begin{array}{c} \delta_{\Lambda}(\Psi x_{n-1}, \Psi x_n, \iota), \Lambda(x_{n-1}, x_n, \iota), \\ \Lambda(\Psi x_{n-1}, x_{n-1}, \iota), \Lambda(\Psi x_n, x_n, \iota) \end{array} \right) \leq 0.$$

Since  $x_n \in \Psi x_{n-1}$  and  $x_{n+1} \in \Psi x_n$ , it is obtained that

$$\begin{aligned} \Lambda(\Psi x_{n-1}, x_{n-1}, \iota) &\geq \Lambda(x_n, x_{n-1}, \iota), \\ \Lambda(\Psi x_n, x_n, \iota) &\geq \Lambda(x_{n+1}, x_n, \iota) \\ \text{and } \delta_{\Lambda}(\Psi x_{n-1}, \Psi x_n, \iota) &\leq \Lambda(x_n, x_{n+1}, \iota). \end{aligned}$$

By  $(\theta_0)$ , it is obtained that

$$\theta \left( \begin{array}{c} \Lambda(x_n, x_{n+1}, \iota), \Lambda(x_{n-1}, x_n, \iota), \\ \Lambda(x_n, x_{n-1}, \iota), \Lambda(x_{n+1}, x_n, \iota) \end{array} \right) \leq 0.$$

By  $(\theta_1)$ , it is obtained that

$$\Lambda(x_{n-1}, x_n, \iota) < \Lambda(x_n, x_{n+1}, \iota), \quad n \in \mathbb{N}^*, \iota > 0.$$

Thus, using the same argument as in Theorem 3.1, it is deduced that  $x_n$  is a Cauchy sequence and is converged to  $x \in \xi$ , the fixed point of  $\Psi$ .

*Unicity.* Suppose that there exists  $y \in \xi$  is another fixed point of  $\Psi$ . Then, it is obtained by (18) that

$$\theta \left( \begin{array}{c} \delta_{\Lambda}(\Psi x, \Psi y, \iota), \Lambda(x, y, \iota), \Lambda(\Psi x, x, \iota), \\ \Lambda(\Psi y, y, \iota) \end{array} \right) \leq 0$$

by  $(\theta_0)$ , it is obtained that

$$\begin{aligned} &\theta \left( \begin{array}{c} \Lambda(x, y, \iota), \Lambda(x, y, \iota), \Lambda(x, x, \iota), \\ \Lambda(y, y, \iota) \end{array} \right) \leq 0 \\ \Rightarrow &\theta(\Lambda(x, y, \iota), \Lambda(x, y, \iota), 1, 1) \leq 0. \end{aligned}$$

By  $(\theta_2)$ , it is obtained that  $\Lambda(x, y, \iota) = 1$ , then  $x = y$ .

As a consequence of Theorem 3.2, if  $\Psi = f$  is single-valued mapping, the following is obtained

**Corollary 3.1.** Let  $(\xi, \Lambda, \star)$  be a complete fuzzy  $b$ -metric space with constant  $s$ . Let  $f : \xi \longrightarrow \xi$  be a continuous mapping and  $\theta \in \Theta$  such that

$$\theta(\Lambda(fx, fy, \iota), \Lambda(x, y, \iota), \Lambda(fx, x, \iota), \Lambda(fy, y, \iota)) \leq 0.$$

Then  $f$  has a unique fixed point  $x \in \xi$ .

**Example 3.6.** Let  $\xi$  be the subset of  $\mathbb{R}^3$  defined by  $\xi = \{A, B, C, D\}$ ,

where  $A = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $C = (0, 0, 1)$  and  $D = (2, 2, 2)$ .  $c \star d = c.d$  for all  $c, d \in [0, 1]$  and  $(\xi, \Lambda, \star)$  is a complete fuzzy  $b$ -metric space such that:

$$\Lambda(x, y, \iota) = e^{\frac{-d(x, y)}{\iota}}, \quad x, y \in \xi, \iota > 0,$$

where  $d(x, y)$  denotes the Euclidean distance of  $\mathbb{R}^3$ .

Let  $\Psi : \xi \rightarrow \xi$  be given by

$$\Psi(A) = \Psi(B) = \Psi(C) = A, \Psi(D) = B.$$

Show that for all  $x, y \in \xi$

$$\theta \left( \Lambda(\Psi x, \Psi y, \iota), \Lambda(x, y, \iota), \Lambda(\Psi x, x, \iota), \Lambda(\Psi y, y, \iota) \right) \leq 0.$$

With  $\theta$  as in Example 3.3, and  $\beta(t) = -\ln(t)$ .

Indeed: If  $x, y \in \{A, B, C\}$ , it is obtained that  $\Lambda(\Psi x, \Psi y, \iota) = \Lambda(A, A, \iota) = 1$ .

So  $\beta(\Lambda(\Psi x, \Psi y, \iota)) = 0 \leq \lambda \beta(\Lambda(x, y, \iota))$ ,  $\lambda \in (0, 1)$ .

If  $x \in \{A, B, C\}$  and  $y = D$ , it is found that

$$\Lambda(\Psi x, \Psi y, \iota) = e^{-\frac{\sqrt{2}}{\iota}} \text{ and } \Lambda(x, y, \iota) = e^{-\frac{3}{\iota}}.$$

So  $\beta(\Lambda(\Psi x, \Psi y, \iota)) \leq \lambda \beta(\Lambda(x, y, \iota))$ ,  $\lambda \in (\frac{\sqrt{2}}{3}, 1)$ .

Now, all the hypotheses of Corollary 3.1 are satisfied and thus  $\Psi$  has a unique fixed point, that is  $x = A$ .

*Remark 3.1.* Theorem 3.1 in [22] is obtained from Corollary 3.1 and Example 3.5.

## 4. Conclusions

In this paper, a new space called a Pompeiu-Hausdorff fuzzy  $b$ -metric space is constructed. Some examples in this space are presented. Additionally, some new fixed-point results in this space are formulated and proven, which extend the results of [22], and the existence and uniqueness of the common fixed point in such a space are demonstrated.

The approach proposed may pave the way for new developments in generalized metrical structures and fixed-point theory. The results obtained can be further used to investigate coincidence, common, and relation theoretic fixed-point results.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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