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A Reaction-diffusion System Modeling the Transmission of Typhoid Fever in a Periodic Environment

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Abstract: Typhoid fever is a life-threatening infection caused by the bacterium Salmonella Typhi, and it is still an important issue in developing countries. There are two infection routes of Typhoid fever, namely, the human-to-human transmission and the environment-to-human transmission. It is evident that people living near rivers may have a higher rate of typhoid infection, and temperature changes also have significant impacts on Typhoid transmission dynamics. In the model, the population of human will be divided into susceptible individuals, infected individuals, carrier individuals, individuals under treatment, and recovered individuals. Then a periodic dispersion-reaction system is used to describe the transport and the interactions between human and bacteria in the environment. The solution maps of the proposed periodic dispersion-reaction system lack the compactness since the population under treatment has no diffusion term, which makes analysis more difficult. After the feasible domain is chosen carefully, the eventually boundedness of the solutions can be established, and the loss of compactness is overcome if the initial data is chosen from the feasible domain. In order to introduce the reproduction number $\mathcal{R}_0$, the linearized system around the disease-free state is constructed, and the basic reproduction number is defined as the spectral radius of the next generation operator. Then the comparison principle and persistence theory can be utilized to establish that the index $\mathcal{R}_0$ completely determines the threshold behavior of the typhoid spread. Brief mathematical and biological interpretations are also presented.

Keywords: Typhoid Fever, Spatial Variations, Seasonality, Basic Reproduction Number, Global Dynamics, Reaction-diffusion Model, Noncompact Solution Maps

1. Introduction

According to the report of World Health Organization, typhoid fever is a life-threatening infection caused by the bacterium Salmonella Typhi (S. Typhi) and it is still an important issue in developing countries. The infection routes of typhoid fever include human-to-human transmission and environment-to-human transmission, and it is usually spread through the ingestion of contaminated food or water. In [17], a mathematical system of ordinary differential equations was proposed to model the spread of typhoid under the situation that the resource of the treatment is restricted. A higher incidence of typhoid fever usually occurs during the rainy season [10, 11], due to the fact that the excreta with pathogenic bacteria pollutes drinking water. On the other hand, it will be more realistic to incorporate spatial variations into the governing model since it is observed that people living near rivers may have a higher infection rate of typhoid [3]. Another observation is that the effects of temperature also play central roles in the transmission of typhoid fever [3], which motivates researchers to include the seasonality in the model.

There are two approaches of modeling the seasonality and spatial homogeneity. One simple approach is the two-patch system in a temporal environment, namely, the environment is divided into two zones, and the gradient of human and bacteria between different zones is modeled by the migration [12]. In this paper, we shall adopt the second approach using a reaction-diffusion model to describe the movements and transmission between human and bacteria in a periodic and bounded environment. The above discussions motivate us to propose the following periodic reaction-diffusion system describing the interactions of populations and bacteria in the environment. Assume that $\Omega \subset \mathbb{R}^2$ is the habitat of populations, and $\frac{\partial}{\partial \nu}$ denotes the differentiation along the outward normal ν to the boundary of Ω, $\partial\Omega$. For the population of human, $S(x,t)$ represents the susceptible class at the location x and time t; $I(x,t)$ stands for the infected class; $C(x,t)$ is the carrier class; $Q(x,t)$ represents the individuals under treatment; $R(x,t)$ represents the recovered individuals. Besides, $B(x,t)$ stands for the density of bacteria at the location x and time t. Then the system takes the following form:

$$\begin{cases}\frac{\partial S}{\partial t} = D_S \Delta S + \Lambda(x) - \frac{\beta_C(x,t)(I+\eta C)S}{S+I+C+Q+R} - \frac{\beta_B(x,t)BS}{B+K_B} - \mu S + \rho R, \ x \in \Omega, \ t>0, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \frac{\beta_C(x,t)(I+\eta C)S}{S+I+C+Q+R} + \frac{\beta_B(x,t)BS}{B+K_B} - (\mu+\sigma+\delta_I+\epsilon_I)I - \theta I, \ x \in \Omega, \ t>0, \\ \frac{\partial C}{\partial t} = D_C \Delta C + \sigma I - (\mu + \delta_C + \epsilon_C)C, \ x \in \Omega, \ t>0, \\ \frac{\partial Q}{\partial t} = \theta I - (\mu+\gamma+\delta_Q)Q, \ x \in \Omega, \ t>0, \\ \frac{\partial R}{\partial t} = D_R \Delta R + \gamma Q + \epsilon_I I + \epsilon_C C - (\mu+\rho)R, \ x \in \Omega, \ t>0, \\ \frac{\partial B}{\partial t} = D_B \Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B, \ x \in \Omega, \ t>0, \\ \frac{\partial u(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t>0, \ u = S,I,C,R,B, \\ u(x,0) = u^0(x), \ x \in \Omega, \ t>0, \ u = S,I,C,Q,R,B. \end{cases} \tag{1}$$

Here the parameter $\Lambda(x)$ represents the recruitment of susceptible population on the location x, and μ stands for the natural mortality of human population. In the following, the abbreviation resp. always means respective. The parameter $\beta_C(x,t)$ (resp. $\beta_B(x,t)$) is the transmission rate (resp. the per capita contact rate) between susceptible individuals and infected/carrier individuals (resp. the environmental bacteria). The parameter η is the measure of the relative infectiousness satisfying $0<\eta<1$ (resp. $\eta>1$) when the infectiousness of carriers is weaker (resp. stronger) than infected individuals; K_B is the saturation constant. Then susceptible population $S(x,t)$ can be either infected by the infected people $I(x,t)$ at the rate $\frac{\beta_C(x,t)(I(x,t)+\eta C(x,t))S(x,t)}{S(x,t)+I(x,t)+C(x,t)+Q(x,t)+R(x,t)}$ or infected through the bacteria $B(x,t)$ at the rate $\frac{\beta_{B(x,t)}(t)B(x,t)S(x,t)}{B(x,t)+K_B}$. People in infected class can become the carrier individuals (resp. treatment class) at the rate σ (resp. θ); ϵ_I (resp. ϵ_C) represents the recovery rate for $I(x,t)$ (resp. $C(x,t)$); δ_I (resp. δ_C) represents disease-related death rate for $I(x,t)$ (resp. $C(x,t)$); $\alpha_I(x,t)$ (resp. $\alpha_C(x,t)$) stands for the bacteria shedding rate by $I(x,t)$ (resp. $C(x,t)$). The disease-related death rate for people under treatment is δ_Q, and γ represents the recovery rate. People in the recovered class can become susceptible at the rate ρ. The generation of bacteria is $gB(x,t)$ with g being a positive constant; the bacteria can become non-infectious at the rate $\mu_B(x)$. The Laplace operator is denoted by Δ; D_S, D_I, D_C, D_R, and D_B are the diffusion coefficients related to human and bacteria, respectively. It is worth pointing out that the class $Q(x,t)$ is supposed to be on treatment, and no diffusion term ΔQ is included in system (1). This will cause some troubles in mathematical analysis.

The parameters $\beta_C(\cdot,t)$, $\beta_B(\cdot,t)$, $\alpha_I(\cdot,t)$, and $\alpha_C(\cdot,t)$ are assumed to be ω-periodic functions, and

$$\mu_B(x) - g > 0, \ \forall \, x \in \bar{\Omega}, \tag{2}$$

which can coincide with the parameters provided in Table 2 of [17]. The organization of the rest of this paper is as follows. The well-posedness of our system (1) is provided in the next section; the basic reproduction number is defined in section 3. Section 4 is devoted to the establishment that the persistence/extinction of typhoid fever can be determined by the basic reproduction number. A brief conclusion is presented in Section 5.

2. Well-posedness

This section is devoted to the study the well-posedness for system (1). Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^6)$ be the Banach space with the supremum norm $\| \cdot \|_{\mathbb{X}}$. Define $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^6_+)$, then $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered space. Let

$$\Lambda^{\min} := \min_{x\in\bar{\Omega}} \Lambda(x), \ \beta_C^{\max} := \max_{(x,t)\in\bar{\Omega}\times[0,\omega]} \beta_C(x,t), \ \beta_B^{\max} := \max_{(x,t)\in\bar{\Omega}\times[0,\omega]} \beta_B(x,t).$$

Assume that ζ is a positive number satisfying

$$0 < \zeta < \frac{\Lambda^{\min}}{(1+\eta)\beta_C^{\max} + \beta_B^{\max} + \mu}. \tag{3}$$

Let $\mathbb{X}_\zeta^+ := \{\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in \mathbb{X}^+ : \zeta \leqslant \phi_1(x), \text{ for all } x \in \bar{\Omega}\}$.

From Corollary 4 in [18] and Lemma 2.2 in [8] (see also [14]), the following result can be proved.

Lemma 2.1. *Assume that ζ is given in* (3). *For every initial value function*

$$u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in \mathbb{X}_\zeta^+,$$

system (1) *admits a unique solution*

$$u(x,t,u^0) := (S(x,t), I(x,t), C(x,t), Q(x,t), R(x,t), B(x,t)) \in \mathbb{X}_\zeta^+$$

on $(0, \tau_{u^0})$ *with* $u(\cdot, 0, u^0) = u^0$, *where* $\tau_{u^0} \leq \infty$.

Proof Suppose that $T_1(t),\ T_2(t),\ T_3(t),\ T_5(t),\ T_6(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ are the C_0 semigroups associated with $D_S\Delta - \mu$, $D_I\Delta - (\mu + \sigma + \delta_I + \epsilon_I + \theta)$, $D_C\Delta - (\mu + \delta_C + \epsilon_C)$, $D_R\Delta - (\mu + \rho)$ and $D_B\Delta - \mu_B(\cdot)$ subject to the Neumann boundary condition, respectively. It then follows that for any $\varphi \in C(\bar{\Omega}, \mathbb{R})$, $t \geqslant 0$,

$$(T_1(t)\varphi)(x) = e^{-\mu t} \int_\Omega \Gamma_1(t,x,y)\varphi(y)dy,$$

$$(T_2(t)\varphi)(x) = e^{-(\mu+\sigma+\delta_I+\epsilon_I+\theta)t} \int_\Omega \Gamma_2(t,x,y)\varphi(y)dy,$$

$$(T_3(t)\varphi)(x) = e^{-(\mu+\delta_C+\epsilon_C)t} \int_\Omega \Gamma_3(t,x,y)\varphi(y)dy,$$

$$(T_5(t)\varphi)(x) = e^{-(\mu+\rho)t} \int_\Omega \Gamma_5(t,x,y)\varphi(y)dy,$$

and

$$(T_6(t)\varphi)(x) = \int_\Omega \Gamma_6(t,x,y)\varphi(y)dy,$$

where Γ_1, Γ_2, Γ_3, Γ_5 and Γ_6 are the Green functions associated with $D_S\Delta$, $D_I\Delta$, $D_C\Delta$, $D_R\Delta$ and $D_B\Delta - \mu_B(\cdot)$ subject to the Neumann boundary conditions, respectively. Let

$$T_4(t)\varphi = e^{-(\mu+\gamma+\delta_Q)t}\varphi,\ \forall\, \varphi \in C(\bar{\Omega}, \mathbb{R}),$$

and

$$T(t) := (T_1(t), T_2(t), T_3(t), T_4(t), T_5(t), T_6(t)).$$

Define $F(t) = (F_1(t), F_2(t), F_3, F_4, F_5, F_6(t)) : \mathbb{X}_\zeta^+ \to \mathbb{X}$, $t \geq 0$, by

$$\begin{cases} F_1(t,\phi)(\cdot) = \Lambda(x) - \frac{\beta_C(x,t)(\phi_2+\eta\phi_3)\phi_1}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6} - \frac{\beta_B(x,t)\phi_6\phi_1}{\phi_6+K_B} + \rho\phi_5, \\ F_2(t,\phi)(\cdot) = \frac{\beta_C(x,t)(\phi_2+\eta\phi_3)\phi_1}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6} + \frac{\beta_B(x,t)\phi_6\phi_1}{\phi_6+K_B}, \\ F_3(\phi)(\cdot) = \sigma\phi_2, \\ F_4(\phi)(\cdot) = \theta\phi_2, \\ F_5(\phi)(\cdot) = \gamma\phi_4 + \epsilon_I\phi_2 + \epsilon_C\phi_3, \\ F_6(t,\phi)(\cdot) = \alpha_I(x,t)\phi_2 + \alpha_C(x,t)\phi_3. \end{cases}$$

Then system (1) can be rewritten as the following integral equation

$$u(t) = T(t)\phi + \int_0^t T(t-s)F(s, u(\cdot,s))ds.$$

Observing that

$$\begin{aligned}\frac{\beta_C(x,t)(\phi_2+\eta\phi_3)\phi_1}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6} &= \beta_C(x,t)\frac{\phi_2}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6}\phi_1+\beta_C(x,t)\eta\frac{\phi_3}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6}\phi_1\\ &\le \beta_C^{\max}\phi_1+\beta_C^{\max}\eta\phi_1,\end{aligned}$$

and

$$\frac{\beta_B(x,t)\phi_6\phi_1}{\phi_6+K_B}=\beta_B(x,t)\frac{\phi_6}{\phi_6+K_B}\phi_1\le\beta_B^{\max}\phi_1.$$

Hence,

$$\begin{aligned}&\phi_1+hF_1(t,\phi)=\phi_1+h[\Lambda(x)-\frac{\beta_C(x,t)(\phi_2+\eta\phi_3)\phi_1}{\phi_1+\phi_2+\phi_3+\phi_4+\phi_6}-\frac{\beta_B(x,t)\phi_6\phi_1}{\phi_6+K_B}+\rho\phi_5]\\ &\ge \phi_1+h[\Lambda^{\min}-(\beta_C^{\max}+\beta_C^{\max}\eta+\beta_B^{\max})\phi_1]\\ &=[1-h(\beta_C^{\max}+\beta_C^{\max}\eta+\beta_B^{\max})]\phi_1+h\Lambda^{\min},\ \forall\, h>0.\end{aligned}$$

Thus, the following relation holds

$$\lim_{h\to 0^+} dist(\phi+hF(t,\phi),\mathbb{X}^+)=0,\ \forall\,\phi\in\mathbb{X}_\zeta^+.$$

By Corollary 4 in [18] or Theorem 7.3.1 in [20], it follows that for every initial value function $u^0(\cdot)\in\mathbb{X}_\zeta^+$, system (1) admits a unique solution $u(x,t,u^0)\in\mathbb{X}^+$ on $(0,\tau_{u^0})$ with $u(\cdot,0,u^0)=u^0$, where $\tau_{u^0}\le\infty$.

On the other hand, it is easy to see that

$$\frac{\beta_C(x,t)(I+\eta C)S}{S+I+C+Q+R}=\beta_C(x,t)S\frac{I}{S+I+C+Q+R}+\beta_C(x,t)\eta S\frac{C}{S+I+C+Q+R}\le\beta_C^{\max}S+\beta_C^{\max}\eta S,$$

and

$$\frac{\beta_B(x,t)BS}{B+K_B}=\beta_B(x,t)\frac{B}{B+K_B}S\le\beta_B^{\max}S.$$

Then the susceptible population $S(x,t)$ satisfies

$$\frac{\partial S}{\partial t}\ge D_S\Delta S+\Lambda^{\min}-(\beta_C^{\max}+\beta_C^{\max}\eta+\beta_B^{\max}+\mu)S.$$

By standard comparison arguments, it is not hard to show that if $S(\cdot,0)\ge\zeta$ then $S(\cdot,t)\ge\zeta$ for $t\ge 0$. The proof is complete.

Consider the following two scalar reaction-diffusion systems

$$\begin{cases}\frac{\partial S}{\partial t}=D_S\Delta S+\Lambda(x)-\mu S,\ x\in\Omega,\ t>0,\\ \frac{\partial S(x,t)}{\partial\nu}=0,\ x\in\partial\Omega,\ t>0,\\ S(x,0)=S^0(x),\ x\in\Omega,\ t>0,\end{cases}\tag{4}$$

and

$$\begin{cases}\frac{\partial B}{\partial t}=D_B\Delta B+gB-\mu_B(x)B,\ x\in\Omega,\ t>0,\\ \frac{\partial B(x,t)}{\partial\nu}=0,\ x\in\partial\Omega,\ t>0,\\ B(x,0)=B^0(x),\ x\in\Omega,\ t>0.\end{cases}\tag{5}$$

The global dynamics of systems (4) and (5) are described in the following lemma.

Lemma 2.2. *([14, Lemma 1])*

The following statements are valid.

(i) *System* (4) *admits a unique positive steady state* $S^*(x)$, *which is globally asymptotically stable in* $C(\bar{\Omega},\mathbb{R}_+)$*;*

(ii) *System* (5) *admits a unique non-negative steady state* 0, *which is globally asymptotically stable in* $C(\bar{\Omega},\mathbb{R}_+)$

provided that the assumption (2) *is imposed.*

We are ready to establish that system (1) has a compact global attractor in $\mathbb{X}_\zeta^+$.

Lemma 2.3. *For every initial value function*

$$u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in \mathbb{X}_\zeta^+,$$

system (1) *has a unique solution*

$$u(x,t,u^0) := (S(x,t), I(x,t), C(x,t), Q(x,t), R(x,t), B(x,t))$$

on $[0,\infty)$ *with* $u(\cdot,0,u^0) = u^0$. *Further, the solution is eventually bounded and the solution maps* $\Phi(t) : \mathbb{X}_\zeta^+ \to \mathbb{X}_\zeta^+$ *associated with system* (1) *are given by* $\Phi(t)u^0(\cdot) = u(\cdot,t,u^0),\ t \geq 0$, *which has a global compact attractor in* $\mathbb{X}_\zeta^+$, $\forall\, t \geq 0$.

Proof Let $\Lambda^{\max} := \max_{x\in\bar{\Omega}} \Lambda(x)$ and

$$U(t) = \int_\Omega [S(x,t) + I(x,t) + C(x,t) + Q(x,t) + R(x,t)]\, dx.$$

Then it follows from (1) that

$$\frac{dU(t)}{dt} = \int_\Omega \Lambda dx - \mu U(t) - \delta_I \int_\Omega I dx - \delta_C \int_\Omega C dx - \delta_Q \int_\Omega Q dx \leq \int_\Omega \Lambda dx - \mu U(t).$$

Thus,

$$\frac{dU(t)}{dt} + \mu U(t) \leq \int_\Omega \Lambda(x) dx \leq |\Omega|\Lambda^{\max},$$

which yields

$$U(t) \leq U(0)e^{-\mu t} + \frac{|\Omega|\Lambda^{\max}}{\mu}(1 - e^{-\mu t}). \tag{6}$$

Then it follows from (6), Theorem 1 in [9] (see also [13]), and the positiveness of solutions that there exists a positive constant M_1 depending on initial data such that the solution (S, I, C, R) of (1) satisfies

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|C(\cdot,t)\|_{L^\infty(\Omega)} + \|R(\cdot,t)\|_{L^\infty(\Omega)} \leq M_1,\ \forall t \geq 0. \tag{7}$$

Furthermore, it follows from (6) that

$$\limsup_{t\to\infty} U(t) \leq \frac{|\Omega|\Lambda^{\max}}{\mu},$$

where $\frac{|\Omega|\Lambda^{\max}}{\mu}$ is independent of initial data. By applying Theorem 1 in [9] to (1) again, it follows that there exists a positive constant M_2 independent of initial data such that

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|C(\cdot,t)\|_{L^\infty(\Omega)} + \|R(\cdot,t)\|_{L^\infty(\Omega)} \leq M_2,\ \forall t \geq \hat{t}, \tag{8}$$

for some large time $\hat{t} > 0$. Let

$$\alpha_I^{\max} := \max_{(x,t)\in\bar{\Omega}\times[0,\omega]} \alpha_I(x,t),\ \alpha_C^{\max} := \max_{(x,t)\in\bar{\Omega}\times[0,\omega]} \alpha_C(x,t),\ \mu_B^{\min} := \min_{x\in\bar{\Omega}} \mu_B(x).$$

Then it follows from (1) and (7) that

$$\frac{\partial Q}{\partial t} \leq \theta M_1 - (\mu + \gamma + \delta_Q)Q,\ x \in \Omega,\ t > 0, \tag{9}$$

and

$$\begin{cases} \frac{\partial B}{\partial t} \leq D_B \Delta B + (\alpha_I^{\max} + \alpha_C^{\max})M_1 - (\mu_B^{\min} - g)B,\ x \in \Omega,\ t > 0, \\ \frac{\partial B(x,t)}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{10}$$

Note that M_1 is independent of initial data. By comparison arguments, together with (9) and (10), it follows that there exists a positive constant $\hat{M}_1$ depending on initial data such that the solution (Q, B) of (1) satisfies

$$\|Q(\cdot,t)\|_{L^\infty(\Omega)} + \|B(\cdot,t)\|_{L^\infty(\Omega)} \le \hat{M}_1,\ \forall t \ge 0.$$

On the other hand, it follows from (1) and (8) that

$$\frac{\partial Q}{\partial t} \le \theta M_2 - (\mu + \gamma + \delta_Q)Q,\ x \in \Omega,\ t \ge \hat{t}, \tag{11}$$

and

$$\begin{cases} \frac{\partial B}{\partial t} \le D_B \Delta B + (\alpha_I^{\max} + \alpha_C^{\max})M_2 - (\mu_B^{\min} - g)B,\ x \in \Omega,\ t \ge \hat{t}, \\ \frac{\partial B(x,t)}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{12}$$

By comparison arguments, together with(11) and (12), it follows that

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} Q(x,t) \le \frac{\theta M_2}{\mu + \gamma + \delta_Q} \text{ and } \limsup_{t\to\infty} \max_{x\in\bar{\Omega}} B(x,t) \le \frac{(\alpha_I^{\max} + \alpha_C^{\max})M_2}{\mu_B^{\min} - g}.$$

From the above discussions, it follows that $\Phi(t)$ is point dissipative on $\mathbb{X}_\zeta^+$, and forward orbits of bounded subsets of $\mathbb{X}_\zeta^+$ for $\Phi(t)$ are bounded. Since the equation of Q in (1) has no diffusion term, system (1) lacks the property of compactness. To overcome this problem, it is clear that the reaction term of Q in (1) takes the form

$$f(I, Q) := \theta I - (\mu + \gamma + \delta_Q)Q,$$

which satisfies

$$\frac{\partial f(I, Q)}{\partial Q} = -(\mu + \gamma + \delta_Q) < 0,\ \forall\, I \ge 0,\ Q \ge 0.$$

Recall that the Kuratowski measure of noncompactness (see, e.g., [2]) is defined by

$$\kappa(B) := \inf\{r : B \text{ has a finite cover of diameter} < r\}$$

for any bounded set $B \subset \mathbb{X}_\zeta^+$. By similar arguments to those in Lemma 4.1 in [7], it is not hard to show that the solution maps $\Phi(t)$ are κ-contracting in the sense that $\lim_{t\to\infty} \kappa(\Phi(t)(B)) = 0$ for any bounded set $B \subset \mathbb{X}_\zeta^+$.

By the continuous-time version of Theorem 2.6 in [19] (see also [5]), $\Phi(t)$ admits a compact global attractor that attracts every point in $\mathbb{X}_\zeta^+$.

3. The Basic Reproduction Number

This section is devoted to the definition of the basic reproduction number for system (1). To this end, one will adopt the theory developed in [16] (with delay $\tau = 0$) to define the basic reproduction number, $\mathcal{R}_0$. It is not hard to see that the disease-free state of system (1) is as follows

$$E_0(x) = (S^*(x), 0, 0, 0, 0, 0).$$

It is not hard to see that the linearized system of (1) around the disease-free state $E_0(x)$ takes the following cooperative system:

$$\begin{cases} \frac{\partial I}{\partial t} = D_I \Delta I + \beta_C(x,t)I + \eta\beta_C(x,t)C + S^*(x)\frac{\beta_B(x,t)}{K_B}B \\ \qquad -(\mu + \sigma + \delta_I + \epsilon_I)I - \theta I,\ x \in \Omega,\ t > 0, \\ \frac{\partial C}{\partial t} = D_C \Delta C + \sigma I - (\mu + \delta_C + \epsilon_C)C,\ x \in \Omega,\ t > 0, \\ \frac{\partial B}{\partial t} = D_B \Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B,\ x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0,\ u = I, C, B. \end{cases} \tag{13}$$

Assume that $\mathbf{E} := C(\bar{\Omega}, \mathbb{R}^3)$ is the Banach space with the norm $\|\cdot\|_{\mathbf{E}}$, and $\mathbf{E}^+ := C(\bar{\Omega}, \mathbb{R}_+^3)$. Let $C_\omega(\mathbb{R}, \mathbf{E})$ be the Banach space which contains all ω-periodic and continuous functions from $\mathbb{R}$ to $\mathbf{E}$, where $\|\varphi\|_{C_\omega(\mathbb{R},\mathbf{E})} = \max_{\theta\in[0,\omega]} \|\varphi(\theta)\|_{\mathbf{E}}$ for any $\varphi \in C_\omega(\mathbb{R}, \mathbf{E})$. From (13), we define $\mathbf{F}(t) : \mathbf{E} \to \mathbf{E}$ by

$$\mathbf{F}(t)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \beta_C(x,t)w_1 + \eta\beta_C(x,t)w_2 + S^*(x)\frac{\beta_B(x,t)}{K_B}w_3 \\ 0 \\ \alpha_I(x,t)w_1 + \alpha_C(x,t)w_2 \end{pmatrix}, \tag{14}$$

and

$$-\mathbf{V}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} D_I\Delta w_1 - (\mu+\sigma+\delta_I+\epsilon_I)w_1 - \theta w_1 \\ D_C\Delta w_2 + \sigma w_1 - (\mu+\delta_C+\epsilon_C)w_2 \\ D_B\Delta w_3 + g w_3 - \mu_B(x)w_3 \end{pmatrix}, \tag{15}$$

for any $(w_1, w_2, w_3) \in \mathbf{E}$. Then system (13) can be written as

$$\frac{dw}{dt} = \mathbf{F}(t)w - \mathbf{V}w,\ w \in \mathbf{E}.$$

It is easy to see that $\mathbf{F}(t) : \mathbf{E} \to \mathbf{E}$ is positive in the sense that $\mathbf{F}(t)\mathbf{E}^+ \subset \mathbf{E}^+$, and hence, the condition (H1) in [16] holds. Assume that $\{\mathbf{Y}(t,s),\ t \geq s\}$ is the evolution family on $\mathbf{E}$ associated with the following system

$$\frac{dw(t)}{dt} = -\mathbf{V}w(t). \tag{16}$$

Clearly, $\mathbf{Y}(t,s)\mathbf{E}^+ \subset \mathbf{E}^+$ for all $t \geq s$, that is, $\mathbf{Y}(t,s)$ is a positive operator on $\mathbf{E}^+$. Let $\omega(\mathbf{Y})$ stand for the exponential growth bound of the evolution family $\{\mathbf{Y}(t,s),\ t \geq s\}$, which is given by

$$\omega(\mathbf{Y}) := \inf\left\{\tilde{\omega} :\ \exists\, M \geq 1 \text{ such that } \parallel \mathbf{Y}(t+s,s) \parallel \leq Me^{\tilde{\omega}t},\ \forall\, s \in \mathbb{R},\ t \geq 0\right\}.$$

In view of Proposition A.2 in [23] and Lemma 2.1 in [15], it follows that

$$\omega(\mathbf{Y}) = \frac{\ln r(\mathbf{Y}(\omega,0))}{\omega} = \frac{\ln r(\mathbf{Y}(\omega+\xi,\xi))}{\omega},\ \forall\, \xi \in [0,\omega]. \tag{17}$$

Motivated by the arguments to those in Lemma 3.4 in [15], the following inequality can be proved.

Lemma 3.1. $\omega(\mathbf{Y}) < 0$.

Proof Note that $\{\mathbf{Y}(t,s),\ t \geq s\}$ is the evolution family on $\mathbf{E}$ associated with system(16), that is,

$$\begin{cases} \frac{\partial w_1}{\partial t} = D_I\Delta w_1 - (\mu+\sigma+\delta_I+\epsilon_I)w_1 - \theta w_1,\ x \in \Omega,\ t > 0, \\ \frac{\partial w_2}{\partial t} = D_C\Delta w_2 + \sigma w_1 - (\mu+\delta_C+\epsilon_C)w_2,\ x \in \Omega,\ t > 0, \\ \frac{\partial w_3}{\partial t} = D_B\Delta w_3 + g w_3 - \mu_B(x)w_3,\ x \in \Omega,\ t > 0, \\ \frac{\partial w_i(x,t)}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0,\ i = 1,2,3. \end{cases}$$

Assume that $\omega(\mathbf{Y}) \geq 0$. Then it follows from (17) that $r(\mathbf{Y}(\omega,0)) \geq 1$. Since $\mathbf{Y}(\omega,0)$ is compact and monotone in $\mathbf{E}$, and the Krein-Rutman theorem (see [6]) implies that $\lambda^* := \frac{1}{\omega}\ln r(\mathbf{Y}(\omega,0)) \geq 0$ is the eigenvalue of

$$\begin{cases} D_I\Delta\psi_1 - (\mu+\sigma+\delta_I+\epsilon_I)\psi_1 - \theta\psi_1 = \lambda\psi_1,\ x \in \Omega, \\ D_C\Delta\psi_2 + \sigma\psi_1 - (\mu+\delta_C+\epsilon_C)\psi_2 = \lambda\psi_2,\ x \in \Omega, \\ D_B\Delta\psi_3 + g\psi_3 - \mu_B(x)\psi_3 = \lambda\psi_3,\ x \in \Omega, \\ \frac{\partial\psi_i(x)}{\partial\nu} = 0,\ x \in \partial\Omega,\ i = 1,2,3, \end{cases} \tag{18}$$

with an eigenvector

$$(\psi_1^*, \psi_2^*, \psi_3^*) \in \mathbf{E}^+ \text{ and } (\psi_1^*, \psi_2^*, \psi_3^*) \not\equiv (0,0,0). \tag{19}$$

After adding the first three equations of (18) and doing integration over the Ω, it follows that

$$\lambda^* \int_\Omega (\psi_1(x) + \psi_2(x) + \psi_3(x))dx = -(\mu + \delta_I + \epsilon_I + \theta) \int_\Omega \psi_1(x)dx \\ - (\mu + \delta_C + \epsilon_C) \int_\Omega \psi_2(x)dx - \int_\Omega (\mu_B(x) - g)\psi_3(x)dx < 0, \tag{20}$$

where the Neuman boundary conditions of $\psi_i^*(x)$, (2), and (19) are used. Then it follows from (20) and (19) that $\lambda^* < 0$, which is a contradiction. Thus, $\omega(\mathbf{Y}) < 0$.

By Lemma 3.1, it is easy to see that the condition (H2) in [16] holds. Thus, the basic reproduction number $\mathcal{R}_0$ can be defined by the developed theory in [16]. Assume that $v \in C_\omega(\mathbb{R}, \mathbf{E})$ and $v(t)$ is the initial distribution of the infectious individuals at time $t \in \mathbb{R}$. For any $s \geq 0$, $\mathbf{F}(t-s)v(t-s)$ represents the rate of new infections produced by the infected individuals who were introduced at time $t - s$. Then $\mathbf{Y}(t, t-s)\mathbf{F}(t-s)v(t-s)$ gives the distribution of those infected individuals who were newly infected at time $t-s$ and remain in the infected compartments at time t, for $t \geq s$. Thus, the integral

$$\int_0^\infty \mathbf{Y}(t, t-s)\mathbf{F}(t-s)v(t-s)ds$$

is the distribution of accumulative new infections at time t produced by all those infectious individuals $v(\cdot)$ introduced at all time previous to t.

The associated linear operators on $C_\omega(\mathbb{R}, \mathbf{E})$ is given by

$$[\mathbf{L}v](t) := \int_0^\infty \mathbf{Y}(t, t-s)\mathbf{F}(t-s)v(t-s)ds, \ \forall\, t \in \mathbb{R}, \ v \in C_\omega(\mathbb{R}, \mathbf{E}).$$

Motivated by the concept of next generation operators (see, e.g., [1, 4, 23, 24, 26]), the basic reproduction number is defined as the spectral radius of $\mathbf{L}$, namely,

$$\mathcal{R}_0 := r(\mathbf{L}). \tag{21}$$

For any given $t \geq 0$, let $P(t)$ be the solution maps of system (13) on $\mathbf{E}$ given by $P(t)\phi = w(\cdot, t, \phi)$, where $w(x, t, \phi)$ is the unique solution of system (13) with $w(\cdot, 0, \phi) = \phi \in \mathbf{E}$. Then $P(\omega)$ is the Poincaré map associated with system (13) on $\mathbf{E}$. Let $r(P(\omega))$ be the spectral radius of $P(\omega)$. By Theorem 3.7 in [16] (see also Lemma 3.5 in [15]), the following relation holds.

Lemma 3.2. *$\mathcal{R}_0 - 1$ and $r(P(\omega)) - 1$ have the same sign.*

By Proposition II.14.4 in [6], the following statement holds, which is crucial in the establishment of extinction/persistence of typhoid fever.

Lemma 3.3. *Let $\lambda = \frac{1}{\omega}\ln r(P(\omega))$. Then there exists a positive, ω-periodic function $v^*(\cdot, t)$ such that $e^{\lambda t}v^*(x, t)$ is a solution of system* (13) *on* $\mathbf{E}$.

4. Global Dynamics

In this section, the global dynamics of system (1) will be investigated.

The following result will play an important role in our subsequent discussions.

Lemma 4.1. *For every initial value function*

$$u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in \mathbb{X}_\zeta^+,$$

assume that system (1) *has a unique solution*

$$u(x, t, u^0) := (S(x, t), I(x, t), C(x, t), Q(x, t), R(x, t), B(x, t))$$

on $[0, \infty)$ *with* $u(\cdot, 0, u^0) = u^0$.

(i) *The following is always valid:*

$$S(x, t, u^0(\cdot)) \geq \zeta > 0, \text{ for } x \in \bar{\Omega}, \ t > 0,$$

and

$$\liminf_{t\to\infty} S(x, t, u^0(\cdot)) \geq \zeta, \text{ uniformly for } x \in \bar{\Omega}.$$

(ii) *If* $(I^0(\cdot), C^0(\cdot), B^0(\cdot)) \not\equiv (0, 0, 0)$, *then*

$$w(x, t, u^0(\cdot)) > 0, \text{ for } x \in \bar{\Omega}, \ t > 0, \ w = S, \ I, \ C, \ Q, \ R, \ B. \tag{22}$$

(iii) *Assume that there is a* $\xi_1 > 0$ *satisfying*

$$\liminf_{t\to\infty} v(x,t,u^0(\cdot)) \ge \xi_1, \textit{ uniformly for } x \in \bar{\Omega},\ \forall\, v = I, C, B. \tag{23}$$

Then there exists a $\xi_2 > 0$ *such that*

$$\liminf_{t\to\infty} w(x,t,u^0(\cdot)) \ge \xi_2, \textit{ uniformly for } x \in \bar{\Omega} \textit{ for},\ \forall\, w = S, I, C, Q, R, B. \tag{24}$$

Proof Part (i). By Lemma 2.1 and and Lemma 2.3, we see that Part (i) is obvious.

Part (ii). In view of the equations I, C, and B in (1), it follows that (I, C, B) satisfies

$$\begin{cases} \frac{\partial I}{\partial t} = D_I \Delta I + a(x,t)I + b(x,t)C + d(x,t)B - (\mu+\sigma+\delta_I+\epsilon_I)I - \theta I,\ x\in\Omega,\ t>0,\\ \frac{\partial C}{\partial t} = D_C\Delta C + \sigma I - (\mu+\delta_C+\epsilon_C)C,\ x\in\Omega,\ t>0,\\ \frac{\partial B}{\partial t} = D_B\Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B,\ x\in\Omega,\ t>0,\\ \frac{\partial u(x,t)}{\partial \nu} = 0,\ x\in\partial\Omega,\ t>0,\ u = I, C, B, \end{cases} \tag{25}$$

where

$$\begin{cases} a(x,t) := \frac{\beta_C(x,t)S(x,t)}{(S+I+C+Q+R)(x,t)},\\ b(x,t) := \frac{\eta\beta_C(x,t)S(x,t)}{(S+I+C+Q+R)(x,t)},\\ d(x,t) := \frac{\beta_B(x,t)S(x,t)}{B(x,t)+K_B}. \end{cases}$$

In view of Par (i), it is easy to see that (22) holds with $w = S$, and

$$a(x,t) > 0,\ b(x,t) > 0,\ d(x,t) > 0,\ \forall\, x\in\bar{\Omega},\ t>0.$$

By the results developed in CH. 7 in [20], one can show that the solution maps associated with system (25) are strongly monotone in $C(\bar{\Omega}, \mathbb{R}^3_+)$. Thus,

$$w(x,t,u^0(\cdot)) > 0,\ \text{for } x\in\bar{\Omega},\ t>0,\ w = I,\ C,\ B.$$

From the strong maximum principle (see, e.g., Theorem 4 of [25] on p. 172) and the Hopf boundary lemma (see, e.g., Theorem 3 of [25] on p. 170), one can further show that (22) holds with $w = B$. The standard comparison arguments implies that (22) holds with $w = Q$. Thus, part (ii) is proved.

Part (iii). In view of (23), there exists a $t_0 > 0$ such that

$$v(x,t,u^0(\cdot)) \ge \frac{1}{2}\xi_1,\ \text{for } x\in\bar{\Omega},\ t\ge t_0,\ \forall\, v = I, C, B,$$

Then the equations of Q and R satisfies the following two equations, respectively,

$$\frac{\partial Q}{\partial t} \ge \frac{1}{2}\xi_1\theta - (\mu+\gamma+\delta_Q)Q,\ x\in\Omega,\ t\ge t_0,$$

and

$$\begin{cases} \frac{\partial R}{\partial t} \ge D_R\Delta R + \frac{1}{2}\xi_1(\gamma+\epsilon_I+\epsilon_C) - (\mu+\rho)R,\ x\in\Omega,\ t\ge t_0,\\ \frac{\partial R(x,t)}{\partial\nu} = 0,\ x\in\partial\Omega,\ t\ge t_0. \end{cases}$$

Then it follows from the standard comparison arguments that

$$\liminf_{t\to\infty} Q(x,t,u^0(\cdot)) \ge \frac{\xi_1\theta}{2(\mu+\gamma+\delta_Q)},\ \text{uniformly for } x\in\bar{\Omega},$$

and

$$\liminf_{t\to\infty} R(x,t,u^0(\cdot)) \ge \frac{\xi_1(\gamma+\epsilon_I+\epsilon_C)}{2(\mu+\rho)},\ \text{uniformly for } x\in\bar{\Omega}.$$

Let

$$\xi_2 := \min\{\zeta, \xi_1, \frac{\xi_1\theta}{2(\mu+\gamma+\delta_Q)}, \frac{\xi_1(\gamma+\epsilon_I+\epsilon_C)}{2(\mu+\rho)}\}.$$

Then (24) holds. The proof of Part (iii) is finished.

In the following, it will be shown that $\mathcal{R}_0$ plays a central role in the extinction/persistence of system (1).

Theorem 4.1. *For every initial value function*

$$u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in \mathbb{X}_\zeta^+,$$

the unique solution of system (1) *is denoted by*

$$u(x,t,u^0) := (S(x,t), I(x,t), C(x,t), Q(x,t), R(x,t), B(x,t))$$

on $[0,\infty)$ *with* $u(\cdot,0,u^0) = u^0$.

(i) *If* $\mathcal{R}_0 < 1$ *and* $\rho = 0$*, then the state* $E_0(x)$ *is globally asymptotically stable in* $\mathbb{X}_\zeta^+$*;*

(ii) *If* $\mathcal{R}_0 > 1$*, then system* (1) *admits at least one (componentwise) positive* ω*-periodic solution* $\hat{u}(x,t)$ *and there exists a* $\xi > 0$ *such that for any* $u^0(\cdot) \in \mathbb{X}_\zeta^+$ *with* $I^0(\cdot) \not\equiv 0$ *or* $C^0(\cdot) \not\equiv 0$ *or* $B^0(\cdot) \not\equiv 0$*, we have*

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} w(x,t,u^0(\cdot)) \geq \xi, \text{ for } w = S, I, C, Q, R, B.$$

Proof Part (i). Assume that $\mathcal{R}_0 < 1$ and $\rho = 0$. In view of Lemma 3.2 and $\mathcal{R}_0 < 1$, it follows that $r(P(\omega)) < 1$. For any given $\varsigma_1 \geq 0$, let $P_{\varsigma_1}(t)$ be the solution maps associated with the following system on $\mathbf{E}$:

$$\begin{cases} \frac{\partial I}{\partial t} = D_I \Delta I + \beta_C(x,t) I + \eta\beta_C(x,t)C + (S^*(x) + \varsigma_1)\frac{\beta_B(x,t)}{K_B}B \\ \qquad -(\mu + \sigma + \delta_I + \epsilon_I)I - \theta I, \ x \in \Omega, \ t > 0, \\ \frac{\partial C}{\partial t} = D_C \Delta C + \sigma I - (\mu + \delta_C + \epsilon_C)C, \ x \in \Omega, \ t > 0, \\ \frac{\partial B}{\partial t} = D_B \Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B, \ x \in \Omega, \ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \ u = I, C, B. \end{cases} \tag{26}$$

By continuity, it follows that $\lim_{\varsigma_1\to 0} r(P_{\varsigma_1}(\omega)) = r(P(\omega)) < 1$. Thus, one can fix a sufficiently small number $\varsigma_1 > 0$ such that $r(P_{\varsigma_1}(\omega)) < 1$. Since $\rho = 0$, it follows from the first equation in (1) that

$$\begin{cases} \frac{\partial S}{\partial t} \leq D_S \Delta S + \Lambda(x) - \mu S, \ x \in \Omega, \ t > 0, \\ \frac{\partial S(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases} \tag{27}$$

In view of (27), Lemma 2.2 and standard comparison arguments (see e. g., [20]), there exists a $t_1 > 0$ such that

$$S(x,t) \leq S^*(x) + \varsigma_1, \ \forall\, x \in \bar{\Omega}, \ t \geq t_1.$$

From the equations of I, C and B in (1), it follows that

$$\begin{cases} \frac{\partial I}{\partial t} \leq D_I \Delta I + \beta_C(x,t) I + \eta\beta_C(x,t)C + (S^*(x) + \varsigma_1)\frac{\beta_B(x,t)}{K_B}B \\ \qquad -(\mu + \sigma + \delta_I + \epsilon_I)I - \theta I, \ x \in \Omega, \ t \geq t_1, \\ \frac{\partial C}{\partial t} = D_C \Delta C + \sigma I - (\mu + \delta_C + \epsilon_C)C, \ x \in \Omega, \ t \geq t_1, \\ \frac{\partial B}{\partial t} = D_B \Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B, \ x \in \Omega, \ t \geq t_1, \\ \frac{\partial u(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t \geq t_1, \ u = I, C, B. \end{cases}$$

Let $\lambda_{\varsigma_1} = \frac{1}{\omega}\ln r(P_{\varsigma_1}(\omega))$. Then it follows from Lemma 3.3 that there exists a positive, ω-periodic function $v_{\varsigma_1}^*(\cdot,t)$ such that $e^{\lambda_{\varsigma_1} t} v_{\varsigma_1}^*(x,t)$ is a solution of the linear system (26) on $\mathbf{E}$. For any given $u^0(\cdot) \in \mathbb{X}_\zeta^+$, there exists a $M_1 > 0$ such that

$$(I(x,t_1,u^0(\cdot)), C(x,t_1,u^0(\cdot)), B(x,t_1,u^0(\cdot))) \leq M_1 e^{\lambda_{\varsigma_1} t_1} v_{\varsigma_1}^*(x,t_1), \ \forall\, x \in \bar{\Omega}.$$

Then the comparison theorem for the parabolic equation (see, e.g., [20]) implies that

$$(I(x,t,u^0(\cdot)), C(x,t,u^0(\cdot)), B(x,t,u^0(\cdot))) \leq M_1 e^{\lambda_{\varsigma_1} t} v_{\varsigma_1}^*(x,t), \ \forall\, x \in \bar{\Omega}, \ t \geq t_1. \tag{28}$$

Since $\lambda_{\varsigma_1} < 0$, it follows from (28) that

$$\lim_{t\to\infty} (I(x,t,u^0(\cdot)), C(x,t,u^0(\cdot)), B(x,t,u^0(\cdot))) = (0,0,0), \text{ uniformly for all } x \in \bar{\Omega}.$$

Thus, $Q(x,t)$ in (1) is asymptotic to the following system

$$\frac{\partial Q}{\partial t} = -(\mu+\gamma+\delta_Q)Q,\ x\in\Omega,\ t>0,$$

and $B(x,t)$ in (1) is asymptotic to system (5). By Lemma 2.2 (ii), the theory of asymptotically periodic semiflows and internally chain transitive sets (see, e.g., Theorem 3.2.1, Lemma 1.2.2 and Theorem 1.2.1 in [27]), it follows that

$$\lim_{t\to\infty} Q(x,t)=0 \text{ and } \lim_{t\to\infty} B(x,t)=0, \text{ uniformly for } x\in\bar{\Omega}.$$

Then $R(x,t)$ in (1) is asymptotic to the following system

$$\begin{cases} \frac{\partial R}{\partial t} = D_R\Delta R-(\mu+\rho)R,\ x\in\Omega,\ t>0,\\ \frac{\partial R(x,t)}{\partial \nu}=0,\ x\in\partial\Omega,\ t>0. \end{cases}$$

By Lemma 2.2 and Theorem 3.2.1, Lemma 1.2.2 and Theorem 1.2.1 in [27]), it is easy to show that

$$\lim_{t\to\infty} R(x,t)=0 \text{ uniformly for } x\in\bar{\Omega}.$$

Thus, $S(x,t)$ in (1) is asymptotic to system (4), and

$$\lim_{t\to\infty} S(x,t)=S^*(x), \text{ uniformly for } x\in\bar{\Omega}.$$

The proof of Part (i) is complete.

Part (ii). Assume that $\mathcal{R}_0>1$.

Let

$$\mathbb{C}=\mathbb{X}_\zeta^+,$$

$$\mathbb{C}_0=\{u^0(\cdot)=(S^0,I^0,C^0,Q^0,R^0,B^0)(\cdot)\in\mathbb{C}: I^0(\cdot)\not\equiv 0,\ C^0(\cdot)\not\equiv 0, \text{ and } B^0(\cdot)\not\equiv 0\},$$

and

$$\partial\mathbb{C}_0:=\mathbb{C}\backslash\mathbb{C}_0=\{u^0(\cdot)\in\mathbb{C}: I^0(\cdot)\equiv 0 \text{ or } C^0(\cdot)\equiv 0 \text{ or } B^0(\cdot)\equiv 0\}.$$

Note that the solution maps of system (1), $\Phi(t):\mathbb{X}_\zeta^+\to\mathbb{X}_\zeta^+$, are defined in Lemma 2.3.

For any $u^0(\cdot)\in\mathbb{C}_0$, it follows from Lemma 4.1 that $I(x,t,u^0(\cdot))>0, \forall\, x\in\bar{\Omega},\ t>0$ and $C(x,t,u^0(\cdot))>0, \forall\, x\in\bar{\Omega},\ t>0$ and $B(x,t,u^0(\cdot))>0, \forall\, x\in\bar{\Omega},\ t>0$. Then $\Phi(\omega)^n\mathbb{C}_0\subset\mathbb{C}_0, \forall n\in\mathbb{N}$. Moreover, Lemma 2.3 implies that $\Phi(\omega)$ has a strong global attractor in $\mathbb{C}$.

Let

$$M_\partial:=\{u^0(\cdot)\in\partial\mathbb{C}_0:\Phi(\omega)^n u^0(\cdot)\in\partial\mathbb{C}_0, \forall n\in\mathbb{N}\},$$

and $\tilde{\omega}(u^0(\cdot))$ be the omega limit set of the orbit $\Gamma^+=\{\Phi(\omega)^n u^0(\cdot):\forall n\in\mathbb{N}\}$. Set

$$\mathcal{M}_0=\{E_0(x)\}=\{(S^*(x),0,0,0,0,0)\}.$$

Let

$$\mathcal{J}_0=\{u^0(\cdot)\in\mathbb{C}: I^0(\cdot)\equiv 0, \text{ and } C^0(\cdot)\equiv 0, \text{ and } B^0(\cdot)\equiv 0\},$$

and

$$\tilde{\mathcal{J}}_0=\{u^0(\cdot)\in\partial\mathbb{C}_0:\Phi(\omega)^n u^0(\cdot)\in\mathcal{J}_0, \forall n\in\mathbb{N}\},$$

where $u^0(\cdot)=(S^0,I^0,C^0,Q^0,R^0,B^0)(\cdot)$.

The following result will be established:

Claim 1. $M_\partial=\tilde{\mathcal{J}}_0$.

One only need to show that $M_\partial\subseteq\tilde{\mathcal{J}}_0$ since it is clear that $\tilde{\mathcal{J}}_0\subseteq M_\partial$.

For any given $u^0(\cdot)=(S^0,I^0,C^0,Q^0,R^0,B^0)(\cdot)\in M_\partial$, one observes that $\Phi(\omega)^n(u^0(\cdot))\in\partial\mathbb{C}_0$, $\forall n\in\mathbb{N}$. Thus, for each $n\in\mathbb{N}$, it follows that $I(\cdot,n\omega,u^0(\cdot))\equiv 0$ or $C(\cdot,n\omega,u^0(\cdot))\equiv 0$ or $B(\cdot,n\omega,u^0(\cdot))\equiv 0$. In view of Lemma 4.1, it can be further shown that for each $t\geq 0$, $I(\cdot,t,u^0(\cdot))\equiv 0$ or $C(\cdot,t,u^0(\cdot))\equiv 0$ or $B(\cdot,t,u^0(\cdot))\equiv 0$.

Assume, by contradiction, that $I(\cdot,t_1,u^0(\cdot))\not\equiv 0$, for some $t_1>0$. By maximum principle (see e.g., [25]), we see that $I(x,t,u^0(\cdot))>0$ for $x\in\bar{\Omega}$ and $t>t_1$. For each $t>t_1$, it then follows that $C(\cdot,t,u^0(\cdot))\equiv 0$ or $B(\cdot,t,u^0(\cdot))\equiv 0$. In case where $C(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>t_1$. Then it follows from the equation of $C(x,t)$ in (1) that $I(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>t_1$. This is a contradiction. Thus, $C(\cdot,t_2,u^0(\cdot))\not\equiv 0$, for some $t_2>t_1$. By maximum principle (see e.g., [25]), it follows that $C(x,t,u^0(\cdot))>0$ for $x\in\bar{\Omega}$ and $t>t_2$. Then one must have $B(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>t_2$. In view of the equation of $B(x,t)$ in (1), it is easy to see that $I(\cdot,t,u^0(\cdot))\equiv 0$ and $C(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>t_2$. This is a contradiction. Therefore, $I(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>0$. Then $C(\cdot,t,u^0(\cdot))\equiv 0$ and $B(\cdot,t,u^0(\cdot))\equiv 0$, for each $t>0$, due to the the equation of $I(x,t)$ in (1). Thus, $u^0(\cdot)\in\tilde{\mathcal{J}}_0$, and hence, $M_\partial\subseteq\tilde{\mathcal{J}}_0$.

Claim 2. For any $u^0(\cdot)\in M_\partial$, the omega limit set $\tilde{\omega}(u^0(\cdot))=\mathcal{M}_0$, and no subset of $\{\mathcal{M}_0\}$ forms a cycle for $\Phi(\omega)$ in M_∂.

For any $u^0(\cdot)\in M_\partial=\tilde{\mathcal{J}}_0$, we have $I(\cdot,n\omega,u^0(\cdot))\equiv 0$ and $C(\cdot,n\omega,u^0(\cdot))\equiv 0$ and $B(\cdot,n\omega,u^0(\cdot))\equiv 0$, for $n\in\mathbb{N}$. By Lemma 4.1, we see that $I(\cdot,t,u^0(\cdot))\equiv 0$ and $C(\cdot,t,u^0(\cdot))\equiv 0$ and $B(\cdot,t,u^0(\cdot))\equiv 0$, for $t\geq 0$. Then $(S(x,t,u^0),Q(x,t,u^0),R(x,t,u^0))$ in (1) satisfies

$$\begin{cases}\frac{\partial S}{\partial t}=D_S\Delta S+\Lambda(x)-\mu S+\rho R,\ x\in\Omega,\ t>0,\\ \frac{\partial Q}{\partial t}=-(\mu+\gamma+\delta_Q)Q,\ x\in\Omega,\ t>0,\\ \frac{\partial R}{\partial t}=D_R\Delta R+\gamma Q-(\mu+\rho)R,\ x\in\Omega,\ t>0,\\ \frac{\partial u(x,t)}{\partial\nu}=0,\ x\in\partial\Omega,\ t>0,\ u=S,R.\end{cases}\tag{29}$$

By Lemma 2.2, the theory of asymptotically periodic semiflows and internally chain transitive sets (see, e.g., Theorem 3.2.1, Lemma 1.2.2 and Theorem 1.2.1 in [27]), it follows that $(S(x,t,u^0),Q(x,t,u^0),R(x,t,u^0))$ in (29) satisfies

$$\lim_{t\to\infty}(S(x,t,u^0),Q(x,t,u^0),R(x,t,u^0))=(S^*(x),0,0),\ \text{uniformly for } x\in\bar{\Omega}.$$

This implies that the omega limit set $\tilde{\omega}(u^0(\cdot))=\mathcal{M}_0$. Obviously, system (29) is cooperative. For the attractivity of $\mathcal{M}_0$ and Lemma 2.2.1 in [27], one concludes that $\mathcal{M}_0$ is locally Liapunov stable. Thus, no subset of $\{\mathcal{M}_0\}$ forms a cycle for $\Phi(\omega)$ in M_∂. Thus, Claim 2 holds.

In view of Lemma 3.2 and $\mathcal{R}_0>1$, it follows that $r(P(\omega))>1$. For any given $\varsigma_2\geq 0$, let $P_{\varsigma_2}(t)$ be the solution maps associated with the following system on $\mathbf{E}$:

$$\begin{cases}\frac{\partial I}{\partial t}=D_I\Delta I+\beta_C(x,t)\frac{S^*(x)-\varsigma_2}{S^*(x)+5\varsigma_2}I+\eta\beta_C(x,t)\frac{S^*(x)-\varsigma_2}{S^*(x)+5\varsigma_2}C+(S^*(x)-\varsigma_2)\frac{\beta_B(x,t)}{K_B+\varsigma_2}B\\ \qquad-(\mu+\sigma+\delta_I+\epsilon_I)I-\theta I,\ x\in\Omega,\ t>0,\\ \frac{\partial C}{\partial t}=D_C\Delta C+\sigma I-(\mu+\delta_C+\epsilon_C)C,\ x\in\Omega,\ t>0,\\ \frac{\partial B}{\partial t}=D_B\Delta B+gB+\alpha_I(x,t)I+\alpha_C(x,t)C-\mu_B(x)B,\ x\in\Omega,\ t>0,\\ \frac{\partial u(x,t)}{\partial\nu}=0,\ x\in\partial\Omega,\ t>0,\ u=I,C,B.\end{cases}\tag{30}$$

By continuity, one can find a sufficiently small value ς_2 with $0<\varsigma_2<\min_{x\in\bar{\Omega}}S^*(x)$ such that $r(P_{\varsigma_2}(\omega))>1$. From the continuous dependence of solutions on the initial value, there is another $\varsigma_2^*>0$ such that for all $u^0(\cdot)$ with $\|u^0(\cdot)-\mathcal{M}_0\|\leq\varsigma_2^*$, it follows that $\|\Phi(t)u^0(\cdot)-\Phi(t)\mathcal{M}_0\|<\varsigma_2$ for all $t\in[0,\omega]$. The following claim will be proved.

Claim 3. For all $u^0(\cdot)\in\mathbb{C}_0$, there holds $\limsup_{n\to\infty}\|\Phi(\omega)^n(u^0(\cdot))-\mathcal{M}_0\|\geq\varsigma_2^*$.

Assume, by contradiction, that there is a $\phi_0\in\mathbb{C}_0$ such that

$$\lim\sup_{n\to\infty}\|\Phi(\omega)^n(\phi_0)-\mathcal{M}_0\|<\varsigma_2^*.$$

Then there is a $n_2\geq 1$ such that $\|Q(\omega)^n(\phi_0)-\mathcal{M}_0\|<\varsigma_2^*$ for $n\geq n_2$. For any $t\geq n_2\omega$, assume $t=n\omega+t'$ with $n=[t/\omega]$ and $t'\in[0,\omega)$. Then it follows that

$$\|\Phi(t)\phi_0-\Phi(t)\mathcal{M}_0\|=\|\Phi(t')(\Phi(\omega)^n(\phi_0))-\Phi(t')\mathcal{M}_0\|<\varsigma_2^*.\tag{31}$$

For $x\in\bar{\Omega},\ t\geq 0$, we see that

$$(\Phi(t)\mathcal{M}_0)(x)=(S^*(x),0,0,0,0,0).$$

In view of (31), for $x \in \bar{\Omega},\ t \geq n_2\omega$, it follows that

$$S^*(x) - \varsigma_2^* < S(x,t,\phi_0) < S^*(x) + \varsigma_2^*,$$

and

$$0 < v(x,t,\phi_0) < \varsigma_2^*,\ \forall\, v = I, C, Q, R, B.$$

Thus, the equations of $I(x,t,\phi_0)$, $C(x,t,\phi_0)$ and $B(x,t,\phi_0)$ in (1) satisfy

$$\begin{cases} \frac{\partial I}{\partial t} \geq D_I \Delta I + \beta_C(x,t)\frac{S^*(x)-\varsigma_2}{S^*(x)+5\varsigma_2} I + \eta\beta_C(x,t)\frac{S^*(x)-\varsigma_2}{S^*(x)+5\varsigma_2} C + (S^*(x)-\varsigma_2)\frac{\beta_B(x,t)}{K_B+\varsigma_2} B \\ \qquad -(\mu+\sigma+\delta_I+\epsilon_I)I - \theta I,\ x\in\Omega,\ t\geq n_2\omega, \\ \frac{\partial C}{\partial t} = D_C\Delta C + \sigma I - (\mu+\delta_C+\epsilon_C)C,\ x\in\Omega,\ t\geq n_2\omega, \\ \frac{\partial B}{\partial t} = D_B\Delta B + gB + \alpha_I(x,t)I + \alpha_C(x,t)C - \mu_B(x)B,\ x\in\Omega,\ t\geq n_2\omega, \\ \frac{\partial u(x,t)}{\partial \nu} = 0,\ x\in\partial\Omega,\ t\geq n_2\omega,\ u = I, C, B. \end{cases}$$

In view of Lemma 3.3, there is a positive, ω-periodic function $v_{\varsigma_2}^*(\cdot,t)$ and $\lambda_{\varsigma_2} := \frac{1}{\omega}\ln r(P_{\varsigma_2}(\omega))$ such that $e^{\lambda_{\varsigma_2}t}v_{\varsigma_2}^*(x,t)$ is a solution of the linear system (30) on $\mathbf{E}$. In view of Lemma 4.1 and $\phi_0 \in \mathbb{C}_0$, one observes that

$$(I(\cdot,t,\phi_0), C(\cdot,t,\phi_0), B(\cdot,t,\phi_0)) \gg (0,0,0),\ \forall\, t > 0.$$

Then there exists a $M_2 > 0$ such that

$$(I(x,n_2\omega,\phi_0), C(x,n_2\omega,\phi_0), B(x,n_2\omega,\phi_0)) \geq M_2 e^{\lambda_{\varsigma_2}n_2\omega} v_{\varsigma_2}^*(x,n_2\omega),\ \forall\, x\in\bar{\Omega}.$$

Then the comparison theorem for the parabolic equation (see, e.g., [20]) implies that

$$(I(x,t,\phi_0), C(x,t,\phi_0), B(x,t,\phi_0)) \geq M_2 e^{\lambda_{\varsigma_2}t} v_{\varsigma_2}^*(x,t),\ \forall\, x\in\bar{\Omega},\ t\geq n_2\omega. \tag{32}$$

Since $\lambda_{\varsigma_2} > 0$, it follows from (32) that $I(x,t,\phi_0) \to \infty$, $C(x,t,\phi_0) \to \infty$, and $B(x,t,\phi_0) \to \infty$ as $t \to \infty$. This contradiction proves Claim 3.

The stable set of $\mathcal{M}_0$ for $\Phi(\omega)$ is denoted by $W^s(\mathcal{M}_0)$. Then the above claims imply that $\mathcal{M}_0$ is an isolated invariant set for $\Phi(\omega)$ in $\mathbb{C}$, and $W^s(\mathcal{M}_0)\bigcap\mathbb{C}_0 = \emptyset$.

From Theorem 3.7 in [19] and [5], it is clear that $\Phi(\omega)$ has a global attractor A_0 in $\mathbb{C}_0$. In view of Theorem 1.3.1 in [27] (see also [21, 22]), it follows that $\Phi(\omega)$ is uniformly persistent with respect to $(\mathbb{C}_0, \partial\mathbb{C}_0)$ in the sense that there is a $\varrho > 0$ such that

$$\liminf_{n\to\infty} \mathrm{d}(\Phi(\omega)^n(u^0(\cdot)), \partial\mathbb{C}_0) \geq \varrho,\ \forall u^0(\cdot)\in\mathbb{C}_0.$$

Since $A_0 = \Phi(\omega)A_0$, we have that $I^0(\cdot) > 0$, $C^0(\cdot) > 0$ and $B^0(\cdot) > 0$, for all $u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in A_0$. Let $B_0 := \bigcup_{t\in[0,\omega]} \Phi(t)A_0$. Then $B_0 \subset \mathbb{C}_0$ and $\lim_{t\to\infty} \mathrm{d}(Q(t)u^0(\cdot), B_0) = 0$, $\forall u^0(\cdot)\in\mathbb{C}_0$. Define a continuous function $p : \mathbb{C} \to \mathbb{R}_+$ by

$$p(u^0(\cdot)) = \min\{\min_{x\in\bar{\Omega}} I^0(x), \min_{x\in\bar{\Omega}} C^0(x), \min_{x\in\bar{\Omega}} B^0(x)\},$$

for any $u^0(\cdot) = (S^0, I^0, C^0, Q^0, R^0, B^0)(\cdot) \in \mathbb{C}$. Since B_0 is compact subset of $\mathbb{C}_0$, it follows that $\inf_{u^0(\cdot)\in B_0} p(u^0(\cdot)) = \min_{u^0(\cdot)\in B_0} p(u^0(\cdot)) > 0$. Thus, there is a $\xi^* > 0$ such that

$$\liminf_{t\to\infty} p(\Phi(t)u^0(\cdot)) = \liminf_{t\to\infty} \min\{\min_{x\in\bar{\Omega}} v(x,t,u^0(\cdot)) : v = I, C, B\} \geq \xi^*,$$

for any $u^0(\cdot)\in\mathbb{C}_0$.

In view of Lemma 4.1, there is another $\xi \in (0, \xi^*)$ such that

$$\liminf_{t\to\infty} v(t, u^0(\cdot)) \geq \xi,\ \forall u^0(\cdot)\in\mathbb{C}_0,\ v = S, I, C, Q, R, B. \tag{33}$$

Furthermore, Theorem 1.3.6 in [27] implies that $\Phi(\omega)$ has a fixed point $\hat{u} \in \mathbb{C}_0$ and hence, system (1) admits an ω-periodic solution $\Phi(t)\hat{u} \in \mathbb{C}_0$. By Lemma 4.1, one can further show that $\hat{u}(x,t) := \Phi(t)\hat{u}$ is a (componentwise) positive ω-periodic solution.

For any $u^0(\cdot) \in \mathbb{X}_\zeta^+$ with $I^0(\cdot) \not\equiv 0$ or $C^0(\cdot) \not\equiv 0$ or $B^0(\cdot) \not\equiv 0$, it follows from Lemma 4.1 that $u^0(\cdot) \notin \tilde{\mathcal{J}}_0$, for each integer n. Thus, one concludes that there is an integer $n_0 = n_0(u^0(\cdot))$ such that $\Phi^{n_0}(\omega)u^0(\cdot) \in \mathbb{C}_0$. Otherwise, $\Phi^n(\omega)u^0(\cdot) \in \partial\mathbb{C}_0$, for each integer n, and hence, $u^0(\cdot) \in \partial\mathbb{C}_0$, due to Lemma 4.1. This implies that $u^0(\cdot) \in M_\partial = \tilde{\mathcal{J}}_0$, a contradiction. Since

$$\Phi(t)u^0(\cdot) = \Phi(t - n_0\omega)\left(\Phi^n(\omega)u^0(\cdot)\right), \ \forall\, t \geq n_0\omega,$$

it follows from (33) that

$$\liminf_{t\to\infty} v(t, u^0(\cdot)) \geq \xi, \ \forall\, v = S, I, C, Q, R, B,$$

where $u^0(\cdot) \in \mathbb{X}_\zeta^+$ with $I^0(\cdot) \not\equiv 0$ or $C^0(\cdot) \not\equiv 0$ or $B^0(\cdot) \not\equiv 0$.

The proof is finished.

5. Conclusion

This paper investigates a reaction-diffusion model describing the transmission of Typhoid fever with spatial homogeneity and seasonality to reflect the effect of population movement in an temporally periodic environment. A mathematical problem comes from the first two equations of system (1), namely, the denominator $S(x,t) + I(x,t) + C(x,t) + Q(x,t) + R(x,t)$ must be positive at any location $x \in \bar{\Omega}$ and any time $t \geq 0$. To overcome this difficulty, the constant ζ in (3) is introduced, and it can be shown that the feasible domain $\mathbb{X}_\zeta^+$ is positively invariant under the solution maps associated with system (1) (see Lemma 2.1). Another mathematical challenge is that the solution maps associated with system (1) lose the compactness since no diffusion term is added on the class $Q(x,t)$ representing the typhoid patients who are quarantined symptomatic. This reflects the fact that $Q(x,t)$ cannot move randomly in the habitat, which makes mathematical analysis more difficult. For this problem the Kuratowski measure of noncompactness (see, e.g., [2]) is introduced, and the solution maps associated with system (1) will be κ-contracting (see Lemma 2.3).

In order to introduce the basic reproduction number, $\mathcal{R}_0$, for the model, one first determines the disease-free state $E_0(x)$ and the linearized system around $E_0(x)$ of (1) is constructed. Basically, the linearized system around $E_0(x)$ can be divided into two parts; one is related to the new infection (see (14)), and the other one is related to the internal evolution (see (15)). Utilizing the concepts in [1, 4, 23, 24, 26], the basic reproduction number, $\mathcal{R}_0$, is defined as the spectral radius of the next generation operator (see (21)). The index $\mathcal{R}_0$ determines not only the local stability of the disease-free state $E_0(x)$ (see Lemma 3.2), but also the extinction/persistence of Typhoid (see Theorem 4.1). Using the comparison principle and the persistent theory, it can be shown that the Typhoid fever will persist if $\mathcal{R}_0 > 1$ (Theorem 4.1 (ii)). On the other hand, the Typhoid fever will die out if $\mathcal{R}_0 < 1$ and the rate of immunity loss is ignored, that is, $\rho = 0$ (Theorem 4.1 (i)). From the study in this paper, it is worth pointing out that incorporating with the immunity loss (i. e. $\rho > 0$), it is challenging to show that the Typhoid fever will die out under the condition $\mathcal{R}_0 < 1$. Although the rate of the immunity loss, ρ, does not appear in the definition of $\mathcal{R}_0$, it seems that ρ also play an important in the extinction of Typhoid fever.

The mobility of the infected population (before they become infectious) during the incubation period could be also a central factor affecting the transmission of Typhoid fever, and the modeling usually involves a delay term with spatial averaging. This leads to a nonlocal and time-delayed system describing the Typhoid fever transmission, in which the incubation in the population is included. It will be also interesting to investigate this reaction-diffusion model with nonlocal delay effect in the near future.

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Conflicts of Interest

The authors declare no conflicts of interest.

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