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Convergence Analysis of Viscosity Implicit Rules of Asymptotically Nonexpansive Mappings in Hilbert Spaces

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Abstract: Viscosity's implicit algorithm for finding a common element of the set of fixed points for nonlinear operators and the set of solutions of variational inequality problems have been investigated by many authors in different settings in Hilbert and Banach space. In most cases, they consider the following study of viscosity implicit double midpoint, generalized viscosity in the class of nonexpansive and asymptotically nonexpansive mappings. The implicit midpoint rule can effectively solve ordinary differential equations. Meanwhile, many authors have used viscosity iterative algorithms for finding common fixed points for nonlinear operators and solutions of variational inequality problems. Recently, the convergence rate and comparison viscosity implicit iterative algorithm has been studied widely. Under suitable conditions imposed on the control parameters, it is shown in this paper that certain two implicit iterative sequences $\{\omega_n\}$ and $\{\xi_n\}$ converge to the same fixed point of an asymptotically nonexpansive mapping in Hilbert spaces without comparison. It is also proven that $\{\omega_n\}$ and $\{\xi_n\}$ converge strongly to the same solution, which also solves the variational inequality problem. The results presented in this paper improve and extend some recent corresponding results in the literature.

Keywords: Viscosity, Hilbert Space, Asymptotically Nonexpansive Mapping, Fixed Point

1. Introduction

Considering H to be a real Hilbert space and E be a nonempty closed convex subset of $H, \mathcal{T} : E \to E$ be a nonexpansive mapping with a nonempty fixed point set F($\mathcal{T}$). The following iteration method is known as the viscosity approximation method: for arbitrarily chosen $\xi_0 \in E$

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1 - \alpha_n)\mathcal{T}\xi_n, n \geq 0, \tag{1}$$

where $\psi : E \to E$ is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Under some certain conditions, the sequence $\{\xi_n\}$ converges strongly to a point $z \in F(\mathcal{T})$ which solves the variational inequality (VI)

$$\langle (I - \psi)z, \xi - z\rangle \geq 0, \xi \in F(\mathcal{T}), \tag{2}$$

where

I is the identity of H. Many authors studied iterative sequence for the implicit midpoint rule because of it's significant for solving ordinary differential equations; see [6]-[2], John T [8], Mendy et al [9], [7] and the references therein. Recently, Xu et al [3], Aibinu et al, [1] proposed the following viscosity implicit midpoint rule (VIMR) for nonexpansive mappings:

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1 - \alpha_n)\mathcal{T}\Big(\frac{\xi_n + \xi_{n+1}}{2}\Big), n \geq 0, \tag{3}$$

In 2015, Ke and Ma [13] proposed the generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces as follows:

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1-\alpha_n)\mathcal{T}(\delta_n \xi_n + (1-\delta_n)\xi_{n+1}), n \geq 0, \tag{4}$$

and

$$\omega_{n+1} = \alpha_n \psi(\omega_n) + \beta_n \omega_n + \gamma_n \mathcal{T}(\delta_n \omega_n + (1-\delta_n)\omega_{n+1}), n \geq 0, \tag{5}$$

They proved that the generalized viscosity implicit rules 4 and 5 converge strongly to a fixed point of $\mathcal{T}$ under certain assumptions, which also solved the V I(2).

In 2016, motivated by the work of Xu [3], Zhao et al [6] proposed the following implicit midpoint rule for asymptotically nonexpansive mappings:

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1-\alpha_n)\mathcal{T}^n\Big(\frac{\xi_n + \xi_{n+1}}{2}\Big), n \geq 0, \tag{6}$$

where $\mathcal{T}$ is an asymptotically nonexpansive mapping. They proved that the sequence $\{\xi_n\}$ converges strongly to a fixed point of $\mathcal{T}$, which, in addition, also solves the V I(2). If in equations 4 and 5, letting $\delta_n = \frac{1}{2}$, then we have equation 6, if $\mathcal{T}^n = \mathcal{T}$ in an nonexpansive mapping, and equation 7 respectively.

$$\omega_{n+1} = \alpha_n \psi(\omega_n) + \beta_n \omega_n + \gamma_n \mathcal{T}^n\Big(\frac{\omega_n + \omega_{n+1}}{2}\Big), n \geq 0, \tag{7}$$

In 2017, He et al [12] studied the following iterative

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1-\alpha_n)\mathcal{T}^n(\beta_n \xi_n + (1-\beta_n)\xi_{n+1}), n \geq 0 \tag{8}$$

in the setting of a Hilbert space and proved that the sequence $\{\xi_n\}$ converges strongly to $\xi^* = P_{F(\mathcal{T})}\psi(\xi^*)$ which is also the unique solution of the following VI

$$\langle (I-\psi)\xi, v-\xi\rangle \geq 0, \forall v \in F(\mathcal{T}) \tag{9}$$

Question 1.1. Its now so important to asked this question. It is possible for the two iterative mentioned in 4 and 5 to be converge to the same fixed point of asymptotically nonexpansive mappings in Hilbert space and will be the solution to the variational inequality?

Under suitable conditions imposed on the control parameters, the analytical proof is given to show that the two sequences converge to the same fixed point of asymptotically nonexpansive mappings.It is therefore, of interest to give affirmative answer to the question. Moreover, it is shown analytically that the sequences 4 and 5 converges to the same fixed point of $\mathcal{T}$.

The proved is been divided into different steps in section three. These results are improvement and extension of some recent corresponding results announced.

2. Preliminaries

In the sequel, we always assume that H is a real Hilbert space and E is a nonempty, closed, and convex subset of H.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is said to be:

a) nonexpansive if

$$\|\mathcal{T}\xi - \mathcal{T}\omega\| \leq \|\xi - \omega\| \forall \xi, \omega \in E; \tag{10}$$

b) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|\mathcal{T}^n\xi - \mathcal{T}^n\omega\| \leq k_n\|\xi - \omega\| \forall \xi, \omega \in E \; and \; \forall n \in \mathbb{N}; \tag{11}$$

c) contraction if there exists the contractive constant $\alpha \in [0, 1)$ such that

$$\|\mathcal{T}\xi - \mathcal{T}\omega\| \leq \lambda\|\xi - \omega\| \; \forall \xi, \omega \in E; \tag{12}$$

Lemma 2.1. (The demiclosedness principle [5]) . Let H be a Hilbert space, E be a nonempty closed convex subset of H, and $\mathcal{T} : E \to E$ be a asymptotically nonexpansive mapping with $Fix(\mathcal{T}) \neq \emptyset$. If $\{\xi_n\}$ is a sequence in E such that $\{\xi_n\}$ weakly converges to u and $\{(I-\mathcal{T})\xi_n\}$ converges strongly to 0, then $\xi = \mathcal{T}(\xi)$

Lemma 2.2. [4]). Assume that $\{\theta_n\}$ is a sequence of nonnegative real numbers such that

$$\theta_{n+1} \leq (1-\eta_n)\theta_n + \delta_n$$

for all $n \in \mathbb{N}$, where $\{\eta_n\} \subseteq (0, 1)$ and $\{\delta_n\} \subseteq \mathbb{R}$ are two sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\eta_n} \leq 0 \quad or \quad \sum_{n=1}^{\infty} |\delta_n| < \infty$

Then $\lim_{n\to\infty} \theta_n = 0$

Then the sequence $\{\theta_n\}$ converges to 0.

3. Main Result

We now prove the following new result.

Theorem 3.1. Let E be a nonempty closed convex subset a real Hilbert space $H, \mathcal{T} : E \to E$ be asymptotically nonexpansive mappings with the same sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1, Fix(\mathcal{T}) \neq \emptyset$ and $\psi : E \to E$ be a contraction mapping with the contractive constant $\lambda \in [0, 1)$. Define two sequences $\{\xi_n\}$ and $\{\omega_n\}$ in E as follows:

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1 - \alpha_n)\mathcal{T}^n(\delta_n \xi_n + (1 - \delta_n)\xi_{n+1}), n \geq 0, \tag{13}$$

and

$$\omega_{n+1} = \alpha_n \psi(\omega_n) + \beta_n \omega_n + \gamma_n \mathcal{T}^n(\delta_n \omega_n + (1 - \delta_n)\omega_{n+1}), n \geq 0, \tag{14}$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n \in (0, 1)$ satisfying the following conditions,

A1 $\alpha_n + \beta_n + \gamma_n = 1$

A2 $\sum_{n=0}^{\infty} \alpha_n = \infty$

A3 $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ for all $n \geq 0$

A4 $\lim_{n\to\infty} \gamma_n = 1 \quad and \quad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = 0$

Then the sequence $\{\xi_n\}$ and $\{\omega_n\}$ strongly converges to a common fixed point p of $\mathcal{T}$, which is also the unique solution of the following variational inequality

$$\langle (I - \psi)\eta, q - \eta \rangle \geq 0 \quad q \in F(\mathcal{T}), \eta \in H. \tag{15}$$

Remark 3.1. The real sequences that satisfies the above conditions are $\alpha_n = \frac{1}{n}, \quad \beta_n = \frac{1}{n}$ and $\gamma_n = 1 - \frac{2}{n}$

Proof. Step 1 Letting $p \in Fix(\mathcal{T})$,then we will show that both the sequence 13 and 14 are both bounded.

$$\begin{aligned} \|\omega_{n+1} - p\| &= \|\alpha_n \psi(\omega_n) + \beta_n \omega_n + \gamma_n \mathcal{T}^n(\delta_n \omega_n + (1 - \delta_n)\omega_{n+1}) - p\| \\ &\leq \alpha_n \|\psi(\omega_n) - \psi(p)\| + \alpha_n \|\psi(p) - p\| + \beta_n \|\omega_n - p\| \\ &+ \gamma_n \delta_n \|\mathcal{T}^n \omega_n - \mathcal{T}^n p\| + \gamma_n (1 - \delta_n)\|\mathcal{T}^n \omega_{n+1} - \mathcal{T}^n p\| \\ &\leq \alpha_n \lambda \|\omega_n - p\| + \alpha_n \|\psi(p) - p\| + \beta_n \|\omega_n - p\| \\ &+ \gamma_n \delta_n k_n \|\omega_n - p\| + \gamma_n (1 - \delta_n) k_n \|\omega_{n+1} - p\| \end{aligned} \tag{16}$$

From the last inequality we have the following

$$\|\omega_{n+1} - p\| \leq \frac{(\lambda\alpha_n + \beta_n + \gamma_n k_n \delta_n)}{1 - \gamma_n k_n (1 - \delta_n)} \|\omega_n - p\| + \frac{\alpha_n}{1 - \gamma_n k_n (1 - \delta_n)} \|\psi(p) - p\| \tag{17}$$

Since $\gamma_n, \delta_n \in (0, 1), 1 - \gamma_n k_n (1 - \delta_n) > 0$ and $\lim_{n\to\infty} k_n = 1$. From the condition $(A1)$, we have

$$\begin{aligned} \|\omega_{n+1} - p\| &\leq 1 - \frac{1 - \lambda\alpha_n - \beta_n - \gamma_n}{1 - \gamma_n (1 - \delta_n)} \|\omega_n - p\| + \frac{\beta_n}{1 - \gamma_n (1 - \delta_n)} \|\psi(p) - p\| \Big] \\ \|\omega_{n+1} - p\| &\leq 1 - \frac{\alpha_n (1 - \lambda)}{1 - \gamma_n (1 - \delta_n)} \|\omega_n - p\| + \frac{\alpha_n (1 - \lambda)}{1 - \gamma_n (1 - \delta_n)} \frac{1}{(1 - \lambda)} \|\psi(p) - p\| \Big] \\ \|\omega_{n+1} - p\| &\leq \max \Big\{ \|\omega_n - p\|, \frac{1}{(1 - \lambda)} \|\psi(p) - p\| \Big\} \end{aligned}$$

Therefore by mathematical induction, we have

$$\|\omega_{n+1} - p\| \leq \max \Big\{ \|\omega_0 - p\|, \frac{1}{(1 - \lambda)} \|\psi(p) - p\| \Big\}$$

Now, from 13, we have

$$\begin{aligned}\|\xi_{n+1}-p\| &= \|\alpha_n\psi(\xi_n)+(1-\alpha_n)\mathcal{T}^n(\delta_n\xi_n+(1-\delta_n)\xi_{n+1})-p\| \\ &\leq \alpha_n\|\psi(\xi_n)-\psi(p)\|+\alpha_n\|\psi(p)-p\| \\ &+ (1-\alpha_n)\delta_n\|\mathcal{T}^n\xi_n-\mathcal{T}^np\|+(1-\alpha_n)(1-\delta_n)\|\mathcal{T}^n\xi_{n+1}-\mathcal{T}^np\| \\ &\leq \alpha_n\lambda\|\xi_n-p\|+\alpha_n\|\psi(p)-p\| \\ &+ (1-\alpha_n)\delta_nk_n\|\xi_n-p\|+(1-\alpha_n)k_n(1-\delta_n)\|\xi_{n+1}-p\|\end{aligned} \tag{18}$$

From the last inequality we have the following

$$\|\xi_{n+1}-p\| \leq \frac{(\lambda\alpha_n+(1-\alpha_n)k_n\delta_n)}{1-(1-\alpha_n)k_n(1-\delta_n)}\|\xi_n-p\|+\frac{\alpha_n}{1-(1-\alpha_n)k_n(1-\delta_n)}\|\psi(p)-p\| \tag{19}$$

Since $\delta_n\in(0,1), [1-(1-\alpha_n)k_n(1-\delta_n)]>0$ and $\lim_{n\to\infty}k_n=1$, we have

$$\begin{aligned}\|\xi_{n+1}-p\| &\leq 1-\frac{((1-\lambda)\alpha_n)}{\delta+(1-\delta_n)\alpha_n}\|\xi_n-p\|+\frac{\alpha_n}{\delta+(1-\delta_n)\alpha_n}\|\psi(p)-p\| \\ &\leq \max\left\{\|\xi_n-p\|,\frac{1}{(1-\lambda)}\|\psi(p)-p\|\right\}\end{aligned} \tag{20}$$

Therefore by mathematical induction, we can see that both the sequences $\{\xi_n\},\{\omega_n\}$ are bounded and also $\{\psi(\omega_n)\}\quad and\quad \{\psi(\xi_n)\}$.

Step 2 We now prove that both the sequence $\{\xi_n\},\{\omega_n\}$ converges p if and only if $\{\xi_{n+1}\},\{\omega_{n+1}\}$ converges to p as $n\to\infty$. That is $\lim_{n\to\infty}\|\xi_{n+1}-p\|=\lim_{n\to\infty}\|\omega_{n+1}-p\|=0$

$$\begin{aligned}\|\omega_{n+1}-p\| &= \|\omega_{n+1}-\mathcal{T}^np+\mathcal{T}^np-p\| \\ &= \|\alpha_n\psi(\omega_n)+\beta_n\omega_n+\gamma_n\mathcal{T}^n(\delta_n\omega_n+(1-\delta_n)\omega_{n+1})-(\beta_n+\alpha_n+\gamma_n)\mathcal{T}^np\|+\|p-\mathcal{T}^np\| \\ &\leq \alpha_n\|\psi(\omega_n)-\mathcal{T}^np\|+\beta_n\|\omega_n-p\|+\beta_n\|p-\mathcal{T}^np\| \\ &+ \gamma_n\delta_nk_n\|\omega_n-p\|+\gamma_n\delta_nk_n\|p-\mathcal{T}^np\|+\gamma_nk_n(1-\delta_n)\|\omega_{n+1}-p\| \\ &+ \gamma_nk_n(1-\delta_n)\|p-\mathcal{T}^np\|+\|p-\mathcal{T}^np\| \\ &\leq \frac{\alpha_n}{1-\gamma_nk_n(1-\delta_n)}\|\psi(\omega_n)-\mathcal{T}^np\|+\frac{(\beta_n+\gamma_n\delta_nk_n)}{1-\gamma_nk_n(1-\delta_n)}\|\omega_n-p\|+\frac{(\beta_n+\gamma_nk_n+1)}{1-\gamma_nk_n(1-\delta_n)}\|p-\mathcal{T}^np\| \\ &\leq \frac{\alpha_n}{1-\gamma_nk_n(1-\delta_n)}\|\psi(\omega_n)-\mathcal{T}^np\|+\frac{(\beta_n+\gamma_n\delta_nk_n)}{1-\gamma_nk_n(1-\delta_n)}\|\omega_n-p\|+\frac{(\beta_n+\gamma_nk_n+1)}{1-\gamma_nk_n(1-\delta_n)}\|p-\mathcal{T}^np\|\end{aligned} \tag{21}$$

Again from 13, we have

$$\begin{aligned}\|\xi_{n+1}-p\| &= \|\xi_{n+1}-\mathcal{T}^np+\mathcal{T}^np-p\| \\ &= \|\alpha_n\psi(\xi_n)+(1-\alpha_n)\mathcal{T}^n(\delta_n\xi_n+(1-\delta_n)\xi_{n+1})-\mathcal{T}^np+\mathcal{T}^np-p\| \\ &\leq \alpha_n\|\psi(\xi_n)-\mathcal{T}^np\|+(1-\alpha_n)\delta_n\|\mathcal{T}^n\xi_n-\mathcal{T}^np\|+(1-\alpha_n)(1-\delta_n)\|\mathcal{T}^n\xi_{n+1}-\mathcal{T}^np\|+\|\mathcal{T}^np-p\| \\ &\leq \alpha_n\|\psi(\xi_n)-\mathcal{T}^np\|+(1-\alpha_n)\delta_nk_n\|\xi_n-p\|+(1-\alpha_n)(1-\delta_n)k_n\|\xi_{n+1}-p\|+\|\mathcal{T}^np-p\| \\ &\leq \frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)k_n}\|\psi(\xi_n)-\mathcal{T}^np\|\frac{(1-\alpha_n)\delta_nk_n}{1-(1-\alpha_n)(1-\delta_n)}\|\xi_n-p\| \\ &+ \frac{1}{1-(1-\alpha_n)(1-\delta_n)k_n}\|\mathcal{T}^np-p\|\end{aligned} \tag{22}$$

Since $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\delta_n=0$, and with the fact that $\|\mathcal{T}^np-p\|\leq k_n\|p-p\|=0$, we then conclude that from 21 and 22, $\lim_{n\to\infty}\|\xi_{n+1}-p\|=\lim_{n\to\infty}\|\omega_{n+1}-p\|=0$. This implies that $\lim_{n\to\infty}\|\xi_n-p\|=\lim_{n\to\infty}\|\omega_n-p\|=0$. Then

its clear that

$$\begin{aligned} \|\omega_{n+1} - \xi_{n+1}\| &= \|\omega_{n+1} - p + p - \xi_{n+1}\| \\ &\leq \|\omega_{n+1} - p\| + \|p - \xi_{n+1}\| \end{aligned} \tag{23}$$

Then taking limits on both sides, we have

$$\lim_{n\to\infty} \|\omega_{n+1} - \xi_{n+1}\| \leq \lim_{n\to\infty} \|\omega_{n+1} - p\| + \lim_{n\to\infty} \|p - \xi_{n+1}\| = 0 \tag{24}$$

This implies that

$$\lim_{n\to\infty} \|\omega_n - \xi_n\| \leq \lim_{n\to\infty} \|\omega_n - p\| + \lim_{n\to\infty} \|p - \xi_n\| = 0 \tag{25}$$

Next, we shown that the implicit iterative sequences 13 and 14 converge to the same fixed point of a asymptotically nonexpansive mapping $\mathcal{T}$.

$$\begin{aligned} \|\omega_{n+1} - \xi_{n+1}\| &= \|\alpha_n\psi(\omega_n) + \beta_n\omega_n + \gamma_n\mathcal{T}^n(\delta_n\omega_n + (1-\delta_n)\omega_{n+1}) \\ &- \Big[\alpha_n\psi(\xi_n) + (1-\alpha)\mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\Big]\| \\ &\leq \|\alpha_n\psi((\omega_n) - \psi(\xi_n)) + \beta_n(\omega_n - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})) \\ &+ \gamma_n[\mathcal{T}^n(\delta_n\omega_n + (1-\delta_n)\omega_{n+1}) - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\| \\ &\leq \alpha_n\|\psi(\omega_n) - \psi(\xi_n)\| + \beta_n\|\omega_n - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\| \\ &+ \gamma_n[\|\mathcal{T}^n(\delta_n\omega_n + (1-\delta_n)\omega_{n+1}) - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\|] \\ &\leq \alpha_n\lambda\|\omega_n - \xi_n\| + \beta_n\|\omega_n - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\| \\ &+ \gamma_n k_n[\|\delta_n\omega_n + (1-\delta_n)\omega_{n+1} - \delta_n\xi_n - (1-\delta_n)\xi_{n+1}\|] \\ &\leq \alpha_n\lambda\|\omega_n - \xi_n\| + \beta_n\|\omega_n - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\| \\ &+ \gamma_n k_n\delta_n\|\omega_n - \xi_n\| + \gamma_n k_n(1-\delta_n)\|\omega_{n+1} - \xi_{n+1}\| \end{aligned} \tag{26}$$

$$\begin{aligned} \|\mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\| &= \|\mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1}) - \mathcal{T}^n p + \mathcal{T}^n p\| \\ &\leq \|\mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1}) - \mathcal{T}^n p\| + \|\mathcal{T}^n p - p\| + \|p\| \\ &\leq \delta_n\|\mathcal{T}^n(\xi_n - p)\| + (1-\delta_n)\|\mathcal{T}^n(\xi_{n+1} - p)\| \\ &+ \|\mathcal{T}^n p - p\| + \|p\| \\ &\leq \delta_n k_n\|\xi_n - p\| + (1-\delta_n)k_n\|\xi_{n+1} - p\| + \|p\| \\ &\leq \delta_n k_n\|\xi_n - p\| + (1-\delta_n)k_n\|\xi_{n+1} - p\| + \|p\| \end{aligned}$$

Again since, from induction 20, the sequence $\{\xi_n\}$ are bounded, then it follows that $\{\omega_n\} \quad and \quad \{\mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\}$ are both bounded. Now setting $Q = \max\|\omega_n - \mathcal{T}^n(\delta_n\xi_n + (1-\delta_n)\xi_{n+1})\|$ in 26, we have the following

$$\begin{aligned} \|\omega_{n+1} - \xi_{n+1}\| &\leq \frac{(\alpha_n\lambda + \gamma_n k_n\delta_n)}{1 - \gamma_n k_n(1-\delta_n)}\|\omega_n - \xi_n\| + \frac{Q\beta_n}{1 - \gamma_n k_n(1-\delta_n)} \\ &\leq \Big(1 - \frac{\beta_n + (1-\lambda)\alpha_n}{1 - \gamma_n k_n(1-\delta_n)}\Big)\|\omega_n - \xi_n\| + \frac{Q\beta_n}{1 - \gamma_n k_n(1-\delta_n)} \\ &\leq \Big(1 - \frac{(1-\lambda)\alpha_n}{1 - \gamma_n k_n(1-\delta_n)}\Big)\|\omega_n - \xi_n\| + \frac{Q\beta_n}{1 - \gamma_n k_n(1-\delta_n)} \\ &\leq \Big(1 - \frac{(1-\lambda)\alpha_n}{1 - \gamma_n k_n(1-\delta_n)}\Big)\|\omega_n - \xi_n\| + \frac{Q\beta_n(1-\lambda)\alpha_n}{1 - \gamma_n k_n(1-\delta_n)(1-\lambda)\alpha_n} \end{aligned}$$

With definition that the $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we then conclude that from 25, that $\lim_{n\to\infty} \|\omega_{n+1} - \xi_{n+1}\| = 0$. This implies that the sequences 13 and 14 converges to the same fixed point.

Step 3 We now prove that the sequence $\{\omega_{n+1}\}$ converges to $\{\omega_n\}$ as $n \to \infty$. That is $\lim_{n\to\infty} \|\omega_{n+1} - \omega_n\| = 0$

$$\begin{aligned}
\|\omega_{n+1} - \omega_n\| &= \|\omega_{n+1} - \mathcal{T}^n\omega_n + \mathcal{T}^n\omega_n - \omega_n\| \\
&\leq \|\alpha_n\psi(\omega_n) + \beta_n\omega_n + \gamma_n\mathcal{T}^n(\delta_n\omega_n + (1-\delta_n)\omega_{n+1}) - (\beta_n + \alpha_n + \gamma_n)\mathcal{T}^n\| + \|\mathcal{T}^n\omega_n - \omega_n\| \\
&\leq \|\alpha_n\psi(\omega_n) - \alpha_n\mathcal{T}^n\omega_n\| + \|\beta_n\omega_n - \beta_n\mathcal{T}^n\omega_n\| + \gamma_n(1-\delta_n)\|\mathcal{T}^n\omega_{n+1} - \mathcal{T}^n\omega_n\| + \|\mathcal{T}^n\omega_n - \omega_n\| \\
&\leq \alpha_n\|\psi(\omega_n) - \mathcal{T}^n\omega_n\| + \beta_n\|\omega_n - \mathcal{T}^n\omega_n\| + \gamma_n(1-\delta_n)k_n\|\omega_{n+1} - \omega_n\| + \|\mathcal{T}^n\omega_n - \omega_n\|
\end{aligned}$$

$$\|\omega_{n+1} - \omega_n\| \leq \frac{\alpha_n}{1-\gamma_n(1-\delta_n)k_n}\|\psi(\omega_n) - \mathcal{T}^n\omega_n\| + \frac{\beta_n + 1}{1-\gamma_n(1-\delta_n)k_n}\|\omega_n - \mathcal{T}^n\omega_n\| \tag{27}$$

$$\|\mathcal{T}^n\omega_n - \omega_n\| \leq k_n\|\omega_n - \omega_n\| = 0 \tag{28}$$

Let $M :> \max\left\{\|\psi(\omega_n) - \mathcal{T}^n\omega_n\|\right\}$, then we have

$$\|\omega_{n+1} - \omega_n\| \leq \frac{\alpha_n M}{1-\gamma_n(1-\delta_n)k_n} + \frac{\beta_n + 1}{1-\gamma_n(1-\delta_n)k_n}\|\omega_n - \mathcal{T}^n\omega_n\| \tag{29}$$

From 29 and with the definition of $\{\alpha_n\}$ in 3.1 $A4$, we can see that $\|\omega_{n+1} - \omega_n\| \to 0$. This means that $\omega_{n+1} \to \omega_n$ *or* $\lim_{n\to\infty} \|\omega_{n+1} - \omega_n\| = 0$

With the same computational, with the definition of $\{\alpha_n\}$ in 3.1 $A4$, we can see that $\|\xi_{n+1} - \xi_n\| \to 0$. This means that $\xi_{n+1} \to \xi_n$ *or* $\lim_{n\to\infty} \|\xi_{n+1} - \xi_n\| = 0$

Again we then show that $\lim_{n\to\infty} \left\|\omega_n - \mathcal{T}(\omega_n)\right\| = 0$. Estimating as follows we have

$$\begin{aligned}
\|\omega_n - \mathcal{T}^n\omega_n\| &= \|\omega_n - \omega_{n+1} + \omega_{n+1} - \mathcal{T}^n\omega_n\| \\
&\leq \|\omega_n - \omega_{n+1}\| + \|\omega_{n+1} - \mathcal{T}^n\omega_n\| \\
&\leq \|\omega_n - \omega_{n+1}\| + \|\alpha_n\psi(\omega_n) + \beta_n\omega_n + \gamma_n\mathcal{T}^n(\delta_n\omega_n + (1-\delta_n)\omega_{n+1}) - \mathcal{T}^n\omega_n\| \\
&\leq \|\omega_n - \omega_{n+1}\| + \alpha_n\|\psi(\omega_n) - \mathcal{T}^n\omega_n\| + \beta_n\|\omega_n - \mathcal{T}^n\omega_n\| + \gamma_n(1-\delta_n)k_n\|\omega_{n+1} - \omega_n\| \\
&\leq \frac{1 + \gamma_n(1-\delta_n)k_n}{1-\beta_n}\|\omega_n - \omega_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\psi(\omega_n) - \mathcal{T}^n\omega_n\|
\end{aligned}$$

From 29 and with the definition of $\{\alpha_n\}$ in $A4$ of 3.1 , we can see that

$$\lim_{n\to\infty} \|\omega_n - \mathcal{T}^n\omega_n\| = 0$$

Using the fact that

$$\begin{aligned}
\lim_{n\to\infty} \|\omega_n - \mathcal{T}(\omega_n)\| &\leq \lim_{n\to\infty} \|\omega_n - \mathcal{T}^n\omega_n\| + \lim_{n\to\infty} \|\mathcal{T}^n\omega_n - \mathcal{T}\omega_n\| \\
&\leq \lim_{n\to\infty} \|\omega_n - \mathcal{T}^n\omega_n\| + \lim_{n\to\infty} k_1\|\mathcal{T}^{n-1}\omega_n - \omega_n\|
\end{aligned} \tag{30}$$

Proving that $\lim_{n\to\infty} \|T^{n-1}\omega_n - \omega_n\| = 0$, we have the following estimation

$$\begin{aligned}
\|\mathcal{T}^{n-1}(\omega_n) - \omega_n\| &= \|\omega_n - \mathcal{T}^{n-1}(\omega_n)\| \\
&= \|\alpha_{n-1}\psi(\omega_{n-1}) + \beta_{n-1}\omega_{n-1} + \gamma_{n-1}\mathcal{T}^{n-1}(\delta_{n-1}\omega_{n-1} + (1-\delta_{n-1})\omega_n \\
&\quad -(\beta_{n-1} + \alpha_{n-1} + \gamma_{n-1})\mathcal{T}^{n-1}\omega_n\| \\
&\leq \alpha_{n-1}\|\psi(\omega_{n-1}) - \mathcal{T}^{n-1}\omega_n\| + \beta_{n-1}\|\omega_{n-1} - \mathcal{T}^{n-1}\omega_n\| \\
&+ \gamma_{n-1}(1-\delta_{n-1})k_1\|\omega_n - \omega_{n-1}\|.
\end{aligned}$$

With the assumption of $\{\alpha_n\}, \{\beta_n\}$ and $\lim_{n\to\infty} \|\omega_{n+1} - \omega_n\| = 0$, we can conclude that

$$\lim_{n\to\infty} \|\mathcal{T}^{n-1}\omega_n - \omega_n\| = 0 \tag{31}$$

Therefore from 30

$$\lim_{n\to\infty} \|\omega_n - \mathcal{T}(\omega_n)\| \leq \lim_{n\to\infty} \|\omega_n - \mathcal{T}^n\omega_n\| + \lim_{n\to\infty} k_1\|\mathcal{T}^{n-1}\omega_n - \omega_n\| = 0$$

With the same computational, and assumption of $\{\alpha_n\}, \{\beta_n\}$ and $\lim_{n\to\infty} \|\xi_{n+1} - \xi_n\| = 0$, we can conclude that

$$\lim_{n\to\infty} \|\mathcal{T}^{n-1}\xi_n - \xi_n\| = 0 \tag{32}$$

and also

$$\lim_{n\to\infty} \|\xi_n - \mathcal{T}(\xi_n)\| \leq \lim_{n\to\infty} \|\xi_n - \mathcal{T}^n\xi_n\| + \lim_{n\to\infty} k_1\|\mathcal{T}^{n-1}\xi_n - \xi_n\| = 0 \tag{33}$$

Step 4 In this step, we will show that $\tau_\omega(x_n) \subseteq Fix(\mathcal{T})$, where
$\tau_\omega(\xi_n) := \{\xi \in H$: there exist a subsequence of $\{\xi_n\}$ converges weakly to $\xi\}$.
Suppose that $\xi \in w_\omega(x_n)$. Then there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} \rightharpoonup \xi$ as $i \to \infty$. From 33, we have

$$\lim_{i\to\infty} \left\|(I - \mathcal{T})\xi_{n_i}\right\| = \lim_{n\to\infty} \left\|\xi_{n_i} - \mathcal{T}\xi_{n_i}\right\| = 0.$$

This implies that $\{(I - \mathcal{T})\xi_{n_i}\}$ converges strongly to 0. By using Lemma 2.1, we have $\mathcal{T}\xi = \xi$, and so $\xi \in Fix(\mathcal{T})$.

Step 5 Now from the inequality 43, we will show that

$$\limsup_{n\to\infty} \langle p - \psi(p), p - \omega_n\rangle \leq 0, \tag{34}$$

where $p \in F(\mathcal{T})$ is the unique fixed point of $P_{F(\mathcal{T})} \circ \psi$, that is, $p = q_{F(\mathcal{T})}(\psi(z))$. Since $\{\omega_n\}$ is bounded, there exists a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ such that $\omega_{n_i} \rightharpoonup \overline{\omega}$ as $i \to \infty$ for some $\overline{\omega} \in H$ and

$$\limsup_{n\to\infty} \langle p - \psi(p), p - \omega_n\rangle = \lim_{i\to\infty} \langle p - \psi(p), p - \omega_{n_i}\rangle \tag{35}$$

From Step 4, we get $\overline{y} \in F(\mathcal{T})$. By using inequality 2.1, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle p - \psi(p), p - \omega_n\rangle &= \lim_{i\to\infty} \langle p - \psi(p), p - \omega_{n_i}\rangle \\ &= \langle p - \psi(p), p - \overline{y}\rangle \leq 0\end{aligned}$$

Again

$$\limsup_{n\to\infty} \langle p - \psi(p), p - \xi_n\rangle \leq 0, \tag{36}$$

where $p \in F(\mathcal{T})$ is the unique fixed point of $P_{F(\mathcal{T})} \circ \psi$, that is, $p = q_{F(\mathcal{T})}(\psi(z))$. Since $\{\xi_n\}$ is bounded, there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} \rightharpoonup \overline{\xi}$ as $i \to \infty$ for some $\overline{\xi} \in H$ and

$$\limsup_{n\to\infty} \langle p - \psi(p), p - \xi_n\rangle = \lim_{i\to\infty} \langle p - \psi(p), p - \xi_{n_i}\rangle \tag{37}$$

From Step 4, we get $\overline{x} \in F(\mathcal{T})$. By using inequality 2.1, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle p - \psi(p), p - \xi_n\rangle &= \lim_{i\to\infty} \langle p - \psi(p), p - \xi_{n_i}\rangle \\ &= \langle p - \psi(p), p - \overline{\xi}\rangle \leq 0\end{aligned}$$

This complete the proof.

Theorem 3.2. Let E be a nonempty closed convex subset a real Hilbert space $H, \mathcal{T} : E \to E$ be asymptotically nonexpansive

mappings with the same sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1, Fix(\mathcal{T}) \neq \emptyset$ and ω be a constant. Define two sequences $\{\xi_n\}$ and $\{\omega_n\}$ in E as follows:

$$\xi_{n+1} = \alpha_n \Xi + (1-\alpha_n)\mathcal{T}^n(\delta_n \xi_n + (1-\delta_n)\xi_{n+1}), n \geq 0, \tag{38}$$

and

$$\omega_{n+1} = \alpha_n \kappa + \beta_n \omega_n + \gamma_n \mathcal{T}^n(\delta_n \omega_n + (1-\delta_n)\omega_{n+1}), n \geq 0, \tag{39}$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n \in (0,1)$ satisfying conditions $A1 - A4$ and $\psi(\xi_n) = \Xi$ and $\psi(\omega_n) = \kappa$, then the sequence $\{\xi_n\}$ and $\{\omega_n\}$ strongly converges to a common fixed point p of $\mathcal{T}$, which is also the unique solution of the following variational inequality

$$\langle (I-\psi)\eta, q-\eta\rangle \geq 0 \quad q \in F(\mathcal{T}), \eta \in H. \tag{40}$$

Taking $\delta_n = 0$

The following corollaries holds:

Corollary 3.1. Let E be a nonempty closed convex subset a real Hilbert space $H, \mathcal{T} : E \to E$ be asymptotically nonexpansive mappings with the same sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1, Fix(\mathcal{T}) \neq \emptyset$ and $\psi : E \to E$ be a contraction mapping with the contractive constant $\lambda \in [0,1)$. Define two sequences $\{\xi_n\}$ and $\{\omega_n\}$ in E as follows:

$$\xi_{n+1} = \alpha_n \psi(\xi_n) + (1-\alpha_n)\mathcal{T}^n(\xi_{n+1}), n \geq 0, \tag{41}$$

and

$$\omega_{n+1} = \alpha_n \psi(\omega_n) + \beta_n \omega_n + \gamma_n \mathcal{T}^n(\omega_{n+1}), n \geq 0, \tag{42}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ satisfying conditions $A1 - A4$ with $\delta_n = 0$, then the sequence $\{\omega_n\}$ and $\{\xi_n\}$ strongly converges to a common fixed point p of $\mathcal{T}$, which is also the unique solution of the following variational inequality

$$\langle (I-\psi)\eta, q-\eta\rangle \geq 0 \quad q \in F(\mathcal{T}), \eta \in H. \tag{43}$$

4. Conclusion

In this paper, we have obtained the strong convergence of a viscosity asymptotically nonexpansive approximation method for finding a common fixed point. Two implicit iterative algorithms were stated, and it was proven that they all converge to the same fixed point through asymptotically nonexpansive mappings in Hilbert space under certain conditions, and an affirmative result was obtained which answered the question. It was also proven that the results obtained here converges to the same unique solution to both the iterative algorithm, which also solved the variational inequality stated in the theorem.

Authors Contribution

Furmose Mendy: conceived idea, the algorithm final approval of the version to be published. John T Mendy: Performed the analysis, proofreading and editing ,and final approval of the version to be published. Jatta Bah: editing, proofreading,and data analysis. Gabriel J Mendy: editing, proofreading, typing and final approval for publication. All authors contributed to interpreting data, drafting the manuscript, and critically revising the manuscript for intellectual content; all authors approved of the published version.

References

[1] Aibinu, M.O. and Kim, J.K., 2020. On the rate of convergence of viscosity implicit iterative algorithms. Nonlinear Funct. Anal. Appl, 25 (1), pp. 135-152.

[2] F.E. Browder, Existence of periodic solutions for nonlinear equations of evolution, Proc. Natl. Acad. Sci. USA 53 (5) (1965), 1100- 1103.

[3] H.K. Xu, M.A. Alghamdi, N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 41 (2015).

[4] H.K. Xu, Iterative algorithms for nonlinear op erators, J. Lond. Math. So c. 66 (2) (2002) 240-256.

[5] K.Go eb el, W.A. Kirk, Topics in Me tric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990).

[6] L.-C. Zhao, S.-S. Chang, C.-F. Wen, Viscosity approximation methods for the implicit midp oint rule of asymptotically nonexpansive mappings in Hilbert spaces, J. Nonlinear Sci. Appl. 9 (6) (2016) 4478-4488.

[7] Mendy, J.T. and Shukla, R., 2022. Viscosity like implicit methods for zeros of monotone operators in Banach spaces. Khayyam Journal of Mathematics, 8 (1), pp. 53-72.

[8] Mendy, J.T., 2020. The viscosity iterative algorithms for the implicit double midpoint rule of nonexpansive mappings in Hilbert spaces. American Journal of Mathematical Analysis, 8 (1), pp. 1-8.

[9] Mendy, S.B., Mendy, J.T. and Jobe, A., 2021. The Generalized Viscosity Implicit Rules of Asymptotically Nonexpansive Mappings in Hilbert Spaces. European Journal of Mathematical Analysis, 1, pp. 19-33.

[10] S.F.A. Naqvi, M.S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, Open J. Math. Sci. 1 (1) (2017) 111-125.

[11] S. He, Y. Mao, Z. Zhou, J.Q. Zhang, The generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces, Applied Mathematical Science 11 (12) (2017) 549-560.

[12] Sijun He, Yingdong Mao, Zheng Zhou, and Jian-Qiang Zhang. The Generalized Viscosity Implicit Rules of Asymptotically Nonexpansive Mappings in Hilb ert Spaces, Applied Mathematical Sciences, Vol. 11, 2017, no. 12, 549-560 HIKARI Ltd, www.m-hikari.com https://doi.org/10.12988/ams.2017.718

[13] Y. Ke, C. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, Fixed Point Theory and Appl. 190 (2015).