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Modification Cross' Theorem on Triangle with Congruence

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Abstract: Cross' theorem states if any triangle *ABC*, on each side constructed a square and vertices of the square are connected, it will form another triangle which has an equal area to triangle *ABC*. In this paper we will discuss the modification Cross' theorem on triangle, which is to construct a square outcase direction on each side triangle produced by Cross' theorem. The purpose of this paper is to modify Cross' theorem and give simple proof even junior or senior high school can answer. The process of proof is done in simple way, that is using a congruence approach. The result obtained are square vertices that are connected is form a trapezoid and have area 5 times the area of triangle *ABC*.

Keywords: Triangle, Square, Congruence, Cross' Theorem

1. Introduction

Geometry has been studied starting from elementary to college, one of the material is about plane. On plane we will know such as triangles, rectangles, squares or trapeziums. At the elementary school level students are just beginning to recognize the nature, elements and determine area, while at the middle and high school levels students have begun to understand concepts and theorems in mathematics.

One of the many theorems in geometry that discusses about triangles is the Cross' theorem. Cross' theorem states if it is any triangle ABC a square constructed on each side of triangle and a squares vertices are connected so it will form another triangle which has an area equal to the triangle *ABC*.

Cross' theorem discovered and named by 14 years old schoolboy David Cross, which was posed by Faux (2004). Cross' theorem can be proved by using congruence [7]. Congruence essentially means that two figures or objects are of the same shape and size which is has been studied at junior high school.

In general, this Cross' theorem only applies to triangles, but several authors have developed this Cross' theorem like Cross' theorem on the triangle using a rectangle and the Cross' Theorem on quadrilateral [4, 11]. The results reveal that there are the relationship between the area area formed from the new triangles constructed and the initial triangle. Various proofs Cross' theorem has also been described [2, 3].

Seeing the relationship of developed Cross' theorem, the authors are interested in modifying the cross theorem on a triangle, which is squares outcase direction constructed on each side triangle Cross' theorem, and a squares vertices connected it will form a trapezoid and has an area 5 times the area triangle ABC, then if each side of trapezoid is extended until intersect with each other it will form another triangle and that triangle is congruent with triangle ABC, shown in Figure 1.

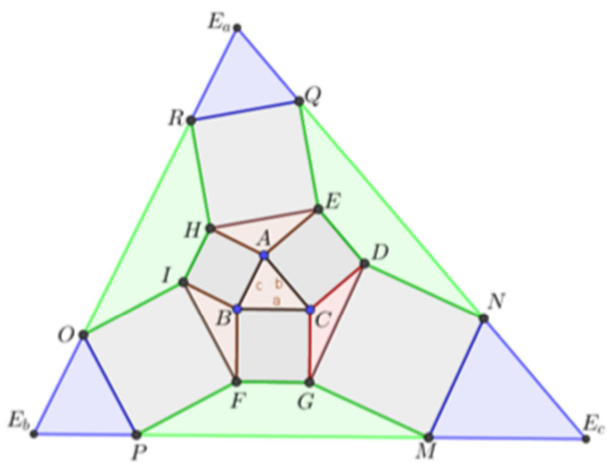

Figure 1. *Illustration modification Cross' theorem.*

Lots of software that can be used in learning mathematics especially in geometry, One software which can be used is Geogebra. Geogebra application is dynamic mathematical software that can be used as a tool in mathematics learning. In this paper using software *Geogebra* which is very helpful in constructing points and lines. *Geogebra* is a versatile software for learning mathematics in schools and colleges, *Geogebra* is used as a media for demonstration and visualization, tools for finding mathematical concepts and preparing materials for teaching.

Based on the description above, the author discusses proof of modification Cross' theorem using easy material and can be discusses by middle or high school students.

2. Cross' Theorem

Cross' theorem was first put forward by Faux (2004) for reader of the *Mathematics Theaching Journal*. Faux states that a triangle *ABC* on each sides triangle a squares are constructed, then a near square vertices connected it will form a new triangles and has an equal area to triangle *ABC* [2].

This Cross' theorem was discovered by David Cross. In general Cross' theorem applies to triangles, but some authors have developed Cross' theorem on triangles using rectangles and Cross' theorem on quadrilateral [4, 11]. Cross' theorem and some of the proofs has been discussed that triangles are formed has the same area from initial triangle [1, 3], as explained in Theorem 2.1.

Theorem 2.1. Let *ABC* denote any triangle, and construct squares on each side of the triangle outward which is *ABIH*, *BCGF* and *ACDE*. If line *EH*, *IF* and *DG* constructed, then will form $\triangle AEH$, $\triangle BIF$ and $\triangle CDG$ which has area same to $\triangle ABC$, shown in Figure 2.

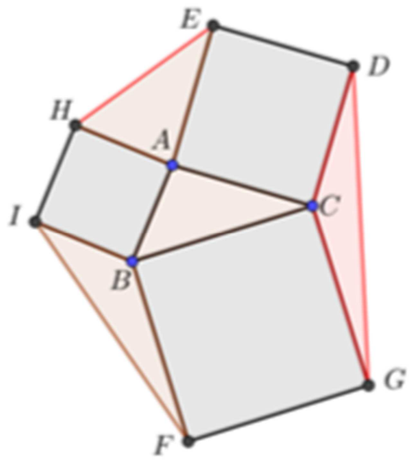

Figure 2. *Illustration Cross' theorem.*

Proof. Suppose that $AB = c$, $BC = a$ and $AC = b$, using trigonometric formula to prove area triangle *ABC*, we have $L\triangle ABC = \frac{1}{2}\, ac \sin \angle ABC = \frac{1}{2}\, ab \sin \angle ACB = \frac{1}{2}\, bc \sin \angle BAC$. Consider $\triangle CDG$, since $\angle DCG = 180 - \angle ACB$, $CD = b$, $CG = a$, hence area $\triangle CDG$

$L\triangle CDG = \frac{1}{2}\, ab \sin \angle DCG$

$L\triangle CDG = \frac{1}{2}\, ab \sin (180 - \angle ACB)$

$L\triangle CDG = L\triangle ABC$

Similar way for $\triangle AEH$ and $\triangle BIF$, then

$L\triangle CDG = L\triangle AEH = L\triangle BIF = L\triangle ABC$

This completes the proof.

Furthermore Cross' theorem using rectangles and not square on each side of the triangle it has been discussed in reference [4]. Because of many of rectangles that can be constructed, so conclude the side belongs to the rectangle must have the same proportion [4]. The result obtain is only new triangle has the same area, shown in Figure 3 and explained in Theorem 2.2.

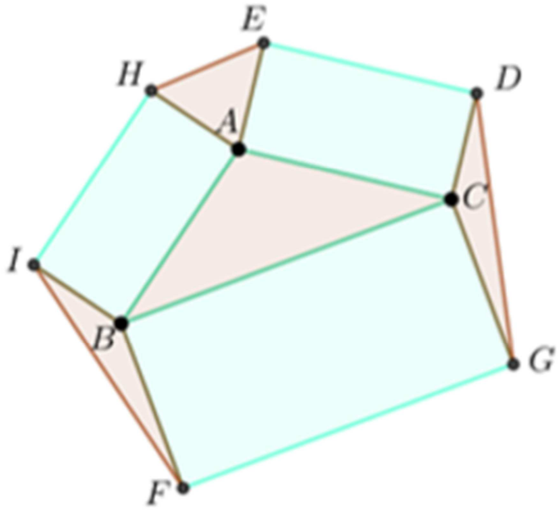

Figure 3. *Illustration Cross' theorem using rectangles.*

Theorem 2.2. Let *ABC* denote any triangle, and consturct rectangles on each side of the triangle outward *ABIH*, *BCGF* and *ACDE* and has a width ½ sides triangle. If line *EH*, *IF* and *DG* constructed, then will form $\triangle AEH$, $\triangle BIF$ and $\triangle CDG$ which has area same each other.

Proof. Let $AB = c$, $BC = a$ and $AC = b$, suppose each rectangle have same *r* proportion, from Theorem 2.1 its easy to show

$L\triangle CDG = L\triangle AEH = L\triangle BIF = L\triangle ABC \times r^2$

This completes the proof.

Furthermore, Cross' theorem on a quadrilateral has been discussed in reeference [11], the theorem on a quadrilateral shows that sum of two pairs triangle at the opposite angle will be the same as the initial quadrilateral area, as explained in Theorem 3.3.

Theorem 3.3 Let *ABCD* denote any quadrilateral, square construction on each side of the quadrilateral outward *ADGH*, *ABFE*, *BCLK* and *CDJI*. If line *EH*, *FK*, *LI* and *JG* constructed, then will form $\triangle AEH$, $\triangle BFK$, $\triangle CLI$ and $\triangle JDG$ which has area $L\triangle AEH + L\triangle CIL = L\ ABCD$ and $L\triangle BFK + L\triangle DGJ = L\ ABCD$, shown in Figure 4.

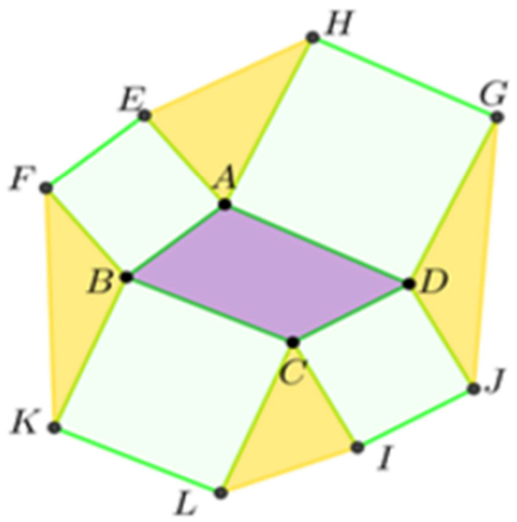

Figure 4. Illustration Cross' theorem on quadlilateral.

Many ideas of congruence concepts are discussed [5-7]. Then in this paper prove the modification Cross' theorem with concepts understood by junior and senior high school students the concept of congruence. The proof pattern in this paper is widely used [8-10]. Based on the Cross' theorem is constructing a square on the sides of a triangle so author is interested in doing modification by constructing a square on a triangle formed from the Cross' theorem.

3. Modification Cross' Theorem

Modification Cross' theorem is based on triangles formed from the Cross' theorem, then construction a square outward direction on each sides of the triangle, if the square vertex is connected it will form new shape, explained in Theorem 3.1.

Theorem 3.1 Let *AEH*, *BIF* and *CDG* are triangles formed from Cross' theorem triangle *ABC*. Furthermore, *EHRQ*, *IFPO* and *DGMN* is constructed on each side triangle Cross' theorem, if vertices a squares are connected then it will form a trapezoid and have area 5 L$\triangle ABC$, shown in Figure 5.

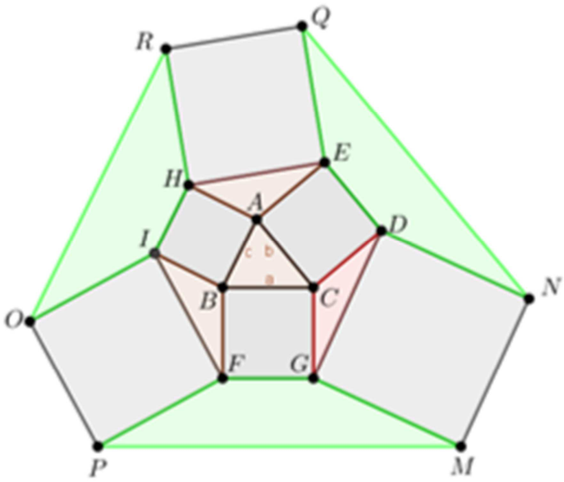

Figure 5. Illustration modification Cross' theorem.

Proof. Will be shown that *FGMP*, *HIOR* and *DEQN* are trapezoid and has an area 5 $L\triangle ABC$. To show *FGMP* trapezoid it will proved *FG*//*PM*. Suppose *A'*, *D'*, *I'*, *P'* and *M'* respectively is projection of point *A* to *BC*, *D* to *GC*, *I* to *FG* and *M* to *FG* so that several triangle are formed its $\triangle ABA'$, $\triangle CDD'$, $\triangle BII'$, $\triangle FPP'$ and $\triangle GMM'$, as in Figure 6.

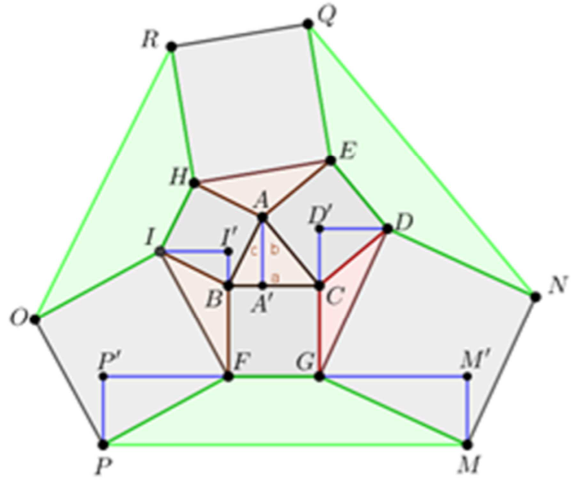

Figure 6. Illustration projection point in modification Cross' theorem.

Consider $\triangle IBF$ on Figure 6, we have

$\angle IBF + \angle ABI + \angle ABC + \angle CBF = 360$

$\angle IBF + 90 + \angle ABC + 90 = 360$

$\angle IBF + \angle ABC + 180 = 360$

$\angle IBF = 360 - 180 - \angle ABC$

$\angle IBF = 180 - \angle ABC$

Thus on $\triangle IBF$ we have

$\angle IBI' + \angle IBF = 180$

$\angle IBF + (180 - \angle ABC) = 180$

$\angle IBI' = 180 - (180 - \angle ABC)$

$\angle IBI' = \angle ABC$

Hence, for $\triangle BAA'$ and $\triangle BII'$, since $\angle IBI' = \angle ABA'$, $\angle BA'A = \angle BII' = 90^\circ$ and $BA = BI$ then $\triangle BAA' \cong \triangle BII'$, so that

$$AA' = II' \tag{1}$$

For $\triangle CD'D$ and $\triangle CAA'$, with similar way obtained $\angle DCG = 180 - \angle ACA'$ and $\angle DCD' = \angle ACA'$, since $\angle DCD' = \angle ACA'$, $\angle CD'D = \angle CA'A = 90^\circ$ and $CD = CA$ then $\triangle CDD' \cong \triangle CAA'$, so that

$$DD' = AA' \tag{2}$$

Next, on $\triangle GMM'$ and $\triangle GDD'$, with similar way obtained $\angle FGM = 180 - \angle CDG$ and $\angle MGM' = \angle CGD$, since $\angle MGM' = \angle CGD$, $\angle GM'M = \angle GD'D = 90^\circ$ and $GM = GD$ then $\triangle GMM' \cong \triangle GDD'$, so that

$$MM' = DD' \tag{3}$$

And then for $\triangle FPP'$ and $\triangle FII'$, with similar way obtained $\angle GFP = 180 - \angle IFI'$ and $\angle PFP' = \angle IFI'$, since $\angle PFP' = \angle IFI'$, $\angle FP'P = \angle FI'I = 90^\circ$ and $FP = FI$ then $\triangle FPP' \cong \triangle FII'$, so that

$$\mathrm{PP'} = \mathrm{II'} \quad (4)$$

Based on (1), (2), (3) and (4) obtained

$$\mathrm{MM'} = \mathrm{DD'} = \mathrm{AA'} = \mathrm{II'} = \mathrm{PP'} \quad (5)$$

Because $MM' = PP'$ then $PM//P'M'$, so that $FG//PM$.

Moreover, to calculate area $FGMP$, consider trapezoid $P'M'MP$ on Figure 6, we have equation obtained

$$\mathrm{PM} = \mathrm{P'M'} = \mathrm{FG} + \mathrm{FP'} + \mathrm{GM'} \quad (6)$$

Consider line FP', since $\triangle FPP' \cong \triangle FII'$ and $\triangle BAA' \cong \triangle BII'$ then

$$FP' = FI' = FB + BI' = FB + BA' \quad (7)$$

Same for line GM', since $\triangle GMM' \cong \triangle GDD'$ and $\triangle CDD' \cong \triangle CAA'$ then

$$\mathrm{GM'} = \mathrm{GD'} = \mathrm{GC} + \mathrm{CD} = \mathrm{GC} + \mathrm{CA'} \quad (8)$$

Substitution (7) and (8) to (6), hence $PM = 4a$.

Next, suppose L, K and S each point that divides line PM to be *4* parts and each one have distance a, shown in Figure 7.

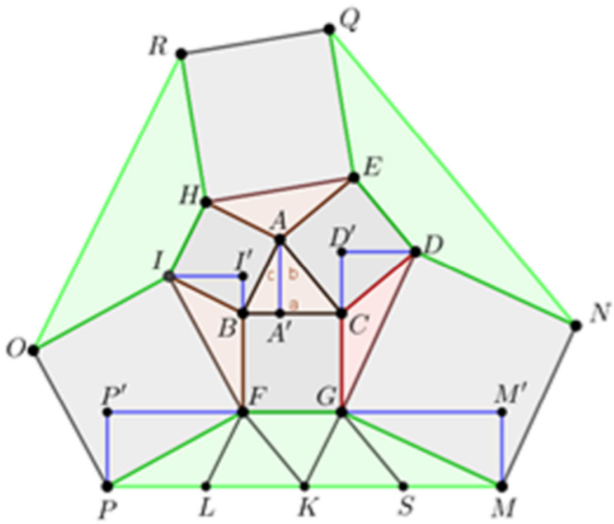

Figure 7. *Illustration triangles on trapezoid FGMP.*

Therefore, $PL = LK = KS = SM = a$, $P'M'//PM$ and $PP' = AA' = MM'$ then obviously

$L\triangle PLF = L\triangle LKF = L\triangle FKG = L\triangle KSG = L\triangle SMG = L\triangle ABC$,

Hence,

$L\ FGMP = 5\ L\triangle ABC$.

In similar way for *HIOR,* that is $HI//RO$ and make that *HIOR* trapezoid, then $RO = 4c$, hence 5 triangles will form in trapezoid *HIOR* and each triangle have same area with triangle *ABC,* shown in Figure 8.

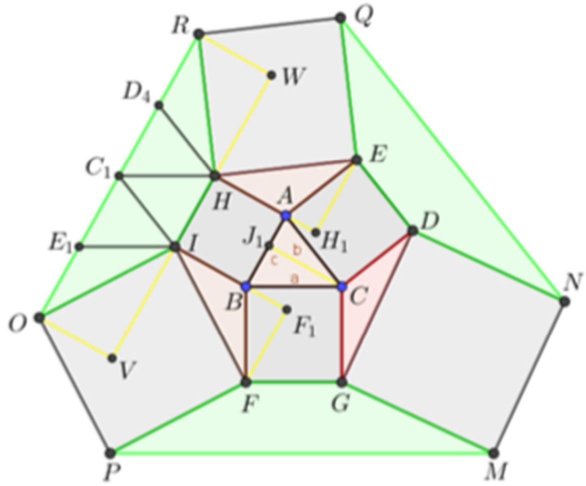

Figure 8. *Illustration proof for HIOR.*

Moreover, similar way apply for *DEQN* which is get *DEQN* is trapezoid and have area $5\ L\ \triangle ABC$, shown in Figure 9.

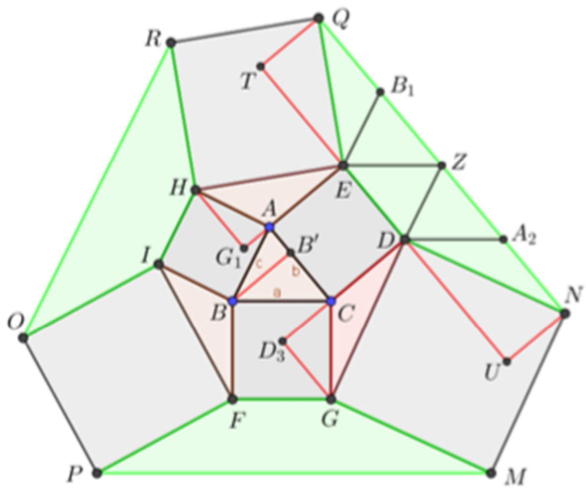

Figure 9. *Illustration proof for DEQN.*

This completes the proof.

Next, will discuss about the extension of line on the sides of expansion trapezoid. In general, if the sides of the triangle produced by the Cross' theorem is extended so that it intersects, there is no relation between the new triangle formed with the initial triangle. Unique to this modification Cross' theorem, if the outer edges of the trapezoid is extended will form a new triangle that is uniform with the initial triangle, as explained in Theorem 3.2.

Theorem 3.2 Let *FGMP*, *HIOR* and *DEQN* is a trapezoid from modification Cross' theorem on triangle *ABC*, then suppose E_a, E_b and E_c are intersects each line *RO* to *NQ*, *RO* to *PM* and *NQ* to *PM*, hence will form $\triangle E_aE_bE_c$ which is uniform to $\triangle ABC$, shown in Figure 10.

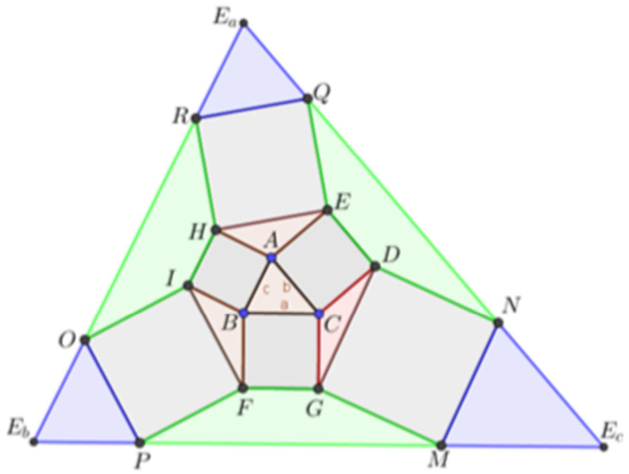

Figure 10. *Illustration extended side of trapezoids.*

Proof. To proof $\Delta E_aE_bE_c \approx \Delta ABC$ will show that each angle on triangle have same size. Therefore from proof Theorem 3.1 we have *FG//PM* so that *BC//PM* then on other side *AB//RO* and *AC//NQ*. Because each line are parallel to each side then obviously the extension will also form a same angle which is $\angle E_aE_bE_c = \angle ABC$, $\angle E_bE_cE_a = \angle BCA$ and $\angle E_cE_aE_b = \angle CAB$. Thus $\Delta E_aE_bE_c \approx \Delta ABC$.

4. Conclusion

In this paper the authors only discuss the shape formed in modification Cross' theorem on triangle and his area. In addition there is relationship betwen trapezoid and initial triangle at modification Cross' theorem on triangle. Therefore, further discussing could be focused on sum of area modification Cross' theorem on triangle include the square, and more expansion on the side of trapezoid.

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