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On the Inverted Gamma Distribution

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Abstract: If a random variable follows a particular distribution then the distribution of the reciprocal of that random variable is called inverted distribution. In this paper we studied some issues related with inverted gamma distribution which is the reciprocal of the gamma distribution. We provide forms for the characteristic function, *r*th raw moment, skewness, kurtosis, Shannon entropy, relative entropy and Rényi entropy function. This paper deals also with the determination of R = P[Y < X] when X and Y are two independent inverted gamma distributions (IGD) with different scale parameters and different shape parameters. Different methods to estimate inverted gamma distribution parameters are studied, Maximum Likelihood estimator, Moments estimator, Percentile estimator, least square estimator and weighted least square estimator. An empirical study is conducted to compare among these methods.

Keywords: Inverted Gamma Distribution, Characteristic Function, Stress-Strength, Shannon Entropy, Relative Entropy, Rényi Entropy, MLE, Percentile Estimator

1. Introduction

The inverted gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution. Perhaps the chief use of the inverted gamma distribution is in Bayesian statistics, where the distribution arises as the marginal posterior distribution for the unknown variance of a normal distribution if an uninformative prior is used; and as an analytically tractable conjugate prior if an informative prior is required.

However, it is common among Bayesians to consider an alternative parameterization of the normal distribution in terms of the precision, defined as the reciprocal of the variance, which allows the gamma distribution to be used directly as a conjugate prior. Other Bayesians prefer to parameterize the inverted gamma distribution differently, as a scaled inverse chi-squared distribution.

Giron and Castillo [4] in 2001 defined the generalized Behrens-Fisher distribution is as the convolution of two Student t distributions and is related to the inverted-gamma distribution by means of a representation theorem as a scale mixture of normals where the mixing distribution is a convolution of two inverted-gamma distributions. One important result in this paper establishes that for odd degrees of freedom the Behrens-Fisher distribution is distributed as a finite mixture of Student t distributions. This result follows from the main theorem concerning the form of the convolution of inverted-gamma distributions with demi-integer shape parameter.

Witkovsky in 2001 [9] presented a formula for evaluation of the distribution of independent inverted Gamma random variables by one dimensional numerical integration. This method is applied to computation of the generalized p-values used for exact significance testing and interval estimation of the parameter of interest in the Behrens-Fisher problem and for variance components in balanced mixed linear model.

Li et al. in 2008 [6] studied the geometric structure of the inverse Gamma manifold from the viewpoint of information geometry and give the Kullback divergence, the J-divergence and the geodesic equations. Also, some applications of the inverse Gamma distribution are provided.

Ali et al. in 2008 [3] defined skew-symmetric distributions based on the double inverted.

Gamma, double inverted Weibull and double inverted compound gamma distributions, all of which have symmetric density about zero. Expressions are derived for the probability density function (pdf), cumulative distribution function (cdf) and the moments of these distributions. They referred that some of these quantities could not be evaluated in closed forms and they used special functions to express them.

Woo in 2012 [10] derived distributions of ratio for two independent gamma variables and two independent inverted gamma variables and then we observe the skewness of two ratio densities. We then consider inference on reliability in two independent gamma random variables and two independent inverted gamma random variables each having known shape parameters.

Abdulah and Elsalloukh in 2012 [1] introduced a new class of asymmetric probability densities, the Epsilon Skew Inverted Gamma (ESIG) distribution. They applied it to analyze skewed and bimodality data. In 2014 [2] the same authors presented basic properties of this distribution, such as the pdf, cdf, and moments are presented. addition, computational forms of parameters estimation of MLE and MME are used. Finally, they illustrated the theory of ESIG distribution by modeling some real data.

Llera and Beckmann in 2016 [7] introduced five different algorithms based on method of moments, maximum likelihood and full Bayesian estimation for learning the parameters of the Inverse Gamma distribution. They also provided an expression for the KL divergence for Inverse Gamma distributions which allows us to quantify the estimation accuracy of each of the algorithms. All the presented algorithms in this paper are novel. The most relevant novelties include the first conjugate prior for the Inverse Gamma shape parameter which allows analytical Bayesian inference, and two very fast algorithms, a maximum likelihood and a Bayesian one, both based on likelihood approximation. In order to compute expectations under the proposed distributions, they used the Laplace approximation. The introduction of these novel Bayesian estimators opens the possibility of including Inverse Gamma distributions into more complex Bayesian structures, e.g. variational Bayesian mixture models. The algorithms introduced in this paper are computationally compared using synthetic data and interesting relationships between the maximum likelihood and the Bayesian approaches are derived.

The probability density function (PDF) and the cumulative distribution function (CDF) for the inverted gamma random variable are respectively,

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \quad , x > 0 \tag{1}$$

$$F(x) = \frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)} \tag{2}$$

Where $\beta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. $\Gamma(u,v) = \int_v^{\infty} t^{u-1} e^{-t}\, dt$, is the upper incomplete gamma function and $\Gamma(u,0) = \Gamma(u)$ is complete gamma function.

In this paper we will refer to Inverted Gamma distribution by $X \sim IG(\alpha,\beta)$, which is mean that the random variable X follow Inverted Gamma distribution with parameters α and β.

The reliability function $R(x)$ and hazard rate function $\lambda(x)$ of $X \sim IG(\alpha,\beta)$ are respectively,

$$R(x) = 1 - F(x) = 1 - \frac{\Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha) - \Gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)}$$

since, $\Gamma(s,\lambda) + \gamma(s,\lambda) = \Gamma(s)$, then

$$R(x) = \frac{\gamma\left(\alpha, \frac{\beta}{x}\right)}{\Gamma(\alpha)} \tag{3}$$

$$\lambda(x) = \frac{f(x)}{R(x)} = \frac{\beta^{\alpha}\, x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)}}{\gamma\left(\alpha, \frac{\beta}{x}\right)} \tag{4}$$

where $\gamma(u,v)$ is the lower incomplete gamma function

The rth raw moment of $X \sim IG(\alpha,\beta)$ can be obtained as,

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx = \int_{x=0}^{\infty} x^r \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{-(\alpha+1)}\, e^{-\frac{\beta}{x}}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{-(\alpha-r+1)}\, e^{-\frac{\beta}{x}}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha-r)}{\beta^{\alpha-r}}\right)$$

$$= \frac{\beta^{r}\, \Gamma(\alpha-r)}{\Gamma(\alpha)} \tag{5}$$

Then, the mean and variance of $IG(\alpha,\beta)$ random variable X are respectively,

$$E(x) = \frac{\beta}{\alpha-1} \quad , \text{for } \alpha > 1 \tag{6}$$

$$V(x) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \text{ for } \alpha > 2 \tag{7}$$

The mode of X is obtained as follows,

$$M_o = \frac{\beta}{\alpha+1} \tag{8}$$

The skewness γ_1 and The excess kurtosis γ_2 are respectively

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{4\sqrt{\alpha-2}}{\alpha-3} \quad \text{for } \alpha > 3 \tag{9}$$

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6(5\alpha-11)}{(\alpha-3)(\alpha-4)} \text{ for } \alpha > 4 \tag{10}$$

The characteristic function of $X \sim IG(\alpha,\beta)$ is,

$$\psi_x(t) = E(e^{itx}) = E\sum_{r=0}^{\infty} \frac{(itx)^r}{r!} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \cdot \frac{\Gamma(\alpha-r)}{\beta^{\alpha-r}}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(it\beta)^r}{r!} \Gamma(\alpha-r) \tag{11}$$

2. Shannon, Renyi and Relative Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. The Shannon entropy of $IG(\alpha,\beta)$ random variable X can be found as follows,

$$H = E(-\mathrm{Ln}(f(x)) = -\int_{-\infty}^{\infty} f(x)\ \mathrm{Ln}(f(x))\,dx$$

So, $-\mathrm{Ln}(f(x)) = \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) + (\alpha+1)\,\mathrm{Ln}(x) + \left(\frac{\beta}{x}\right)$, then

$$E(-\mathrm{Ln}(f(x)) = \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) + \frac{(\alpha+1)\beta^{\alpha}}{\Gamma(\alpha)}\int_{x=0}^{\infty}\mathrm{Ln}(x)\,x^{-(\alpha+1)}\,e^{-\left(\frac{\beta}{x}\right)}\,dx + \frac{\beta^{\alpha+1}}{\Gamma(\alpha)}\int_{x=0}^{\infty} x^{-(\alpha+1+1)}\,e^{-\left(\frac{\beta}{x}\right)}\,dx$$

$$\text{since, } \int_{x=0}^{\infty} x^{-(\alpha+1)}\,e^{-\left(\frac{\beta}{x}\right)}\,dx = \frac{\Gamma(\alpha)}{\beta^{\alpha}}, \text{ then,}$$

$$H = \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) - \frac{(\alpha+1)\beta^{\alpha}}{\Gamma(\alpha)}\int_{x=0}^{\infty}\mathrm{Ln}\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\alpha+1} e^{-\left(\frac{\beta}{x}\right)}\,dx + \alpha$$

$$\text{Let } \left(\frac{1}{x}\right) = y \text{ , then,}$$

$$H = \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) - \frac{(\alpha+1)\beta^{\alpha}}{\Gamma(\alpha)}\int_{x=0}^{\infty}\mathrm{Ln}(y)\,y^{\alpha-1}\,e^{-\beta y}\,dy + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$\text{since } \int_{x=0}^{\infty}\mathrm{Ln}(y)\,y^{\alpha-1}\,e^{-\beta y}\,dy = \beta^{-\alpha}\{\acute{\Gamma}(\alpha) - \mathrm{Ln}(\beta(\Gamma(\alpha))\}$$

$$= \beta^{-\alpha}\,\Gamma(\alpha)\{\psi(\alpha) - \mathrm{Ln}(\beta)\}$$

And $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, then

$$H = \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) - \frac{(\alpha+1)\beta^{\alpha}}{\Gamma(\alpha)}\ \beta^{-\alpha}\,\Gamma(\alpha)\{\psi(\alpha) - \mathrm{Ln}\beta\} + \alpha$$

$$= \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) - (\alpha+1)\{\psi(\alpha) - \mathrm{Ln}(\beta)\} + \alpha$$

$$= \mathrm{Ln}(\Gamma(\alpha)) - \alpha\,\mathrm{Ln}(\beta) + (\alpha+1)(\ln(\beta) - \psi(\alpha)) + \alpha$$

$$= \alpha + \ln(\beta\,\Gamma(\alpha)) - (\alpha+1)\psi(\alpha) \tag{12}$$

Rènyi entropy for a random variable $X \sim IG(\alpha,\beta)$ can be derived as,

$$\theta_{\omega} = \frac{1}{1-\omega}\mathrm{Ln}\int_{-\infty}^{\infty} f^{\omega}(x)\,dx$$

Now, since $f^{\omega}(x) = \frac{\beta^{\omega\alpha}}{\Gamma^{\omega}(\alpha)}\ x^{-\omega(\alpha+1)}\ e^{-\left(\frac{\omega\beta}{x}\right)}$, then,

$$\theta_{\omega} = \frac{1}{1-\omega}\ln\int_{x=0}^{\infty}\frac{\beta^{\omega\alpha}}{\Gamma^{\omega}(\alpha)}\ x^{-\omega(\alpha+1)}\,e^{-\frac{\omega\beta}{x}}\,dx$$

$$= \frac{1}{1-\omega}\ \mathrm{Ln}\left(\frac{\beta^{\omega\alpha}}{\Gamma^{\omega}(\alpha)}\int_{x=0}^{\infty} x^{-\omega\alpha-\omega}\,e^{-\frac{\omega\beta}{x}}\,dx\right)$$

$$= \frac{1}{1-\omega}\ \mathrm{Ln}\left(\frac{\beta^{\omega\alpha}}{\Gamma^{\omega}(\alpha)}\int_{x=0}^{\infty} x^{-(\omega\alpha+\omega-1+1)}\,e^{-\frac{\omega\beta}{x}}\,dx\right)$$

$$= \frac{1}{1-\omega}\mathrm{Ln}\left(\frac{\beta^{\omega\alpha}}{\Gamma^{\omega}(\alpha)}\cdot\frac{\Gamma(\omega\alpha+\omega-1)}{(\omega\beta)^{(\omega\alpha+\omega-1)}}\right)$$

$$= \frac{1}{1-\omega}\left((\omega\alpha)\,\mathrm{Ln}(\beta) - \omega\,\mathrm{Ln}(\Gamma(\alpha)) + \mathrm{Ln}(\Gamma((\omega\alpha+\omega-1)) - (\omega\alpha+\omega-1)\,\mathrm{Ln}(\omega\beta)\right) \tag{13}$$

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions G and G^*. It is not symmetric in G and G^*. In applications, G typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while G^* typically represents a theory, model, description, or approximation of G. Specifically, the Kullback–Leibler divergence of G^* from G, denoted $D_{KL}(G \parallel G^*)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution G^* to the posterior probability distribution G. More exactly, it is the amount of information that is *lost* when G^* is used to approximate G ,defined operationally as the expected extra number of bits required to code samples from G using a code optimized for G^* rather than the code optimized for G.

So, relative entropy for a random variable $X \sim IG(\alpha,\beta)$ can be found as follows,

$$D = E\left(\mathrm{Ln}\left(\frac{f(x)}{g(x)}\right)\right) = \int_{-\infty}^{\infty} f(x)\,\mathrm{Ln}\left(\frac{f(x)}{g(x)}\right)\,dx$$

Since, $f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-(\alpha+1)}\,e^{-\left(\frac{\beta}{x}\right)}$ and $g(x) = \frac{b^{a}}{\Gamma(a)}\ x^{-(a+1)}\,e^{-\left(\frac{b}{x}\right)}$, then,

$$\ln\left(\frac{f(x)}{g(x)}\right) = \ln\left(\frac{\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-(\alpha+1)}\,e^{-\left(\frac{\beta}{x}\right)}}{\frac{b^{a}}{\Gamma(a)}\,x^{-(a+1)}\,e^{-\left(\frac{b}{x}\right)}}\right) \text{let } k = \ln\left(\frac{\beta^{\alpha}\Gamma(\alpha)}{b^{a}\Gamma(\alpha)}\right)\text{, then,}$$

$$D = \int_{x=0}^{\infty} f(x)\left\{k - (\alpha+1)\ln(x) + (a+1)\ln(x) - \frac{\beta}{x} + \frac{b}{x}\right\}dx$$

$$= k - (\alpha-a)\int_{x=0}^{\infty}\ln(x)\,f(x)dx - (\beta-b)\int_{x=0}^{\infty}\frac{1}{x}\ f(x)dx$$

$$= k - (\alpha-a)(\ln(\beta) - \psi(\alpha)) - (\beta-b)\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty} x^{-(\alpha+1+1)}\,e^{-\left(\frac{\beta}{x}\right)}\,dx$$

$$= k - (\alpha-a)(\ln(\beta) - \psi(\alpha)) - (\beta-b)\frac{\beta^{\alpha}}{\Gamma(\alpha)}\ \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}$$

$$= k - (\alpha-a)(\ln(\beta) - \psi(\alpha)) - (\beta-b)\frac{\alpha}{\beta} \tag{14}$$

3. Stress-Strength Reliability

Inferences about R = P[Y < X], where X and Y are two independent random variables, is very common in the reliability literature. For example, if X is the strength of a component which is subject to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength.

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $IG(\theta,\lambda)$ and $IG(\alpha,\beta)$, then,

$$R = p(y < x) = \int_{x=0}^{\infty} f_x(x)\, F_y(x)\, dx$$

$$= \int_{x=0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \frac{\Gamma\left(\theta,\frac{\lambda}{x}\right)}{\Gamma(\theta)}\, dx$$

since, $\Gamma(s,\lambda) + \gamma(s,\lambda) = \Gamma(s)$, then, $\Gamma(s,\lambda) = \Gamma(s) - \gamma(s,\lambda)$ and

$$R = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \frac{\left[\Gamma(\theta) - \gamma\left(\theta,\frac{\lambda}{x}\right)\right]}{\Gamma(\theta)}\, dx$$

since, $\gamma\left(\theta,\frac{\lambda}{x}\right) = \left(\frac{\lambda}{x}\right)^{\theta} \Gamma(\theta)\, e^{-\left(\frac{\lambda}{x}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{x}\right)^{k}}{\Gamma(\theta+k+1)}$ then,

$$R = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \frac{\left[\Gamma(\theta) - \left(\frac{\lambda}{x}\right)^{\theta} \Gamma(\theta)\, e^{-\left(\frac{\lambda}{x}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{x}\right)^{k}}{\Gamma(\theta+k+1)}\right]}{\Gamma(\theta)}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \left[1 - \left(\frac{\lambda}{x}\right)^{\theta} e^{-\left(\frac{\lambda}{x}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{x}\right)^{k}}{\Gamma(\theta+k+1)}\right] dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)}\, dx -$$

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \left(\frac{\lambda}{x}\right)^{\theta} e^{-\left(\frac{\lambda}{x}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{x}\right)^{k}}{\Gamma(\theta+k+1)}\, dx$$

$$= 1 - \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\lambda^{k+\theta}}{\Gamma(\theta+k+1)} \int_{x=0}^{\infty} x^{-(\alpha+\theta+k+1)}\, e^{-\frac{(\beta+\lambda)}{x}}\, dx$$

since, $\int_{x=0}^{\infty} x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} dx = \frac{\Gamma(\alpha)}{\beta^{\alpha}}$, then,

$$R = 1 - \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\lambda^{k+\theta}}{\Gamma(\theta+k+1)} \cdot \frac{\Gamma(\alpha+\theta+k)}{(\beta+\lambda)^{\alpha+\theta+k}}$$

$$= 1 - \frac{\left(1+\frac{\lambda}{\beta}\right)^{-\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \left[\frac{\Gamma(\alpha+\theta+k)}{\Gamma(\theta+k+1)\left(\frac{\beta}{\lambda}+1\right)^{(\theta+k)}}\right] \tag{15}$$

when θ is an integer, R can be found as follows as a second way,

$$R = p(y < x) = \int_{x=0}^{\infty} f_x(x)\, F_y(x)\, dx$$

$$= \int_{x=0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \frac{\Gamma\left(\theta,\frac{\lambda}{x}\right)}{\Gamma(\theta)}\, dx$$

Since, $\frac{\Gamma\left(\theta,\frac{\lambda}{x}\right)}{\Gamma(\theta)} = \sum_{i<\theta}^{\infty} e^{-\left(\frac{\lambda}{x}\right)} \frac{\left(\frac{\lambda}{x}\right)^{i}}{i!}$, then,

$$R = \int_{x=0}^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\, x^{-(\alpha+1)}\, e^{-\left(\frac{\beta}{x}\right)} \cdot \sum_{i<\theta}^{\infty} e^{-\left(\frac{\lambda}{x}\right)} \frac{\left(\frac{\lambda}{x}\right)^{i}}{i!}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{i<\theta}^{\infty} \frac{\lambda^{i}}{i!} \int_{x=0}^{\infty} x^{-(\alpha+1)}\, x^{-i}\, e^{-\left(\frac{\beta}{x}\right)}\, e^{-\left(\frac{\lambda}{x}\right)}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{i<\theta}^{\infty} \frac{\lambda^{i}}{i!} \int_{x=0}^{\infty} x^{-(\alpha+i+1)}\, e^{-\left(\frac{\beta+\lambda}{x}\right)}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \sum_{i<\theta}^{\infty} \frac{\lambda^{i}}{i!} \cdot \frac{\Gamma(\alpha+i)}{(\beta+\lambda)^{\alpha+i}} \tag{16}$$

4. Parameters Estimation of Inverse Gamma Distribution

The main aim of this section is to study different estimators of the unknown parameters of IG distribution.

4.1. The Maximum Likelihood Estimator (MLE)

If $x_1, x_2, .., x_n$ is a random sample from $IG(\alpha,\beta)$, then the likelihood and log likelihood functions are respectively,

$$L = \frac{\beta^{\alpha n}}{(\Gamma(\alpha))^{n}} \prod_{i=1}^{n} (x_i)^{-(\alpha+1)}\, e^{-\sum_{i=1}^{n} \frac{\beta}{x_i}}$$

$$Ln(L) = n\alpha\, Ln(\beta) - n\, Ln(\Gamma(\alpha)) - (\alpha+1) \sum_{i=1}^{n} Ln(x_i) - \sum_{i=1}^{n} \frac{\beta}{x_i}$$

$$\text{Now, since } \frac{\partial Ln(L)}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} \left(\frac{1}{x_i}\right) \tag{17}$$

$$\text{and } \frac{\partial Ln(L)}{\partial \alpha} = n\, Ln(\beta) - n\psi(\alpha) - \sum_{i=1}^{n} Ln(x_i) \tag{18}$$

$$\text{from (a.1), } \hat{\beta}_{MLE} = \frac{n\hat{\alpha}}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \tag{19}$$

substituting in (a. 2) we get:

$$n\left(Ln(n\hat{\alpha}) - Ln\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\right) - n\psi(\hat{\alpha}) = \sum_{i=1}^{n} Ln(x_i)$$

$$n\, Ln(n) + n\, Ln(\hat{\alpha}) - n\psi(\hat{\alpha}) = \sum_{i=1}^{n} Ln(x_i) + nLn\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)$$

$$Ln(\hat{\alpha}) - \psi(\hat{\alpha}) = \frac{\sum_{i=1}^{n} Ln(x_i)}{n} + Ln\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right) - Ln(n)$$

$$h(\hat{\alpha}) = \frac{\sum_{i=1}^{n} Ln(x_i)}{n} + Ln\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right) - Ln(n) \tag{20}$$

once we get α^_MLE numerically from (20) we substitute it' s value in (19) to get β^_(MLE)

4.2. The Exact Method of Moments Estimator (EMME)

Here we provide the method of moments estimators of the parameters of a (IG) distribution when both are unknown.

Since the mean and variance of X which is follow IG (α,β) are defined in (6) and (7) respectively, then the coefficient of variation is,

$$CV = \frac{\sqrt{var(x)}}{E(x)} = \frac{\beta}{(\alpha-1)\sqrt{\alpha-2}} \cdot \frac{\alpha-1}{\beta} = \frac{1}{\sqrt{\alpha-2}}$$

The CV is independent of scale parameter β,

By equating the sample CV with population CV , we obtain:

$\frac{s}{\bar{x}} = \frac{1}{\sqrt{\alpha-2}}$, and then can get:

$$\hat{\alpha}_{EMME} = \left(\frac{\bar{x}^2}{s^2}\right) + 2 \tag{21}$$

where $S^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{(n-1)}$ and $\bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}$.
by substituting (21) in (6) we get the EMME of β as follows,

$$\hat{\beta}_{EMME} = \bar{x}\left(\frac{\bar{x}^2}{s^2}+1\right) \tag{22}$$

4.3. The Approximate Method of Moments Estimators (AMME)

Since the mean and mode of X which is follow IG (α,β) are defined in (6) and (8) respectively, then

$$\frac{\bar{x}}{Mo} = \frac{\alpha+1}{\alpha-1} \tag{23}$$

$\rightarrow \frac{\alpha+1}{\alpha-1} = \frac{\sum_{i=1}^{n}\frac{x_i}{n}}{Mo} \rightarrow (\alpha+1)Mo = \alpha - 1\sum_{i=1}^{n}\frac{x_i}{n} \rightarrow \alpha Mo + Mo = \alpha\bar{x} - \bar{x}$, then,

$$\alpha(\bar{x} - Mo) = Mo + \bar{x} \tag{24}$$

is independent of the scale parameter β , then, after calculating the sample mode and the sample mean and substituting their values in (24), One can get the AMME of α , say $\hat{\alpha}_{AMME}$ as follows,

$$\hat{\alpha}_{AMME} = \frac{\bar{x}+Mo}{\bar{x}-Mo} \tag{25}$$

by substituting (25) in (6) we get the AMME of β as follows:

$$\hat{\beta}_{AMME} = \bar{x}\left(\frac{\bar{x}+Mo}{\bar{x}-Mo}-1\right) \tag{26}$$

4.4. Estimators Based on Percentiles (PE)

Kao in (1959) [5] originally explored this method by using the graphical approximation to the best linear unbiased estimators. The estimators can be obtained by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. In the case of a IG distribution, it is possible to use the same concept to obtain the estimators of α and β based on percentiles because of the structure of its distribution function.

Firstly, we find numerically the value of x=F^ (-1) (pi, α, β), where F is defined in (2) and pi is the estimate of $F(x_{(i)},\alpha,\beta)$, then $\hat{\alpha}_{PE}$ and $\hat{\beta}_{PE}$, can be obtained by minimizing

$$\sum_{i=1}^{n}\left[x_{(i)} - F^{-1}(pi,\alpha,\beta)\right]^2 \tag{27}$$

with respect to α and β, where Equation (d.1) is a nonlinear function of α and β .It is possible to use some nonlinear regression techniques to estimate α and β simultaneously, where $E\left(F(x_{(i)})\right) = pi = \frac{i}{n+1}$ is the most used estimator of $F(x_{(i)})$.

4.5. Least Squares Estimator (LSE)

This method was originally suggested by Swain, Venkatraman and Wilson (1988) [8] to estimate the parameteres of beta distribution. Therefore in the case of IG distribution, the least squares estimators of α and β, Say $\hat{\alpha}_{LSE}$ and $\hat{\beta}_{LSE}$ respectively, can be obtained by minimizing,

$$\sum_{i=1}^{n}\left(\frac{\Gamma\left(\alpha,\frac{\beta}{x_{(i)}}\right)}{\Gamma(\alpha)} - \frac{i}{n+1}\right)^2 \tag{28}$$

With respect to α and β .

4.6. Weighted Least Squares Estimators (WLSE)

The weighted least squares estimators of α and β say $\hat{\alpha}_{WLSE}$ and $\hat{\beta}_{WLSE}$ can be obtained by minimizing,

$$\sum_{i=1}^{n}\omega i\left(\frac{\Gamma\left(\alpha,\frac{\beta}{x_{(i)}}\right)}{\Gamma(\alpha)} - \frac{i}{n+1}\right)^2 \tag{29}$$

with respect to α and β where, $\omega i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$

5. The Empirical Study and Discussions

We conduct extensive simulations to compare the performances of the different methods, stated in section 4, for estimating unknown parameters of Inverted Gamma distribution, mainly with respect to their mean square errors (MSE) for different sample sizes and for different parameters values.

The experiments are conducted according to run size $K = 1000$. We reported the results for $n = 10$ (small sample), $n = 20$ (moderate sample) and $n = 50, 100$ (large sample) and for the following different values of a and b,

β	0.6	1	0.9	1.2	0.3
α	1	0.6	0.9	0.3	1.2

The results are reported in table (1). From the table, we observe that,

1) The MSE's decrease as sample size increases in all methods of estimation. It verifies the asymptotic unbiasedness and consistency of all the estimators.
2) It can be said that the estimation of scale parameters are more accurate for the smaller values of those parameters whereas the estimation of shape parameters are more accurate for the larger values of those parameters. in other words, MSE's increase as scale parameter increases whereas MSE's increase as shape parameter decreases.
3) The performances of *LSE*, *EMME* and *AMME* are according to their order.
4) The performances of *EMME*'s and *AMME*'s are close to each other.
5) For small (n=10) sample size and moderate (n=20) sample size, it is observed that *PE* works the best for both of the two parameters whereas the second best method is *MLE*.
6) For large (n=50, 100) sample size, it is observed that *MLE* works the best from all other methods to estimate

the shape parameter whereas the second best method is *PE*. *PE* works the best from all other methods to estimate the scale parameter whereas the second best method is *MLE*.

Table 1. *Empirical MSE to estimate the IG distribution parameters α and β.*

case		1		2		3		4		6	
parameters		α	β	α	β	α	β	α	β	α	β
sample size	the method	0.6	1	1	0.6	0.9	0.9	1.2	0.3	0.3	1.2
10	MLE	10.04613	1.610114	9.668391	1.413262	9.796981	1.571422	9.145992	1.117305	12.47327	1.638623
	EMME	12.56167	1.969878	12.15179	1.773704	12.29646	1.931186	11.58921	1.477069	14.97274	1.997708
	AMME	12.65008	2.003139	12.28038	1.801535	12.43308	1.96309	11.77406	1.508972	15.12545	2.033685
	PE	7.892236	1.596538	7.522538	1.392219	7.667203	1.551737	6.927808	1.098298	10.34349	1.620296
	LSE	12.25627	1.953586	11.83032	1.748589	11.99105	1.906749	11.28381	1.45399	14.66734	1.978023
	WLSE	11.36418	1.825293	10.93822	1.618259	11.11503	1.780492	10.4319	1.326375	13.80739	1.848372
20	MLE	7.868125	1.58364	7.410022	1.382037	7.594871	1.53884	6.87155	1.084722	10.27116	1.605362
	EMME	12.34468	1.959696	11.90265	1.754698	12.07142	1.916931	11.3481	1.462814	14.75575	1.984132
	AMME	12.49738	1.987526	12.11965	1.779814	12.22412	1.945441	11.53295	1.489287	14.90845	2.011284
	PE	7.691313	1.508294	7.23321	1.303975	7.442169	1.467566	6.807254	1.01277	10.1506	1.537482
	LSE	12.15179	1.925077	11.77406	1.718722	11.88658	1.882991	11.21951	1.42548	14.56286	1.95155
	WLSE	11.35614	1.807644	10.95429	1.602647	11.08289	1.767595	10.40779	1.311442	13.77525	1.835475
50	MLE	7.305542	1.582283	6.847439	1.380679	7.032288	1.542234	6.341114	1.087438	9.660354	1.61215
	EMME	12.24824	1.933222	11.86246	1.72551	12.00713	1.888422	11.26773	1.434304	14.66734	1.959017
	AMME	12.43308	1.969878	12.0232	1.769632	12.16787	1.930507	11.42847	1.47639	14.81201	2.001102
	PE	7.586834	1.484536	7.176952	1.277502	7.321616	1.441771	6.598295	0.98969	9.997904	1.51033
	LSE	11.46866	1.823936	11.07485	1.616223	11.21951	1.779814	10.56049	1.327733	13.91991	1.849051
	WLSE	11.28381	1.79882	10.86589	1.597216	11.03466	1.755377	10.35956	1.297187	13.69488	1.825293
100	MLE	7.016214	1.563276	6.622406	1.358958	6.742959	1.519833	6.035712	1.066395	9.427284	1.58975
	EMME	12.15983	1.877561	11.7821	1.6746	11.88658	1.833439	11.24362	1.379322	14.5709	1.902676
	AMME	12.33664	1.875524	11.92676	1.675957	12.5456	1.832081	11.38025	1.38	14.77182	1.903355
	PE	7.442169	1.407831	7.056398	1.202834	7.225173	1.367103	6.485778	0.912986	9.837166	1.434983
	LSE	11.29184	1.81036	10.89	1.606041	11.02663	1.765559	10.30331	1.311442	13.73506	1.834118
	WLSE	11.17933	1.789317	10.81767	1.584319	10.92215	1.743837	10.25508	1.304654	13.65469	1.814432

6. Summary and Conclusions

In view of the great importance of Gamma distributions in statistical analysis, the inverted gamma distribution (IGD) is considered here. For IGD we derived exact formulas of hazard function, characteristic function, *r*th raw moment, skewness, kurtosis, Shannon entropy function, relative entropy, quantile function and stress-strength reliability. Different methods to estimate inverted gamma distribution parameters are studied, Maximum Likelihood estimator, Moments estimator, Percentile estimator, least square estimator and weighted least square estimator. An empirical study was conducted to compare among these methods. It seemed to us that the Percentile estimator is the best one for small and moderate samples and it is also the best to estimate the scale parameter for large samples, whereas the maximum likelihood estimator is the best to estimate the shape parameter for large samples.

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