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Some Random Coefficient Models with Laplace Marginals

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Abstract: In this paper, we study a first order random coefficient autoregressive model with Laplace distribution as marginal. A random coefficient moving average model of order one with Laplace as marginal distribution is introduced and its properties are studied. By combining the two models, we develop a first order random coefficient autoregressive moving average model with Laplace marginal and discuss its properties. A first order random coefficient moving average process with generalized Laplace innovations is also obtained.

Keywords: Autoregressive Process, Laplace Distribution, Moving Average Process, Random Coefficient Models

1. Introduction

Laplace distribution, also known as double exponential distribution, is one of the widely used symmetric distributions for modeling data with heavier tails than normal distribution. This distribution that arises as the difference of two exponential random variables, has found applications in a variety of areas such as image and speech recognition, dynamics of manufacturing companies, ecosystem respiration, ocean engineering, vision, image and signal processing, dynamics of electricity prices, fracture problems, information theory, steam generator inspection, inventory management and quality control, financial data, geographical information systems etc.

The probability density function (pdf) of classical Laplace distribution defined in [8], denoted by L (θ, σ) is

$$f(x)=\frac{1}{2\sigma}e^{\frac{-|x-\theta|}{2\sigma}}, -\infty<x<\infty, -\infty<\theta<\infty, \sigma>0.$$

and the characteristic function is

$$\varphi_X(t)=\frac{e^{i\theta t}}{1+\sigma^2t^2};\ \sigma>0.$$

Mathai (see [10]) introduced generalized Laplace distribution as the distribution of difference of two Gamma variables. This distribution have applications in different contexts such as input – output processes, growth- decay mechanism, residual effect, gain or loss, formation of sand tunes, growth of melatonin in human body, formation of solar neutrinos etc.

The pdf of GL$(\theta,\kappa,\sigma,\tau)$ is

$$h(x)=\frac{\sqrt{2}e^{\frac{\sqrt{2}\,(\frac{1}{\kappa}-\kappa)(x-\theta)}{2\sigma}}}{\sqrt{\pi}\sigma^{\tau+\frac{1}{2}}\Gamma\tau}\left[\frac{\sqrt{2}|x-\theta|}{\kappa+\frac{1}{\kappa}}\right]^{\tau-\frac{1}{2}}K_{\tau-\frac{1}{2}}\left(\frac{\sqrt{2}}{2\sigma}(\kappa+\frac{1}{\kappa})|x-\theta|\right),$$
$$x\neq\theta.$$

where K_λ is the modified Bessel function of third kind with index λ, see [8]. A standard GL density is obtained for θ=0, σ=1 and for θ=0, κ=1 we get a symmetric Laplace distribution.

The characteristic function of GL$(\theta,\kappa,\sigma,\tau)$ defined in [8] is

$$\varphi_X(t)=\frac{e^{i\theta t}}{(1+\sigma^2t^2-i\mu t)^\tau};$$

$$-\infty<t<\infty, -\infty<\mu<\infty, \tau>0, \sigma>0.$$

The characteristic function of GL (0, σ, τ) introduced by Mathai (see [10]) is

$$\varphi_X(t)=\frac{1}{(1+\sigma^2t^2)^\tau};\ \tau>0, \sigma>0.$$

Modelling can contribute to understanding the physical system by revealing something about the physical process that builds persistence into the series. Models for time series data have many forms and represent different stochastic processes. The commonly used models in time series are autoregressive (AR) models, moving average (MA) models and autoregressive moving average (ARMA) models.

The AR model describes how an observation directly depends upon one or more previous measurements plus a white noise term.

Autoregressive time series models of order p, denoted by AR (p) is defined as

$$X_n = \sum_{j=1}^{p} \alpha_j X_{n-j} + Z_n.$$

The MA model describes how an observation depends upon the current white noise term as well as one or more previous innovations.

Finite Moving Average processes of order q, denoted by MA (q) is defined as

$$X_n = \sum_{j=0}^{q} \beta_j Z_{n-j}.$$

If a process consists of both AR and MA parameters, then it is called an ARMA process.

Autoregressive moving average processes of order (p, q), denoted by ARMA (p, q) is defined as

$$X_n = \sum_{j=1}^{p} \alpha_j X_{n-j} + Z_n + \sum_{j=0}^{q} \beta_j Z_{n-j}.$$

where Z_t, $-\infty < t < \infty$ is assumed to be independent and identically distributed random variables with common distribution function F.

Many time series models have been introduced and studied by various authors in recent years. Starting with the pioneering work in [3], non-Gaussian autoregressive models with different stationary marginal distribution are being developed by various researchers. Lawrance and Lewis (see [9]) introduced a first order moving average model with exponential marginal. Jacob and Lewis (see [4]) linked the two models into a first order autoregressive moving average model with exponential marginals. Andel (see [1]) and Dewald and Lewis (see [2]) developed and studied the autoregressive models for real valued variables using Laplace marginal distribution. Jayakumar et al. (see [5]) introduced autoregressive processes with α-Laplace (Linnik) distribution as marginal distribution. Mathew and Jayakumar (see [12]) developed the autoregressive processes associated with generalized Linnik distribution. Jayakumar and Kuttykrishnan and Jayakumar et al. (see [6], [7]) developed autoregressive models with asymmetric Laplace distribution as marginals and discussed various applications in modeling currency exchange rate, interest rate, stock price changes etc. Even though there are a lot of literatures on time series models with heavy tailed distributions, the random coefficient autoregressive models with heavy tailed distributions as marginals have not studied much.

Nicholls and Quinn generalized the autoregressive model in [11] as

$$X_n = \rho_1 X_{n-1} + \rho_2 X_{n-2} + \cdots + \rho_n X_{n-p} + \varepsilon_n$$

by allowing $\rho_i's$ to be random variables to define a random coefficient autoregressive model.

The sequence $\{X_n\}$ is said to follow the pth order random coefficient autoregressive model if

$$X_n = \sum_{i=1}^{p} \left(b_i + V_{i,n}\right) X_{n-i} + \varepsilon_n, n = 1, 2, 3, \dots$$

The following assumptions are made on this model:

(i)$\{\varepsilon_n\}$ is a sequence of independent and identically distributed (iid) random variables with mean 0 and variance σ^2.

(ii) $b = \left(b_1, b_2, \dots b_p\right)$ is a vector of real constants.

(iii) $\left\{Z_n, = \left(V_{1,n}, V_{2,n}, \dots, V_{p,n}\right)\right\}$ is a sequence of iid random vectors with mean zero and dispersion matrix Γ.

(iv) $\{\varepsilon_n\}$ and $\{V_n\}$ are statistically independent.

Hence the first order random coefficient autoregressive model is given by

$$X_n = (b_1 + V_{1,n}) X_{n-1} + \varepsilon_n, n = 1, 2, 3, \dots$$

If $b_1 = 0 \; and \; V_{1,n} = V_n$, then

$$X_n = V_n X_{n-1} + \varepsilon_n, n = 1, 2, 3, \dots \quad (1)$$

or

$$X_n = V_n (X_{n-1} + \varepsilon_n), n = 1, 2, 3 \dots \quad (2)$$

The paper is organized as follows. In Section 2, we define a first order random coefficient autoregressive model with Laplace distribution as marginal and study its properties. In Section 3, we introduce a first order random coefficient moving average Laplace process and discuss its basic properties. A first order random coefficient autoregressive moving average model is also developed in Section 4. The paper ends with the concluding remarks in Section 5.

2. First Order Random Coefficient Laplace Autoregressive Process

The first order random coefficient model (1) defined in [11] has correlation coefficient $\rho_j = [E(V_n)]^j$ and the joint characteristic function is

$$\varphi_{X_n, X_{n+1}}(t_1, t_2) = \varphi_{\varepsilon_{n+1}}(t_2) \int_V \varphi_{X_n}(t_1 + v t_2) dG(v).$$

where V_n is considered as a power function random variable with distribution function (df)

$$F_{V_n}(v) = v^2, 0 < v < 1 \quad (3)$$

Also for the model (2) we have the same correlation coefficient, but the joint characteristic function is $\varphi_{X_n, X_{n+1}}(t_1, t_2) = \int_V \varphi_{\varepsilon_{n+1}}(t_2 v) \, \varphi_{X_n}(t_1 + v t_2) dG(v)$.

Theorem 2.1

Let the process $\{X_n\}$ be defined as the model (2) and

$X_0 = \varepsilon_1$ where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d random variables such that $\{V_n\}$ has the pdf

$$f_{V_n}(v) = 2v;\ 0 < v < 1. \qquad (4)$$

Then the process $\{X_n\}$ is stationary if and only if

$$\varepsilon_1 \sim L(\sigma).$$

Proof

Let $\varphi_{X_n}(t)$ and $\varphi_{\varepsilon_n}(t)$ be the characteristic function of X_n and ε_n respectively. Then

$$\varphi_{X_n}(t) = \int_0^1 \varphi_{X_{n-1}}(tv)\, \varphi_{\varepsilon_n}(tv) f_{v_n}(v)dv.$$

For n=1,

$$\varphi_{X_1}(t) = \int_0^1 \varphi_{X_0}(tv)\, \varphi_{\varepsilon_1}(tv) 2vdv.$$

Since $X_0 = \varepsilon_1$,

$$\varphi_{X_1}(t) = \int_0^1 \left[\varphi_{X_0}(tv)\right]^2 2vdv.$$

Assume the process $\{X_n\}$ is stationary,

$$\varphi_X(t) = \int_0^1 [\varphi_X(tv)]^2\, 2vdv$$

$$= \frac{2}{t^2}\int_0^t z[\varphi_X(z)]^2\, dz,$$ by the substitution $tv = z$.

Then, differentiating $t^2\varphi_X(t) = 2\int_0^t z[\varphi_X(z)]^2\, dz$ with respect to t, we get,

$$\frac{\varphi'_X(t)}{[\varphi_X(t)]^2} + \frac{2}{t}\frac{1}{\varphi_X(t)} = \frac{2}{t}.$$

This reduces to the linear equation, $\frac{du}{dt} - \frac{2}{t}u = -\frac{2}{t}$, by the substitution $\frac{1}{\varphi_X(t)} = u$.

Solving, we get $\varphi_X(t) = \frac{1}{1+ct^2}, c > 0$.

Conversely, let $\varphi_{X_n}(t) = \int_0^1 \varphi_{X_{n-1}}(tv)\, \varphi_{\varepsilon_n}(tv) f_{v_n}(v)dv$.

Since $X_0 \sim L(\sigma)$, for n=1,

$$\varphi_{X_1}(t) = \int_0^1 \varphi_{X_0}(tv)\, \varphi_{\varepsilon_1}(tv) 2vdv$$

$$= \int_0^1 \left[\frac{1}{1+\sigma^2 t^2 v^2}\right]^2 2v\, dv$$

which reduces to

$$\varphi_{X_1}(t) = \frac{1}{\sigma^2 t^2}\int_1^{1+\sigma^2 t^2} \frac{1}{z^2} dz$$

$$= \frac{1}{1+\sigma^2 t^2}$$

Assuming $X_{n-1} \sim L(\sigma)$, we can prove that $X_n \xrightarrow{d} L(\sigma)$.

Hence the process $\{X_n\}$ is stationary with Laplace marginals, which completes the proof.

Definition 2.1

The first order random coefficient autoregressive symmetric Laplace process is defined as follows:

Let $X_0 = \varepsilon_1$ and for $n = 1, 2, ...; X_n = V_n(X_{n-1} + \varepsilon_n)$,

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of iid random variables with pdf

$$f_{V_n}(v) = 2v;\ 0 < v < 1$$

and ε_1 is distributed as symmetric Laplace $L(\sigma)$.

Properties

The simulated sample path using 100 observations generated from first order autoregressive Laplace process with σ=0.2 and 0.4 are presented in Figure 1 and Figure 2 respectively.

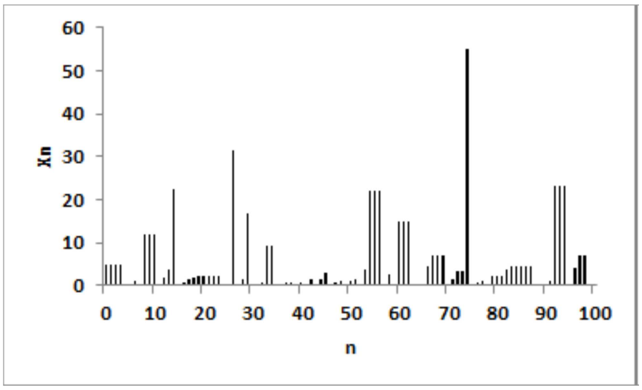

Figure 1. *Sample Path of Random coefficient AR (1) model for σ=0.2.*

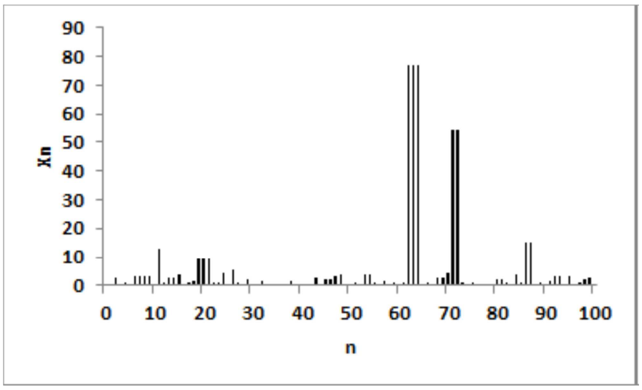

Figure 2. *Sample Path of Random coefficient AR (1) model for σ=0.4.*

The joint distribution of (X_n, X_{n+1}) for first order autoregressive symmetric Laplace is

$$\varphi_{X_n,X_{n+1}}(t_1,t_2) = \int_V \varphi_{\varepsilon_{n+1}}(vt_2)\varphi_{X_n}(t_1 + vt_2)2vdv.$$

$$= \int_0^1 \left[\frac{1}{1+\sigma^2 {t_2}^2 v^2}\right]^2 \left[\frac{1}{1+\sigma^2 (t_1+t_2 v)^2}\right]^2 2v\, dv$$

which is not symmetric in t_1 *and* t_2. Hence the process is not time reversible.

3. First Order Random Coefficient Laplace Moving Average Process

The first order random coefficient moving average model defined in [11] takes the form

$$X_n = W_n \varepsilon_{n-1} + \varepsilon_n, n = 1, 2, 3 \ldots. \quad (5)$$

or

$$X_n = W_n(\varepsilon_{n-1} + \varepsilon_n), n = 1, 2, 3 \ldots. \quad (6)$$

Theorem 3.1

Let the process $\{X_n\}$ be defined as the model (6) where $\{W_n\}$ is a sequence of iid random variables with distribution function,

$$F_{W_n}(w) = w^2, 0 < w < 1 \quad (7)$$

and $\{\varepsilon_n\}$ is a sequence of iid Laplace random variables independent of $\{W_n\}$. Then the process $\{X_n\}$ defines a first order Laplace moving average process.

Proof

$$\varphi_{X_n}(t) = \int_0^1 \varphi_{\varepsilon_{n-1}}(tw)\, \varphi_{\varepsilon_n}(tw) f_{w_n}(w) dw.$$

Since $\{\varepsilon_n\}$ follows Laplace distribution with characteristic function $\frac{1}{1+\sigma^2 t^2}$,

$$\varphi_{X_n}(t) = \int_0^1 \left[\frac{1}{1+\sigma^2 t^2 w^2}\right]^2 2w dw.$$

On the substitution $z = 1 + \sigma^2 t^2 w^2$, the integral reduces to

$$\varphi_{X_n}(\mathrm{t}) = \frac{1}{\sigma^2 t^2} \int_1^{1+\sigma^2 t^2} \frac{dz}{z^2}$$

$$= \frac{1}{1+\sigma^2 t^2}.$$

Hence the theorem.

Properties

The simulated sample path using 100 observations generated from first order moving average Laplace process with σ=0.2 and 0.4 are presented in Figure 3 and Figure 4 respectively.

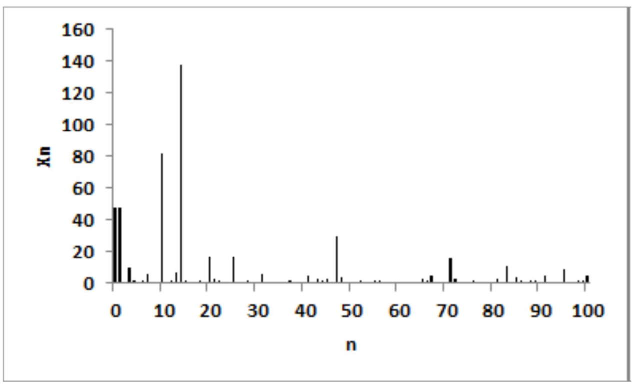

***Figure 3.** Sample Path of Random coefficient MA (1) model for σ=0.2.*

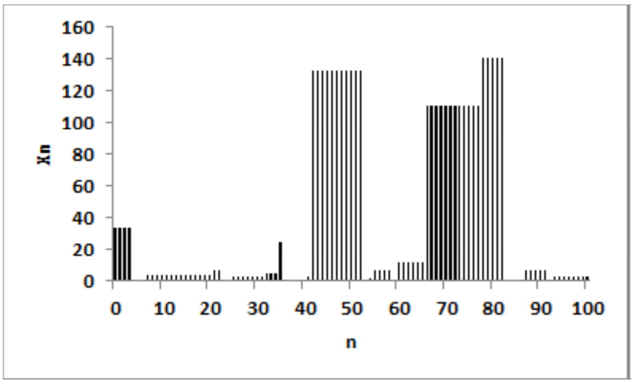

***Figure 4.** Sample Path of Random coefficient MA (1) model for σ=0.4.*

Similar to the joint distribution of autoregressive random coefficient model, the joint distribution of (X_n, X_{n+1}) for first order moving average symmetric Laplace process is also not symmetric in t_1 *and* t_2. Hence the process is not time reversible.

Theorem 3.2

Let the process $\{X_n\}$ be defined as the model (6) where $\{W_n\}$and $\{\varepsilon_n\}$ are independent sequences such that $\{W_n\}$ has the probability density function,

$$f_{W_n}(w) = 2w;\ 0 < w < 1. \quad (8)$$

Then X_n has Laplace distribution if X_n and ε_n are identically distributed random variables.

Proof

We have

$$\varphi_{X_n}(t) = \int_0^1 \varphi_{\varepsilon_{n-1}}(tw)\, \varphi_{\varepsilon_n}(tw) 2w dw.$$

On the substitution $tw = z$, the integral reduces to

$$\varphi_{X_n}(\mathrm{t}) = \frac{2}{t^2} \int_0^t z\left[\varphi_{\varepsilon_n}(z)\right]^2 dz.$$

Assuming X_n and ε_n follows same distribution, we get $t^2 \varphi_X(t) = 2 \int_0^t [\varphi_X(z)]^2\ dz$.

On differentiation with respect to t, the above equation becomes

$$\frac{\varphi'_X(t)}{[\varphi_X(t)]^2} + \frac{2}{t}\frac{1}{\varphi_X(t)} = \frac{2}{t}.$$

This reduces to the linear equation $\frac{du}{dt} - \frac{2}{t}u = -\frac{2}{t}$, by the substitution $\frac{1}{\varphi_X(t)} = u$.

Solving, we get $\varphi_X(t) = \frac{1}{1+ct^2}, c > 0$.

Hence the theorem.

Remark 3.1

If X_n and ε_n are identically distributed random variables such that $\{X_n\}$ is defined as (5), then also X_n has Laplace distribution, where $\{W_n\}$ and $\{\varepsilon_n\}$ are independent sequences such that $\{W_n\}$ has pdf (8).

Theorem 3.3

Let the process $\{X_n\}$ be defined as (5) and $\{W_n\}$ is a sequence of iid random variables with pdf

$$f_{W_n}(w) = 2\tau w^{2\tau-1};\ 0 < w < 1, \tau > 0$$

and $\{\varepsilon_n\}$ is a sequence of iid Generalized Laplace GL (0, σ,τ) random variables independent of $\{W_n\}$. Then the process $\{X_n\}$ defines a first order Laplace moving average process provided $X_0 \overset{d}{\to} \varepsilon_1$.

Proof

We have

$$\varphi_{X_n}(t) = \int_0^1 \varphi_{\varepsilon_{n-1}}(tw)\, \varphi_{\varepsilon_n}(t) 2\tau w^{2\tau-1} dw$$

$$= \int_0^1 \frac{2\tau}{[1+\sigma^2 t^2]^{\tau+1}} \left[\frac{1}{1+\sigma^2 t^2 w^2}\right]^{\tau+1} w^{2\tau-1} dw$$

$$= \int_0^1 \frac{\tau}{[1+\sigma^2 t^2]^{\tau+1}} \left[\frac{1}{1+\sigma^2 t^2 u}\right]^{\tau+1} u^{\tau-1} du$$

$$= \int_0^{t^2} \frac{\tau}{[1+\sigma^2 t^2]^{\tau+1}} \frac{1}{t^{2\tau}} z^{\tau-1} \left[\frac{1}{1+\sigma^2 z}\right]^{\tau+1} dz$$

This reduces to

$\varphi_X(t) = \frac{1}{1+\sigma^2 t^2}$, on the substitution $\frac{z}{1+\sigma^2 z} = u$ and on simplification.

Hence the theorem.

4. First Order Random Coefficient Laplace Autoregressive Moving Average Process

By combining the models (2) and (6) and using Theorems (2.1) and (3.1), we have the following theorem.

Theorem 4.1

Let the process $\{X_n\}$ be defined by $X_n = W_n(\varepsilon_n + Y_{n-1})$, $Y_n = V_n(Y_{n-1} + Z_n)$,

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of iid random variables and $\{W_n\}$ and $\{V_n\}$ are also two independent sequences of iid random variables with cdf

$$F_{W_n}(w) = w^2, 0 < w \leq 1 \text{ and } F_{V_n}(v) = v^2, 0 < v \leq 1 \text{ with } \varepsilon_1 \overset{d}{\to} Y_0 \overset{d}{\to} Z_1.$$

Then the process $\{X_n\}$ is stationary if and only if

$$\varepsilon_1 \overset{d}{\to} L(\sigma).$$

Proof

Proof follows easily by same argument as in Theorems (2.1) and (3.1).

5. Conclusion

Laplace distribution and its generalizations have proved to be a successful alternative for Gaussian distribution with a wide range of applications in financial modeling, communication engineering, time series modeling, image source modeling, gene expression data modeling etc. Laplace laws have appeared in literature on modeling stock market returns, currency exchange returns and interest rate. This distribution is a commonly used model for heavy tailed data such as image and speech comparison data, microarrays etc. As Laplace distributions is considered as the first choice for modeling whenever data exhibit heavier tails than Gaussian tail, the random coefficient Laplace time series models discussed in this paper could be the good models for fitting of various time series data.

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