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Placement of M-Sequences over the Field *Fp* in the Space R^n

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Abstract: Spread spectrum communication systems are widely used today in a variety of applications for different purposes such as access of same radio spectrum by multiple users (multiple access), anti-jamming capability (so that signal transmission cannot be interrupted or blocked by spurious transmission from enemy), interference rejection, secure communications, multi-path protection, etc. Several spreading codes are popular for use in practical spread spectrum systems. one of these important codes is Maximal Sequence (M-sequence) length codes, These are the longest codes that can be generated by a shift register of a specific length, The number of 1-s in the complete sequence and the number of 0-s will differ by one, Further, the auto-correlation of an m-sequence is -1, another interesting property of an M-sequence is that the sequence, when added (modulo-2) with a cyclically shifted version of itself, results in another shifted version of the original sequence. Hence, the M-sequences are also known as, pseudo-noise or PN sequences. Current article study placement M-Sequences over the finite field Fp or Mp-Sequences (where p is a prime number) in the space R^n, these sequences can be generated as a closed set under the addition. These sequences form additive groups with the corresponding null sequence that was generated by the feedback shift registers. Such Mp-Sequences see a great application in the forward links of communication channels. Furthermore, they form coders and decoders that combine the information by p during the connection process with the backward links of these channels. These sequences scrutinize the transmitted information to enable it to reach the receivers in an accurate form. This study has defined eminent surfaces in the vector space 'R' having dimensions of 'n' as quadratic forms, spheres, and planes that contain these sequences.

Keywords: M-Sequences, Coefficient of Correlation, Orthogonal Sequences, Additive Group, Code, Field *Fp*, Space R^n

1. Introduction

The sequence of order k that is homogeneously linear has been supported by recurring:

$$a_{n+k} + \sum_{i=0}^{k-1} \lambda_i a_{n+i} = 0 \ , \lambda_i \in F_p = \{0,1,2,\dots,p-1\}, p \text{ is prime} \quad (1)$$

With the characteristic polynomial;

$$f(x) = x^k + \lambda_{k-1}x^{k-1} + \lambda_{k-2}x^{k-2} + \dots + \lambda_1 x + \lambda_0 \quad (2)$$

If $f(x)$ is prime and $a_1, a_2, \dots a_k$ are not equal to zero, then the sequence $a_1, a_2, a_3, \dots$ is periodic with the maximum length or period $p^k - 1$. The calculations preformed using mod p is called Mp-Sequence over *Fp*, and the number of all these non-zero sequences corresponding to all cases of $a_1, a_2, \dots a_k$ is p^k-1. These Mp-Sequences can be widely used in communication channels, and coders, and decoders by mod p as. The Mp-Sequences have zero sequence form an additive group, when the addition is pre-formed at mod p. [1, 2]

Example 1. The sequence over F_3 of Linear Recurring;

$$a_{n+2} + a_{n+1} + 2a_n = 0 \ or \ a_{n+2} = 2a_{n+1} + a_n \quad (3)$$

The prime factor with the characteristic equation $x^2 + x + 2 = 0$ as well as the characteristic polynomial $f(x) = x^2 + x + 2$ produce F_{3^2}. If α is a root of $f(x)$ and generates F_{3^2}, then this characteristic equation can be solved as $\{\alpha^n, \alpha^{3n}\}$. While equation (3) has a general solution that is given by; [3]

$$a_n = 2\alpha . \alpha^n + (1+\alpha) . \alpha^{3n} \quad (4)$$

while the sequence is periodic with the period $3^2 - 1 = 8$.

Initially it has position $a_1 = 1, a_2 = 0$, then $S_1 = (1\,0\,1\,2\,2\,0\,2\,1)$; whereas, by the cyclic permutations on S_1 it has \$ = $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ where;

$$S_2 = (1\,1\,0\,1\,2\,2\,0\,2);\ S_3 = (2\,1\,1\,0\,1\,2\,2\,0);\ S_4 = (0\,2\,1\,1\,0\,1\,2\,2);\ S_5 = (2\,0\,2\,1\,1\,0\,1\,2)$$

$$S_6 = (2\,2\,0\,2\,1\,1\,0\,1);\ S_7 = (1\,2\,2\,0\,2\,1\,1\,0);\ S_8 = (0\,1\,2\,2\,0\,2\,1\,1) \quad (5)$$

The first two digits represent the initial position for each of the sequences for the feedback register. In the given example the resulting sequences is: 10122021 10122….

Example 2: The Sequence over F_5 as the Linear Recurring is;

$$a_{n+2} + a_{n+1} + 2a_n = 0 \ or \ a_{n+2} = 4a_{n+1} + 3a_n \quad (6)$$

The characteristic equation of this sequence is $x^2 + x + 2 = 0$ and its characteristic polynomial is $g(x) = x^2 + x + 2$ which is a prime polynomial generates F_{5^2}, for the initial position (0 1) we have the following sequence;

$$01443402331304112103224 2\ 014\ldots \quad (7)$$

Example 3: The Sequence over F_2 of Linear Recurring is given by;

$$a_{n+3} = a_{n+1} + a_n \text{ or } a_{n+3} + a_{n+1} + a_n = 0 \quad (8)$$

The prime factor with the characteristic equation $x^3 + x + 1 = 0$ and the polynomial characteristic $f(x) = x^3 + x + 1$ produces the field F_{2^3} by considering sequence that is periodic within the period $2^3 - 1 = 7$. Initially, $a_1 = 1, a_2 = 0, a_3 = 0$, thereby, S_1 = (1001011), while with the cyclic permutations of S_1 has \$=$\{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ where;

$$S_2 = (1100101), S_3 = (1110010), S_4 = (0111001) \ (9)$$

$$S_5 = (1011100), S_6 = (0101110), S_7 = (0010111) \ (10)$$

There are first three digits that represent the initial position for each of these sequences for the feedback register [1-10].

Some of the famous surfaces are in the vector space R with n dimension. The quadratic forms are between the family of famous surfaces and their equation is as follows;

$$f(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + d = 0 \quad (11)$$

The original point can shift and rotate the axes; therefore, we can write equation (11) as symmetric equations for some symmetric axes and symmetric planes;

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j = k, a_{ij} = a_{ji}; i,j = 1,2..,n \quad (12)$$

The above equation (11 or 12) represents some of the eminent surfaces in the space R^n that are Ellipsoids, Hiperbolasoids, Spheres, or Lines [11-14].

The equation of the Ellipsoid in the space R^n with the center *o* is;

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1 \quad (13)$$

The equation of the sphere in the space R^n with the center *o* and the radius *r* is;

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \quad (14)$$

The equation of the plane in the space R^n is;

$$\sum_{i=1}^{n} a_ix_i = b \quad (15)$$

2. Research Method and Materials

The Ultimately Periodic Sequence $a_0, a_1, \ldots.$ in $F_p = \{0, 1, 2, \ldots, p-1\}$ along with *r* which, represents periodic or the smallest period given as; $a_{n+r} = a_n$; $n = 0,1,\ldots.$ [1-4]. Euler function $\varphi(n)$represents relative prime number of the natural numbers through *n*. [5-8]. The code *C* of the form [*n*, *k*, *d*] is used, if each element (Codeword) has the length *n*. The Rank *k* is the number of information components and minimum distance is *d* [9]. Two vectors are; $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ on F_p having *n* length. The coefficient of correlations functions of *x* and *y* is indicated by $R_{x,y}$; [4,5]

$$R_{x,y} = \sum_{i=0}^{n-1}(-1)^{x_i+y_i} \quad (16)$$

where, $x_i + y_i$ is computed mod *p*. The periodic sequence $(a_i)_{i\in N}$ over *Fp* with *r* period comprises of the properties within "Ideal Auto Correlation" but only in the case when there is a "periodic auto Correlation".

$R_a(\tau)$ as given below for $p > 2$:

$$R_a(\tau) = \begin{cases} r\,;\ for\ \tau \equiv 0 \\ 0;\ otherwise\ and\ \tau \neq \frac{r}{2} \end{cases} \quad (17)$$

When [6, 7];

$$R_a(\tau) = \sum_{t=0}^{r-1}(-1)^{a(\tau+t)+a(t)} \quad (18)$$

Any Periodic Sequence $a_0, a_1, \ldots.$ over F_p having prime polynomial, represents an ideal autocorrelation and orthogonal cyclic code [8, 9].

Theorem 1.

i. If $a_0, a_1, \ldots.$ is a sequence of order *k* in F_p, which is homogeneously linear recurring and satisfying equation (1) and $\lambda_0 \neq 0$, then it is a periodic sequence.
ii. If the current sequence is homogeneous and linear recurring then it will be periodic, and if its characteristic polynomial $f(x)$ is prime and the period of the sequence is *r* then $r|ord\ f(x)$.
iii. If $f(x)$ is prime characteristic polynomial of the sequence then *r* the period of the sequence is given by $r = p^k - 1$, known as M_p – Sequence over F_p or briefly M_p-Sequence. [10-11]

Theorem 2 (Fermat's theorem).

If there is the finite field *F*, having *q* elements, then all the elements of *F* will satisfy this equation [12];

$$x^q = x \tag{19}$$

Theorem 3.

g(x) representing the prime characteristic polynomial over Fp of the (H. L. R. S.) $a_0, a_1, \dots$ for k degree, while the root of $g(x)$ is α within Fq considering any splitting field Fp, then the general term of the sequence is;

$$a_n = \sum_{i=1}^{k} C_i (\alpha^{p^{i-1}})^n \tag{20}$$

And the coefficients C_i determined in Fq uniquely. [13]

Theorem 4.

For any prime p and positive integers m and n,

$$(q^m - 1) \mid (q^n - 1) \Leftrightarrow m \mid n \tag{21}$$

If F_q represents a field within the $q = p^n$ is the number of elements in it then the order of any subfields of it is of the form p^m, *where m is positive* and $m \mid n$, or m is positive divisor of n, Also, through inverse, if $m \mid n$ and m is positive, then there is a subfield F in the field F_q of Order p^m. In any field, if $\beta^2 = \beta$ then $\beta = 0$ *or* 1. [14]

Theorem 5.

The number of unitary irreducible polynomials of degree m and order e in $F_q[x]$ equals to $\varphi(e)/m$ if $e \geq 2$, where m is the order of q *mod e*, it equals 2 in case $m = e = 1$, and equals to zero elsewhere. [15, 16]

Theorem 6.

i. The necessary and sufficient condition for the square matrix $A = [a_{ij}]_n$ to be similar to the diagonal matrix $D = [d_i]_n$ (T. e: There is a regular matrix $B = [b_{ij}]_n$ such, $A = B^{-1}DB$ *or* $D = BAB^{-1}$) is the matrix A is symmetric and has n independent characteristic vectors.

ii. If the independent characteristic vectors of the matrix A are $V_1 = (v_{11}, v_{12}, \dots, v_{1n})^T$, ….., $V_n = (v_{n1}, v_{n2}, \dots, v_{nn})^T$ corresponding to the characteristic values $\lambda_1, \dots\dots, \lambda_n$ then:

$$B = [V_1, V_2, \dots, V_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

And if the axis Y_1, Y_2, …, Y_n corresponding to the independent characteristic vectors V_1, V_2, …, V_n, then the quadratic form $\sum_{i,j=1}^{n} a_{ij} X_i X_j = X^T A X$ can be shown as:

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = \sum_{i=1}^{n} \lambda_i Y_i^2 = Y^T D Y \tag{22}$$

Where, $Y^T = [Y1, Y2, \dots, Yn]$. [17-19] *Corollary 7.*

All non-zero elements in $GF(p^n)$ satisfy the equation $x^{\frac{p^n - 1}{2}} = 1$, or the equation $x^{\frac{p^n - 1}{2}} = p - 1$ *mod p*. The corresponding matrix of a linear recurring sequence generated by prime polynomial or any power of the matrix satisfies the following condition;

$$R_j + R_{j + \frac{p^n - 1}{2}} = 0 \; mod \; p : \; j = 1, 2, 3, \dots, \frac{p^n - 1}{2} \tag{23}$$

When R_j is the j^{th} row.

Sum of all the entries in any row (or any one period) of the Mp- Sequence is equal to zero *mod p*. Practical examples show that the repetition of any non-zero element in one period is p^{n-1}, but the repetition of zero in one period is p^{n-1}-1. [20]

$$\sum_{i=1}^{p^n - 1} a_i^2 = 0, mod \; p \tag{24}$$

3. Results and Discussions (Findings)

3.1. Quadratic Forms

The general form of second degree equations is;

$$f(x_1, x_2, \dots, x_n) \equiv \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d = 0 \tag{25}$$

If $a_{ij} = a_{ji}$ it can write (25) by the form:

$$f(x_1, x_2, \dots, x_n) \equiv \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d = 0 \tag{26}$$

By translation of the axes, the first degree terms can be removed; or

$$f(X_1, X_2, \dots, X_n) \equiv \sum_{i=1}^{n} a_{ii} X_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} a_{ij} X_i X_j + k' = 0 \tag{27}$$

Or,

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j + k' = 0 \tag{28}$$

Or,

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = k \tag{29}$$

The form,

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j, \tag{30}$$

When, $a_{ij} = a_{ji}$ is called as "Quadratic Form" and can be written as;

$$= [X_1 X_2 \dots X_n] \begin{bmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \tag{31}$$

Where;

$$A = \begin{bmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \tag{32}$$

And the equation (29) becomes:

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = X^T A X = k \tag{33}$$

The equation (33) is symmetric for the origin of coordinate and if there are symmetric axes it pass from the origin coordinates and the equation (29) becomes;

$$X^T AX = k \quad (34)$$

The equation of the sphere in the space R^n having a center as the origin point and r as the radius;

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \; or \; \sum_{i=1}^{n} x_i^2 = r^2 \quad (35)$$

The equation of the plane in the space R^n is;

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + d = 0 \; or \sum a_i x_i + d = 0 \quad (36)$$

3.2. Return a Quadratic Form to Its Regular Form

In theorem 6, if the matrix A of the quadratic form contains n independent characteristic vectors $V_1, V_2, \ldots, V_n$ consequently to the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, and if the axis $Y_1, Y_2, \ldots, Y_n$ corresponding to these independent characteristic vectors, then the quadratic form;

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = X^T AX \quad (37)$$

Can be shown as;

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = \sum_{i=1}^{n} \lambda_i Y_i^2 = Y^T DY \; when \; D = \begin{bmatrix} \lambda_1 & 0 & \vdots & 0 \\ 0 & \lambda_2 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \lambda_n \end{bmatrix} \quad (38)$$

Moreover, the equation;

$$\sum_{i,j=1}^{n} a_{ij} X_i X_j = k \quad (39)$$

becomes;

$$\sum_{i=1}^{n} \lambda_i Y_i^2 = k \quad (40)$$

3.3. Mp-Sequences as a Points on the Quadratic Surface

Each of the M*p*-Sequences $(a_i)_N$ over Fp generated by the prime polynomial $f(x)$ of degree k has the period $r = p^k - 1$; where as, each of these non-zero elements has repetition p^{k-1} and its zero element have the repetition p^{k-1}-1.

One family of the functions of the quadratic forms;

$$F(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i=1, j>i}^{n-1} a_{ij} x_i x_j = C' \quad (41)$$

As a surface of second degree in the space R^n, the coefficients were determined as $a_{ii}, i = 1, 2, \ldots. n, a_{ij}, i = 1, 2, \ldots., n-1 \& j > i$ and *C'* for the Mp-Sequences, when $n = p^k - 1 \; and x_i = a_i$, i=1,…, n, will be on the surface. The function;

$$F(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} x_i^2 + \sum_{i=1, j>i}^{n-1} x_i x_j = C \quad (42)$$

The constant C was determined for the sequences which will lie on the surface (42).

It is known that;

$$\sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1, j>i}^{n-1} x_i x_j = (x_1 + x_2 + \cdots + x_n)^2 \quad (43)$$

Or;

$$\sum_{i=1, j>i}^{n-1} x_i x_j = \frac{(x_1 + x_2 + \cdots + x_n)^2 - \sum_{i=1}^{n} x_i^2}{2} \quad (44)$$

Or;

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1, j>i}^{n-1} x_i x_j = \frac{(x_1 + x_2 + \cdots + x_n)^2 - \sum_{i=1}^{n} x_i^2}{2} + \sum_{i=1}^{n} x_i^2 \quad (45)$$

Or;

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1, j>i}^{n-1} x_i x_j = \frac{(x_1 + x_2 + \cdots + x_n)^2 + \sum_{i=1}^{n} x_i^2}{2} \quad (46)$$

It is known that;

$$\begin{cases} 1 + 2 + \ldots + n = (n(n+1))/2 \\ 1^2 + 2^2 + \ldots + n^2 = (n(n+1)(2n+1))/6 \\ 1^3 + 2^3 + \ldots + n^3 = [(n(n+1))/2]^2 \end{cases} \quad (47)$$

Thus:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= p^{(k-1)}\big(1 + 2 + \cdots + (p-1)\big) \\ &= p^{(k-1)} \frac{(p-1)p}{2} \\ x_1 + x_2 + \cdots + x_n &= \frac{p^k (p-1)}{2} \\ (x_1 + x_2 + \cdots + x_n)^2 &= \frac{p^{2k}(p-1)^2}{4} \\ \sum_{i=1}^{n} x_i^2 &= p^{(k-1)}(1^2 + 2^2 + \cdots + (p-1)^2) \\ &= p^{(k-1)} \frac{(p-1)p(2p-1)}{6} \end{aligned} \quad (48)$$

$$= \frac{p^k(p-1)(2p-1)}{6} \tag{49}$$

$$\frac{(x_1+x_2+\cdots+x_n)^2+\sum_{i=1}^n x_i^2}{2} = \frac{\frac{p^{2k}(p-1)^2}{4}+\frac{p^k(p-1)(2p-1)}{6}}{2} \tag{50}$$

$$\frac{(x_1+x_2+\cdots+x_n)^2+\sum_{i=1}^n x_i^2}{2} = \frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \tag{51}$$

The Mp-Sequences lie on the surface if;

$$\frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \tag{52}$$

Or the equation (42) will be;

$$F(x) \equiv \sum_{i=1}^n x_i^2 + \sum_{i=j,j>i}^{n-1} x_i x_j = \frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \tag{53}$$

Or;

$$\sum_{i=1}^n x_i^2 + 2\sum_{i=1,j>i}^{n-1} \frac{1}{2} x_i x_j = \frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \tag{54}$$

The equation can be written as;

$$[x_1, x_2, \dots, x_n]\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \tag{55}$$

The Matrix can be determined;

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{bmatrix} \tag{56}$$

For matrix *A* characteristic polynomial is represented as I_n and given by;

$$I_n = \begin{vmatrix} \lambda-1 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & \lambda-1 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & \lambda-1 \end{vmatrix} = 0 \tag{57}$$

Subtracting the last column from the all columns results in

$$I_n = \begin{vmatrix} \lambda-\frac{1}{2} & 0 & 0 & \cdots & -\frac{1}{2} \\ 0 & \lambda-\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}-\lambda & \frac{1}{2}-\lambda & \frac{1}{2}-\lambda & \cdots & \lambda-1 \end{vmatrix} = 0 \tag{58}$$

The determinant is computed according to the first row;

$$I_n = \left(\lambda-\frac{1}{2}\right) I_{n-1} + (-1)^{n+1}\left(-\frac{1}{2}\right)\begin{vmatrix} 0 & \left(\lambda-\frac{1}{2}\right) & 0 & \cdots & 0 \\ 0 & 0 & \left(\lambda-\frac{1}{2}\right) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{1}{2}-\lambda\right) & \left(\frac{1}{2}-\lambda\right) & \left(\frac{1}{2}-\lambda\right) & \cdots & \left(\frac{1}{2}-\lambda\right) \end{vmatrix}_{(n-1)} \tag{59}$$

Computation of the determinant according to the first column;

$$I_n = \left(\lambda - \tfrac{1}{2}\right) I_{n-1} + (-1)^{n+1}\left(-\tfrac{1}{2}\right)(-1)^n\left(\tfrac{1}{2} - \lambda\right)\left(\lambda - \tfrac{1}{2}\right)^{n-2} \tag{60}$$

Thus;

$$I_n = \left(\lambda - \tfrac{1}{2}\right) I_{n-1} + (-1)^{2n+3}\left(\tfrac{1}{2}\right)\left(\lambda - \tfrac{1}{2}\right)^{n-1} \tag{61}$$

$$I_n = \left(\lambda - \tfrac{1}{2}\right)^{n-1}\left[\left(\lambda - \tfrac{1}{2}\right) - n.\tfrac{1}{2}\right] \tag{62}$$

First the rule is to be proved true for $n = 3$ is;

$$I_3 = \begin{vmatrix} (\lambda-1) & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & (\lambda-1) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & (\lambda-1) \end{vmatrix} = \begin{vmatrix} \left(\lambda-\frac{1}{2}\right) & 0 & -\frac{1}{2} \\ 0 & \left(\lambda-\frac{1}{2}\right) & -\frac{1}{2} \\ \left(\frac{1}{2}-\lambda\right) & \left(\frac{1}{2}-\lambda\right) & (\lambda-1) \end{vmatrix}$$

$$I_3 = \left(\lambda-\frac{1}{2}\right)\begin{vmatrix} \left(\lambda-\frac{1}{2}\right) & -\frac{1}{2} \\ \left(\frac{1}{2}-\lambda\right) & (\lambda-1) \end{vmatrix} + (-1)^4\left(-\frac{1}{2}\right)\begin{vmatrix} 0 & \left(\lambda-\frac{1}{2}\right) \\ \left(\frac{1}{2}-\lambda\right) & \left(\frac{1}{2}-\lambda\right) \end{vmatrix}$$

$$I_3 = \left(\lambda-\tfrac{1}{2}\right)\left[\left(\lambda-\tfrac{1}{2}\right)(\lambda-1) - \tfrac{1}{2}\left(\lambda-\tfrac{1}{2}\right)\right] - (-1)^4\left(\tfrac{1}{2}\right)\left(\lambda-\tfrac{1}{2}\right)^2 = \left(\lambda-\tfrac{1}{2}\right)^2\left(\lambda - 1 - \tfrac{1}{2} - \tfrac{1}{2}\right) = \left(\lambda-\tfrac{1}{2}\right)^2\left(\left(\lambda-\tfrac{1}{2}\right) - 3.\tfrac{1}{2}\right) \tag{63}$$

Therefore, the rule is proven true for $n = 3$. Secondly, the rule is true on n-1 and also true for n, which means:

$$I_{n-1} = \left(\lambda - \tfrac{1}{2}\right)^{n-2}\left[\left(\lambda - \tfrac{1}{2}\right) - (n-1).\tfrac{1}{2}\right] \tag{64}$$

It is true and will prove that;

$$I_n = \left(\lambda-\frac{1}{2}\right)^{n-1}\left[\left(\lambda-\frac{1}{2}\right) - n.\frac{1}{2}\right]$$

$$I_n = \left(\lambda-\frac{1}{2}\right) I_{n-1} + (-1)^{2n+3}\left(\frac{1}{2}\right)\left(\lambda-\frac{1}{2}\right)^{n-1}$$

$$I_n = \left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{2}\right)^{n-2}\left[\left(\lambda-\frac{1}{2}\right) - (n-1).\frac{1}{2}\right] + (-1)^{2n+3}\left(\frac{1}{2}\right)\left(\lambda-\frac{1}{2}\right)^{n-1}$$

$$I_n = \left(\lambda-\frac{1}{2}\right)^{n-1}\left[\left(\lambda-\frac{1}{2}\right) - (n-1).\frac{1}{2}\right] + (-1)^{2n+3}\left(\frac{1}{2}\right)\left(\lambda-\frac{1}{2}\right)^{n-1}$$

$$I_n = \left(\lambda-\tfrac{1}{2}\right)^{n-1}\left[\left(\lambda-\tfrac{1}{2}\right) - (n-1).\tfrac{1}{2} - \tfrac{1}{2}\right] \tag{65}$$

Thus;

$$I_n = \left(\lambda-\tfrac{1}{2}\right)^{n-1}\left[\left(\lambda-\tfrac{1}{2}\right) - n.\tfrac{1}{2}\right] \tag{66}$$

Or;

$$I_n = \left(\lambda-\tfrac{1}{2}\right)^{n-1}\left[\lambda - (n+1)\tfrac{1}{2}\right] \tag{67}$$

The characteristic equation becomes;

$$\left(\lambda-\tfrac{1}{2}\right)^{n-1}\left[\lambda - (n+1)\tfrac{1}{2}\right] = 0 \tag{68}$$

The characteristic values are;

$$\lambda_i = \tfrac{1}{2}, i = 1,2,\ldots,(n-1) and \lambda_n = (n+1)\tfrac{1}{2} \tag{69}$$

The corresponding characteristic vectors were determined as;

$For \lambda_i = \frac{1}{2}$:

$$\begin{cases} -\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \cdots + (-\frac{1}{2}x_n) = 0 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \cdots + (-\frac{1}{2}x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \cdots + (-\frac{1}{2}x_n) = 0 \end{cases}$$

$$\Rightarrow x_1 + x_2 + x_3 + \cdots + x_n = 0 \tag{70}$$

And the characteristic vectors include;

$$\begin{cases} \vec{V}_1 = (1,-1,0,\dots,0)_n \\ \vec{V}_2 = (1,0,-1,\dots,0)_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \vec{V}_{n-1} = (1,0,0,\dots,-1)_n \end{cases} \tag{71}$$

And the vectors are not orthogonal.
For $\lambda_n = (n+1)\frac{1}{2}$;

$$\begin{cases} \frac{n-1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \cdots + (-\frac{1}{2}x_n) = 0 \\ -\frac{1}{2}x_1 + \frac{n-1}{2}x_2 - \frac{1}{2}x_3 + \cdots + (-\frac{1}{2}x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \cdots + \frac{n-1}{2}x_n = 0 \end{cases} \tag{72}$$

There is only one solution that is; $x_1 = x_2 = \cdots = x_n$. While, the characteristic vector is; $\vec{V}_n = (1,1,\dots,1)_n$. The vectors: $\{\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n\}$ are linearly independent because their determinant not equal to zero;

$$\Delta = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}_n \tag{73}$$

By adding all columns to the first column;

$$\Delta = \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -1 \\ n & 1 & 1 & 1 & 1 \end{vmatrix}_n = (-1)^{n+1}n.(-1)^{n-1} \neq 0 \tag{74}$$

The vectors: $\{\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n\}$ are linearly independent. According to the new coordinates $X_1X_2 \dots X_n$ corresponding of the potential vectors $\{\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n\}$ with the vectors unit $\{\vec{I}_1, \vec{I}_2, \dots, \vec{I}_n\}$;

$$\begin{cases} \vec{I}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0\right)_n \\ \vec{I}_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, \dots, 0\right)_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \vec{I}_{n-1} = \left(\frac{1}{\sqrt{2}}, 0, 0, \dots, -\frac{1}{\sqrt{2}}\right)_n \\ \vec{I}_n = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)_n \end{cases} \tag{75}$$

While,

$$\overrightarrow{oM} = x_1\vec{\imath}_1 + x_2\vec{\imath}_2 + \cdots + x_n\vec{\imath}_n = X_1\vec{I}_1 + X_2\vec{I}_2 + \cdots + X_n\vec{I}_n \tag{76}$$

Thus;

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2 + \frac{1}{\sqrt{2}}X_3 + \cdots + \frac{1}{\sqrt{2}}X_{n-1} + \frac{1}{\sqrt{n}}X_n \\ x_2 = -\frac{1}{\sqrt{2}}X_1 + 0 + 0 + \cdots + 0 + \frac{1}{\sqrt{n}}X_n \\ x_3 = 0 - \frac{1}{\sqrt{2}}X_2 + 0 + \cdots + 0 + \frac{1}{\sqrt{n}}X_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n = 0 + 0 + \cdots - \frac{1}{\sqrt{2}}X_{n-1} + \frac{1}{\sqrt{n}}X_n \end{cases} \xRightarrow[Addition]{After}$$

$$(x_1 + \cdots + x_n) = \sqrt{n}X_n \; or \; X_n = \frac{p^k(p-1)}{2\sqrt{n}} \tag{77}$$

And;

$$\left(\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{2}}X_i\right) + \frac{1}{\sqrt{n}}X_n\right)^3 + \left(-\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{n}}X_n\right)^3 + \cdots + \left(-\frac{1}{\sqrt{2}}X_n + \frac{1}{\sqrt{n}}X_n\right)^3 =$$

$$= x_1^3 + \cdots + x_n^3 \tag{78}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{n}} \\ -\frac{1}{\sqrt{2}} & 0 & \cdots & \frac{1}{\sqrt{n}} \\ 0 & -\frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{n}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \tag{79}$$

And the equation (54 or 55) using theorem 6, becomes;

$$\frac{1}{2}X_1^2+\frac{1}{2}X_2^2+\cdots+\frac{1}{2}X_{n-1}^2+\frac{(n+1)}{2}X_n^2=\frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{24} \quad (80)$$

Or;

$$X_1^2+X_2^2+\cdots+X_{n-1}^2+(n+1)X_n^2=\frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{12} \quad (81)$$

The sphere with the center is considered $o(0,0,\dots,0)_n$ and the radius *r;*

$$x_1^2+x_2^2+\cdots+x_n^2=r^2 \ or \ \sum_{i=1}^{n}x_i^2=r^2 \quad (82)$$

The M_p-Sequences of the period p^k-1 lie on the surface of the sphere (The sphere has the same center for both the coordinates) but only in the following case;

$$r^2=p^{(k-1)}\left[\frac{(p-1)p(2p-1)}{6}\right]=\frac{p^k(p-1)(2p-1)}{6} \quad (83)$$

Or;

$$x_1^2+x_2^2+\cdots+x_n^2=\frac{p^k(p-1)(2p-1)}{6} \quad (84)$$

Or;

$$X_1^2+X_2^2+\cdots+X_n^2=\frac{p^k(p-1)(2p-1)}{6} \quad (85)$$

We can get the same result from equation (81) after subtraction $nX_n^2=n\frac{p^{2k}(p-1)^2}{4n}$.

The intersection of (81) and (85) gives;

$$nX_n^2=\frac{p^k(p-1)[3.p^k(p-1)+2(2p-1)]}{12}-\frac{p^k(p-1)(2p-1)}{6} \quad (86)$$

Or;

$$nX_n^2=\frac{p^{2k}(p-1)^2}{4} \quad (87)$$

Or;

$$X_n=\mp\frac{p^k(p-1)}{2\sqrt{n}}; \text{(the – Of Quadrature)} \quad (88)$$

Also the M$_p$-Sequences of the period p^k-1 lie on the plane;

$$x_1+x_2+\cdots+x_n=p^{k-1}\big(1+2+\cdots+(p-1)\big) \quad (89)$$

Or;

$$x_1+x+\cdots+x_n=\frac{p^k(p-1)}{2} \quad (90)$$

The intersection of (53) and (84) is the plan (90), and the intersection of (81) and (85) is the plan (88). And

$$x_1^3+x_2^3+\cdots+x_n^3=p^{k-1}(1^3+2^3+\cdots+(p-1)^3)$$

$$x_1^3+x_2^3+\cdots+x_n^3=p^{k-1}(1^3+2^3+\cdots+(p-1)^3)$$

$$=p^{k-1}\frac{[p(p-1)]^2}{4}$$

$$=p^{k+1}\frac{(p-1)^2}{4} \quad (91)$$

Thus:

$$\left(\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{2}}X_i\right)+\frac{1}{\sqrt{n}}X_n\right)^3+\left(-\frac{1}{\sqrt{2}}X_1+\frac{1}{\sqrt{n}}X_n\right)^3+\cdots+\left(-\frac{1}{\sqrt{2}}X_n+\frac{1}{\sqrt{n}}X_n\right)^3=p^{k+1}\frac{(p-1)^2}{4} \quad (92)$$

3.4. The Regular Surfaces for Mp-Sequences

In the original coordinate $ox_1x_2\dots x_n$, the system of equations of the surfaces is;

$$\{x_1+x_2+\cdots+x_n=\frac{p^k(p-1)}{2} \quad (93)$$

The equations of the system is;

$$\begin{cases}\sum_{i=1}^{n}x_i^2+\sum_{i=1,j>i}^{n-1}x_ix_j=\frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24}\\ x_1^2+x_2^2+\cdots+x_n^2=\frac{p^k(p-1)(2p-1)}{6}\\ x_1^3+x_2^3+\cdots+x_n^3=\frac{p^{k+1}(p-1)^2}{4}\end{cases} \quad (94)$$

These are pending with the equation (93) and cannot give any new information about other variables $x_{k+1},\dots,x_n$. The first equation (93) can define the variables by using the equations of recurring linear sequence. The equations (93) and (94) become for the new coordinate $oX_1X_2\dots X_n$ as;

$$\begin{cases} X_n = \dfrac{p^k(p-1)}{2\sqrt{n}} \end{cases}$$

$$\begin{cases} X_1^2 + X_2^2 + \cdots + (n+1)X_n^2 = \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{12} \\ X_1^2 + X_2^2 + \cdots + X_n^2 = \frac{p^k(p-1)(2p-1)}{6} \\ \left(\left(\sum_{i=1}^{n-1} \frac{1}{\sqrt{2}} X_i\right) + \frac{1}{\sqrt{n}} X_n\right)^3 + \left(-\frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{n}} X_n\right)^3 + \cdots + \left(-\frac{1}{\sqrt{2}} X_n + \frac{1}{\sqrt{n}} X_n\right)^3 = \frac{p^{k+1}(p-1)^2}{4} \end{cases} \tag{95}$$

The system (94) and system (95) are pending in the equation (93). For the sequence $x_1, x_2, x_3, \dots, x_n$ over F_p with the period $n = p^k - 1$ the first $x_1, x_2, \dots, x_k$ (the initial position) are given and if it can find the other $x_{k+1} to\ x_{\frac{n}{2}}$, then finding the other $x_{\frac{n}{2}+1}\ to\ x_n$ is very easy from the relations;

$$\begin{cases} if\ x_i = 0\ then\ x_{\frac{n}{2}+\mathrm{i}} = 0, i = 1,2, \dots, \frac{n}{2} \\ if\ x_i \neq 0\ then\ x_{\frac{n}{2}+i} = p - x_i, \mathrm{i} = 1,2, \dots, \frac{n}{2} \end{cases} \tag{96}$$

Thus it needs other $\frac{n}{2} - 1 - k$ independent equations (surfaces) for all one period of the sequence from the Linear Recurring Sequence;

$$a_{n+k} + \sum_{i=1}^{k-1} \lambda_i a_{n+i} = 0, \lambda_i \in F_p = \{0,1,2, \dots, (p-1)\}, p\ is\ prime \tag{97}$$

The solution of this systems calculated by mod p.

Example 4: In the space $R^8 (for\ n = 8)$ with the coordinate $ox_1x_2 \dots x_n$, the M$_3$-Sequences (over F_3) are generated by the recurring equations;

$$a_{n+2} + a_{n+1} + 2a_n = 0\ or\ a_{n+2} = 2a_{n+1} + a_n \tag{98}$$

With the characteristic equation $x^2 + x + 2 = 0$ and $f(x) = x^2 + x + 2$, that represents a prime characteristic polynomial generates the field F_{3^2} and the periodic sequence comprising of the period $3^2 - 1 = 8$. For the initial position: $a_1 = 1, a_2 = 0$,; therefore, $S_1 = (1\ 0\ 1\ 2\ 2\ 0\ 2\ 1)$ and by the cyclic permutations on S$_1$

$\$ = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$; where,

$$S_2 = (1\ 1\ 0\ 1\ 2\ 2\ 0\ 2);\ S_3 = (2\ 1\ 1\ 0\ 1\ 2\ 2\ 0);\ S_4 = (0\ 2\ 1\ 1\ 0\ 1\ 2\ 2);\ S_5 = (2\ 0\ 2\ 1\ 1\ 0\ 1\ 2);$$
$$S_6 = (2\ 2\ 0\ 2\ 1\ 1\ 0\ 1);\ S_7 = (1\ 2\ 2\ 0\ 2\ 1\ 1\ 0);\ S_8 = (0\ 1\ 2\ 2\ 0\ 2\ 1\ 1) \tag{99}$$

It can be seen that the sequence satisfies the equations of systems (94) and (95) For $S_1 = (1\ 0\ 1\ 2\ 2\ 0\ 2\ 1)$, and further for initial position: $a_1 = 1 = x_1, a_2 = 0 = x_2$ and system (94) it can find all other $x_i, i = 3, \dots, 8$ as following;

From Corollary 6. (2.); If $x_i = 0$ then $x_{i+4} = 0$ and if $x_i \neq 0$ then.

$x_{i+4} = 3 - x_i$, thus: $x_5 = 3 - x_1 = 2$ and $x_6 = x_2 = 0, x_7 = 3 - x_3$, and

$x_8 = 3 - x_4$, and the first equation becomes:

$$x_8 + x_4 = 3 \tag{100}$$

From the recurring equations;

$$a_{n+2} + a_{n+1} + 2a_n = 0\ or\ a_{n+2} = 2a_{n+1} + a_n \tag{101}$$

When the calculation by mod 3 is;

$$x_3 = 2x_2 + x_1 = 1\ then\ x_7 = 3 - 1 = 2\ \&\ x_4 = 2a_3 + a_2 = 2 \tag{102}$$

From $x_4 = 2$ give as $x_8 = 3 - 2 = 1$

The equation (93) and system (94) became;

$$\begin{cases} \sum_{i=1}^{8} x_i = 9 \\ \sum_{i=1}^{8} x_i^2 + \sum_{i=1, j>i}^{8-1} x_i x_j = 48 \\ x_1^2 + x_2^2 + \cdots + x_8^2 = 15 \\ x_1^3 + x_2^3 + \cdots + x_8^3 = 27 \end{cases} \tag{103}$$

Example 5: In the space R^{24} (for *n* = 24) with the coordination of $ox_1x_2 \dots x_n$, the M$_5$-Sequences (over F_5) are generated by the recurring equations;

$$a_{n+2} + a_{n+1} + 2a_n = 0\ or\ a_{n+2} = 4a_{n+1} + 3a_n \tag{104}$$

With the characteristic equation $x^2 + x + 2 = 0$ and the prime characteristic polynomial $g(x) = x^2 + x + 2$, it generates F_{5^2} with the initial position (0 1) which results in sequence; 014434023313041121032242 014... With the period $5^2 - 1 = 24$, the sequence is satisfied by;

$$\begin{cases} \sum_{i=1}^{24} x_i = 50 \\ \sum_{i=1}^{24} x_i^2 + \sum_{i=1,j>1}^{24-1} x_{ni}x_j = 1325 \\ x_1^2 + x_2^2 + \cdots + x_{24}^2 = 150 \\ x_1^3 + x_2^3 + \cdots + x_{24}^3 = 500 \end{cases} \quad (105)$$

The regular quadratic Surfaces for M_2-Sequences

In F_2, the equation (93) and system (94) becomes;

$$\{x_1 + x_2 + \cdots + x_n = 2^{k-1} \quad (106)$$

$$\begin{cases} \sum_{i=1}^{n} x_i^2 + \sum_{i=1,j>i}^{n-1} x_i x_j = \frac{2^k[2^k+2]}{8} \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 2^{k-1} \\ x_1^3 + x_2^3 + \cdots + x_n^3 = 2^{k-1} \end{cases} \quad (107)$$

The equations of systems (105) and (107) are pending in the first equation and for the sequence $x_1, x_2, x_3,\ldots, x_n$ over F_2 with the period $2^k - 1$. While, the first $x_1, x_2, x_3,\ldots, x_k$ (the initial position) are given and it can find the other x_{k+1} to x_n. Therefore, other $n-1-k$ independent equation (surface) is needed to find all one period of the sequence and it can get from the Linear Recurring Sequence;

$$a_{n+k} + \sum_{i=1}^{k-1} \lambda_i a_{n+i} = 0, \lambda_i \in F_2 = \{0,1\} \quad (108)$$

Example 6: Suppose the Sequence of Linear Recurring is;

$$a_{n+3} = a_{n+1} + a_n \text{ or } a_{n+3} + a_{n+1} + a_n = 0 \quad (109)$$

With the characteristic equation $x^3 + x + 1 = 0$ and the polynomial characteristic $f(x) = x^3 + x + 1$ represent a prime and generate the field F_{2^3}. This sequence represents a periodic within the period $2^3 - 1 = 7$. Initial position: $a_1 = 1, a_2 = 0,\ a_3 = 0$, then $S_1 = (1\ 0\ 0\ 1\ 0\ 1\ 1)$ and when it consider the cyclic per mutations on S_1, it has $\$ = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$. Where;

$$S_2 = (1\ 1\ 0\ 0\ 1\ 0\ 1), S_3 = (1\ 1\ 1\ 0\ 0\ 1\ 0), S_4 = (0\ 1\ 1\ 1\ 0\ 0\ 1)$$
$$S_5 = (1\ 0\ 1\ 1\ 1\ 0\ 0), S_6 = (0\ 1\ 0\ 1\ 1\ 1\ 0), S_7 = (0\ 0\ 1\ 0\ 1\ 1\ 1) \quad (110)$$

There are first three digits that represent the initial position for each of these sequences and generates these sequences for the feedback register. The sequences satisfies the equations (94) and (95) for $p = 2$ and $k = 3$, and further for the first sequence $S_1 = (1\ 0\ 0\ 1\ 0\ 1\ 1)$;

$$\begin{cases} \sum_{i=1}^{7} x_i = 4 \\ \sum_{i-1}^{7} x_i^2 + \sum_{i=1,j>i}^{6} x_i x_j = 10 \\ x_1^2 + x_2^2 + \cdots + x_7^2 = 4 \\ x_1^3 + x_2^3 + \cdots + x_7^3 = 4 \end{cases} \quad (111)$$

New system can be developed from the first equation in latest system and from the linear recurring sequence;

$$\begin{cases} x_4 + x_5 + x_6 + x_7 = 3 \\ x_4 + x_2 + x_1 = 0 \\ x_5 + x_3 + x_2 = 0 \\ x_6 + x_4 + x_3 = 0 \end{cases} \quad (112)$$

The solution of this system (when the calculation by mod 2) is;

$$x_4 = 1, x_5 = 0, x_6 = 1, x_7 = 1 \quad (113)$$

4. Conclusion

a) The above theorems have concluded that the finding of the surfaces (plains) that contains Mp- Sequences is;

$$\sum_{i=1}^{n} x_i = \frac{p^k(p-1)}{2} \quad (114)$$

b) The formula of finding the (Ellipsoids 2., Spheres 3., Others 4.) the surfaces contains the Mp-Sequences that are shown below;

$$\begin{cases} \sum_{i=1}^{n} x_i^2 + \sum_{i=1,j>i}^{n-1} x_i x_j = \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{24} \\ x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{p^k(p-1)(2p-1)}{6} \\ x_1^3 + x_2^3 + \cdots + x_n^3 = \frac{p^{k+1}(p-1)^2}{4} \end{cases} \quad (115)$$

a) The finding of the surface (plain) for the symmetric axes $\vec{V}_n$, which contains Mp-Sequences is;

$$X_n = \frac{p^k(p-1)}{2\sqrt{n}} \quad (116)$$

b) The finding of the (Ellipsoids 2., Spheres 3., others 4.) surfaces for the symmetric axes containing Mp-Sequences;

$$\begin{cases} X_1^2 + X_2^2 + \cdots + (n+1)X_n^2 = \frac{p^k(p-1)[3p^k(p-1)+2(2p-1)]}{12} \\ X_1^2 + X_2^2 + \cdots + X_n^2 = \frac{p^k(p-1)(2p-1)}{6} \\ \left(\left(\sum_{i=1}^{n-1} \frac{1}{\sqrt{2}} X_i\right) + \frac{1}{\sqrt{n}} X_n\right)^3 + \left(-\frac{1}{\sqrt{2}} X_1 + \frac{1}{\sqrt{n}} X_n\right)^3 + \cdots + \left(-\frac{1}{\sqrt{2}} X_n + \frac{1}{\sqrt{n}} X_n\right)^3 = \frac{p^{k+1}(p-1)^2}{4} \end{cases} \quad (117)$$

c) The equations are satisfied by the given examples of sequences on F_2, F_3 and F_5.

Limitation: This method of compose sequences is useful for only binary sequences and the addition on the sequences computed by "*mod* 2 " also used Microsoft Word 2010 and the Microsoft equation 3.0 for written the math equations.

The method for reading a page which has a block will be according to the following direction as in figure1.

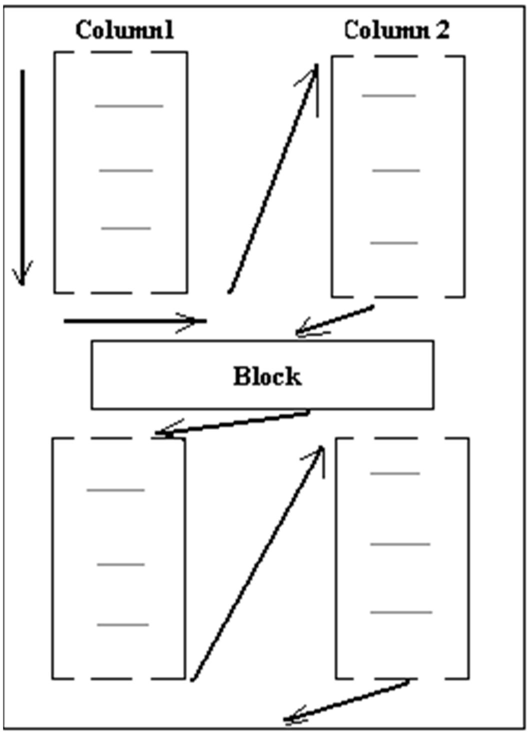

Figure 1. *Method reading page with block.*

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