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Topological Methods in Geometric Foundations of Teleparallel Fused Quantum Gravity

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Abstract: We have studied the gravitation in the context of the noncommutative manifold $M_4 \times Z_2$ where Z_2 is not the two point space but corresponds to a direction-vector attached to a space-time point. A local field theory, noncommutative Yang - Mills fields is limited to obtain the return thatsuch a symmetry group differences are. Noncommutative gauge symmetry of space - time and theinternal symmetry of the mixer is a very natural and clear perception that the gravitational forcegauge a characteristic feature of the theory. The gauge fields of the dimensionally reduced noncommutativeYang-Mills theory map onto a Weitzenböck spacetime and a teleparallel theory of gravity arisesas the zero curvature reduction of a Poincare gauge theory which induces an Einstein-Cartanspace-time characterized by connections with both nonvanishing torsion and curvature. This analysissuggests that noncommutative Yang-Mills theory naturally induces gravitation through a torsioned space-time. Thus as in the case of a noncommutative manifold whereZ_2 is a two-point space there appears to be aconnection between gravity and electroweak theory in this formalism this is achieved through therealization of chiral anomaly and torsion. It is noted to be that that Weitzenböck geometry thatEinstein's General Relativity with the teleparallel gravity equivalent, provoked by her some notablefeatures are. This show has been that the geometry torsioned space-time at which the the chiralanomaly inconsistencies in the torch made is. This show is that it naturally Weitzenböck geometryof the moves that the gravity of a teleparallel formula birth towards.

Keywords: Field Theory, Einstein Gravity, Four Dimensional Manifold, Gauge Symmetries

1. Introduction

In recent times general relativity in noncommutative geometry has been studied by several authors [1, 2]. Chamseddine, Felder and Fröhlich [1] considered gravity in a space-time which isthe product of a four dimensional manifold by a two-point space. It has been found that whenthe Riemannian metric is the same on the two copies of the manifold, one obtains a modelof scalar field coupled to Einstein gravity. The scalar field is geometrically represented as describing the distance between the two points in the internal space. Langmann and Szabo [2] considered noncommutative gauge theory in flat space and showed that dimensional reductions lead to Weitzenböck geometry on space-time and the induced diffeomorphism invariant fieldtheory can be made equivalent to a teleparallel formulation of gravity which macroscopicallydescribes general relativity. These authors have elegantly shown that noncommutative U(1) Yang-Mills theory on flat space $R^n \times R^n$ can generate a theory of gravitation on R^n . The basicobservation is that the algebra of functions on R^{2n} with Lie bracket defined in terms of thedeformed product of the noncommutative theory contains the Lie algebra of vector fields on R^n . A local field theory can be obtained by restricting noncommutative Yang-Mills fields such thatthe symmetry group contains diffeomorphism invariance. Noncommutative gauge symmetriesgive a very natural and explicit realizations of the mixing of space-time and internal symmetrieswhich is a characteristic featureof the conventionalgaugetheoryofgravity. Thegaugefieldsofthedimensionallyreducednoncommutative Yang-Mills theory map onto a Weitzenböck spacetime and a teleparalleltheory of gravity arises as the zero curvature reduction of a Poincaregauge theory which induces an Einstein-Cartan space-time characterized by connections withboth nonvanishing torsion and curvature. This analysis suggests that noncommutative Yang-Mills theory naturally induces gravitation through a torsioned space-time.

In section 2, we shall study gravity in the space-time manifold $M_4 \times Z_2$ where Z_2 is a discrete space described by a 'direction-vector' and show how this leads to torsion. In section 3 we shallconsider chiral anomaly, torsion and topological invariants.

2. Noncommutative Space $M_4 \times Z_2$ and Torsion

In an earlier paper [3, 4] it has been pointed out that noncommutative geometry having the space-time manifold leads to the quantization of a fermion when the discrete space-timeis incorporated as an internal variable. Indeed this leads to the introduction of an anisotropicfeature in the internal space so that we can consider the space-time coordinate in complex-time as $Z_\mu = x_\mu + i\zeta_\mu$ where ζ_μ represents a 'direction vector' attached to the space-time point x_μ. The two orientations of the 'direction vector' give rise to two internal helicitiescorresponding to fermion and antifermion. The complex space-time exhibiting the internalhelicity states can be written in terms of a two-component spinorial variable $\theta(\bar{\theta})$ when the' direction vector' ζ_μ is associated with θ through the relation $\zeta_\mu = \frac{1}{2}\lambda_\mu^\alpha \theta_\alpha (\alpha = 1,2)$. This helpsus to write the relevant metric as $g_{\mu\nu}(x,\theta,\bar{\theta})$. This metric gives rise to the $SL(2,C)$ gaugetheory of gravity [5] when the gauge fields A_μ are matrix-valued having the $SL(2,C)$ group structure and the curvature is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{1}$$

It is noted that in this framework, the noncommutative space $M_4 \times Z_2$ appears as if the space-time point q_μ in Minkowski space is extended by a gauge field A_μ which is matrix-valued andbelongs to the group structure $SL(2,C)$. Indeed we can identify the position and momentumoperators as

$$\frac{Q_\mu}{\omega} = -i(\frac{\partial}{\partial p_\mu} + A_\mu)$$

$$\frac{P_\mu}{\omega} = -i(\frac{\partial}{\partial q_\mu} + A_\mu) \tag{2}$$

where ω is the dimensionless variable, $\omega = \frac{\hbar}{lmc}$. Here q_μ is the space-time point in Minkowski space M_4 and p_μ its conjugate representing the momentum variable. To study the effect of this geometry in gravitation following Carmeli and Malin [6] we choose the simplest Lagrangian density in spinor affine space

$$L = \frac{1}{4} Tr(\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}) \tag{3}$$

Now writing

$$F_{\mu\nu} = \vec{f}_{\mu\nu} . \vec{g} \ , \ F_{\mu\nu} = \vec{f}_{\mu\nu} . \vec{g}$$

Where $\vec{g} = (g_1, g_2, g_3)$ are the generators of the $SL(2,C)$ group in tangent space given by

$$g_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we can define the current density

$$\vec{J}_\theta^\mu = \varepsilon^{\mu\nu\alpha\beta} \vec{a}_\nu \times \vec{f}_{\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta} \partial_\nu \vec{f}_{\alpha\beta} \tag{4}$$

satisfying the relation

$$\partial_\mu \vec{J}_\theta^\mu = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu \partial_\nu \vec{f}_{\alpha\beta} = 0 \tag{5}$$

When the space is Riemannian with metric structure, the conserved current may be written as

$$\vec{J}^\mu = \vec{J}^\mu + \frac{1}{\chi} \varepsilon^{\mu\nu\alpha\beta} \vec{a}_\nu \times \vec{f}_{\alpha\beta} \tag{6}$$

where $\chi = -\frac{8\pi G}{c^4}$ [5]. $\vec{J}^\mu$ is the contribution to the conserved current due to the energy momentum tensor, $\vec{J}^\mu = \tilde{J}^\mu . \vec{n}$, $\vec{n}$ being a unit vector. The second part of the right hand sideof eqn. (6) is the contribution of the spinorial variable θ which is associated with the directionvector. We can now write the action

$$S = S_1 + S_2 - A\int \vec{J}_\mu \vec{J}_\mu d^4x \tag{7}$$

where $A = \frac{1}{k^2} = \frac{1}{16\pi G}$; k =Planck length. Using the relations [6]

$$R_{\alpha\beta\gamma} = R_{\alpha\beta\gamma} V^\delta \tag{8}$$

$$\text{Where } R_{\alpha\beta\gamma} = R_{\alpha\beta\gamma} V^\delta \tag{9}$$

V^δ being an arbitrary vector and taking

$$R_{\alpha\beta\gamma\delta} = R^{(0)}{}_{\alpha\beta\gamma\delta} V^\delta - e_{\gamma\alpha} e_{\delta\beta} \tag{10}$$

With

$$R^{(0)}{}_{\alpha\beta\gamma\delta} = -\partial_\alpha \omega_{\beta\gamma\delta} + \partial_\beta \omega_{\alpha\gamma\delta} + \partial^a_{\alpha\gamma} \omega_{\beta a\delta} - \partial^a_{\alpha\gamma} \omega_{\alpha a\delta} \tag{11}$$

denoting Riemannian curvature related to rotation whereas the second term $e_{\gamma\alpha} e_{\delta\beta}$ in equation (10) corresponds to translation, we have

$$S_1 = \frac{1}{k^2} \int \vec{\tilde{J}}_\mu \vec{\tilde{J}}_\mu d^4x = -\frac{1}{k^2} \int \mathrm{Re}\, d^4x \tag{12}$$

where R is the scalar curvature and e is given by the relation

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\lambda} = -e \quad (13)$$

Againwriting

$$\vec{a}_\nu \times \vec{f}_{\alpha\beta} = k^2 S_{\nu\alpha\beta}\vec{n} \quad (14)$$

where $\vec{n}$ is unit vector, the second part of the action becomes

$$S_2 = -\frac{4}{k^2}\int S_{\nu\alpha\beta}S^{\nu\alpha\beta}d^4x \quad (15)$$

giving rise to the torsion term. Thus we find that the topological Lagrangian given by equation (3) which is associated with the 'direction-vector' represented in spinorial variable leads to torsion. Indeed the gauge theoretical extension of a space-time point in the noncommutative manifold $M_4 \times Z_2$ may be taken to be responsible for torsion and eventually helps us to realize the Einstein-Cartan space-time.

3. Chiral Anomaly, Torsion and Topological Invariants in Noncommutative Manifold $M_4 \times Z_2$

We may observe here that the gauge-theoretical extension of the space-time point xμ in thenoncommutative manifold $M_4 \times Z_2$ as discussed above will lead to the deformation of symplecticstructure. Indeed from equation (2) we note that the symplectic form of the phase space willnow be given by

$$\Omega = \frac{1}{2}g^{ij}dp_i\Lambda dq_j \quad (16)$$

With

$$g^{ij} = j^{ij} + \hbar\Delta^{ij} \quad (17)$$

Where

$$j^{ij} = \begin{bmatrix} 0 & -I \\ I & o \end{bmatrix} \quad (18)$$

is associated with the usual symplectic structure and Δ^{ij} is the curvature tensor comingfrom the induced vector potential A_μ. When we associate a gauge field A_i with the phase space variable p_i and q_i, the phase space variable ϕ^a will be modified $\phi^a + i\hbar A^a(\phi)$ as where $\phi^a = (p^1, p^2, p^3, q^1, q^2, q^3)$, $a = 1,...,6$ are canonical coordinates in $\Re^6$. Thus for the functions $f(\phi^a + i\hbar A^a)$ and $g(\phi^a + i\hbar A^a)$ we can consider the product upto the order of $\hbar$ and suppressing the index a over ϕ.

$$f(\phi^a + i\hbar A^a)g(\phi^a + i\hbar A^a) = [f(\phi + i\hbar A^a\partial_a f(\phi)][g(\phi + i\hbar A^a\partial_a g(\phi)$$

$$= f(\phi)g(\phi) + i\hbar A^a[(\partial_a f(\phi))g(\phi) + \partial_a g(\phi))f(\phi)] + o(\hbar^2) \quad (19)$$

Now since A^a is taken to be a gauge field to extract a physically meaningful result for theexpression of the product given by eqn. (19) we express it in terms of the gauge invariant quantity F_{ab}, the field strength defined by $F_{ab} = \partial_a A_b - \partial_b A_a = \varepsilon_{abc}\mathrm{B}^c$ where B^c represents the magnetic field. Indeed writing the extended coordinates as operators we substitute the expression for $\phi^a + iA^a$ by $\phi^a + iF^{ab}(\phi)\partial_b$, where $\partial_b \equiv \frac{\partial}{\partial\phi^b}$ Now neglecting the second order derivative terms and higher order terms $O(k^2)$, the expression (19) can be written as

$$f(\phi)g(\phi) + \frac{i}{2}F_{ab}\partial_a f(\phi)\partial_b g(\phi) \quad (20)$$

The curvature F_{ab} will deform the symplectic structure. It may be remarked here that when the curvature F_{ab} is trivial such that $\partial_a F_{ab} = 0$, F_{ab} becomes a constant matrix and by normalization we can relate F_{ab} with the matrix

$$j^{ij} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

in eqn. (18) changing the indices a, b to i, j associated with the usual symplectic structure. In that case we note that the product given by eqn. (20) is just the star product

$$(f * g)(\phi) + f(\phi)g(\phi) + \frac{i\hbar}{2}\partial_a f(\phi)\partial_b g(\phi) + O(\hbar^2) \quad (21)$$

With F_{ab} being the usual symplectic matrix. Comparing this with eqn. (20) we note that the product deformation may be viewed as to be caused by the induced background magnetic field. It is observed that when F_{ab} is nontrivial we will have the deformation of the symplectic structure so that the noncommutativity parameter becomes a function of φ (φ=p, q). This deformation of the symplectic structure gives rise to the Berry phase which is related to chiral anomaly [7, 8].

In the background of the $SL(2,C)$ gauge fields, the Lagrangian for a Dirac spinor field maybe written (neglecting the mass term)

$$L = -\bar{\psi}\gamma^\mu D_\mu\psi - \frac{1}{4}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad (22)$$

where D_μ is the $SL(2,C)$ gauge covariant derivative given by $D_\mu \equiv \partial_\mu - igA_\mu$ whereg is some coupling constant and $A_\mu \in SL(2,C)$. It has been shown in an earlier paper [9] that if we split the Dirac massless spinor into chiral forms and identify the internal helicity as determined by the direction vector with left (right) chirality, we have the standard conservation laws.

$$\partial_\mu[\frac{1}{2}(-ig\overline{\psi}_R\gamma_\mu\psi_R)+j_\mu^1]=0$$

$$\partial_\mu[\frac{1}{2}(-ig\overline{\psi}_L\gamma_\mu\psi_L+ig\overline{\psi}_R\gamma_\mu\psi_R)+j_\mu^2]=0$$

$$\partial_\mu[\frac{1}{2}(-ig\overline{\psi}_L\gamma_\mu\psi_L)+j_\mu^2]=0 \quad (23)$$

Where $\vec{j}_\mu$ is given by equation (4). These three equations represent a consistent set of equationsif we choose

$$j_\mu^1=-\frac{1}{2}j_\mu^2,\ j_\mu^3=+\frac{1}{2}j_\mu^2 \quad (24)$$

which evidently gurantees the vector current conservation. Then we can write

$$\partial_\mu(\overline{\psi}_R\gamma_\mu\psi_R+j_\mu^2)=0$$

$$\partial_\mu(\overline{\psi}_L\gamma_\mu\psi_L-j_\mu^2)=0 \quad (25)$$

$$\partial_\mu(\overline{\psi}\gamma_\mu\gamma_5\psi)=\partial_\mu J_\mu^5=-2\partial_\mu j_\mu^2 \quad (26)$$

Thus the chiral anomaly is expressed here in terms of the second $SL(2,C)$ component of the gauge field current j_μ^2.

Now from eqn. (4) and (14) in the previous section, we note that the current j_μ^2 associated with the chiral anomaly gives rise to torsion.

It may be mentioned here that when a vector ξ_μ is attached to a space-time point x_μ, the space-time manifold can be taken to correspond to the de Sitter space $M^{4,1}$. Indeed when we consider ξ_μ as an internal variable the wave function of this extended particle may be written as $\phi(x_\mu,\xi_\mu)$. This suggests that with $\xi=0$ we can relate $\gamma=\sqrt{\xi_\mu^2}$ with the radius of thisextension.

If we consider x_μ as the centre of mass coordinate and γ as its extension, this leads to the extension of the Lorentz group $SO(3,1)$ to the de Sitter group $SO(4,1)$. When the attached vector ξ_μ appears as the 'direction vector' the space-time manifold corresponds to thenoncommutative space $M_4\times Z_2$ which has the fermionic field as the underlying field. In thecase of a massive spinor we can define a plane D^- where for the coordinate $z_\mu=x_\mu+i\xi_\mu$, ξ_μ belongs to the interior of the forward light cone $\xi\rangle\rangle 0$ and represents the upper half plane. The lower half plane D^+ is given by the set of coordinates $z_\mu=x_\mu-i\xi_\mu$ with ξ_μ in the interior of the backward light cone $\xi\langle\langle 0$. The map $z\to z^*$ sends the upper half plane to the lowerhalf plane. The space M of the null plane ($\xi_\mu^2=0$) is the Shilov boundary so that a function holomorphic in $D^-(D^+)$ is determined by the boundary value. If we consider that any function holomorphic in $D^-(D^+)$ the helicity $+\frac{1}{2}(-\frac{1}{2})$ in the null plane may be taken to be the limitingvalue of the internal helicity in the upper (lower) half plane. In the manifold $M_4\times Z_2$ the inherent anisotropy in the discrete space suggests that we should consider the space or timereversal noninvariant current associated with the field strength given by

$$j^\mu=\varepsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} \quad (27)$$

From eqn. (4) we find that this is associated with the axial vector current as given by relation (26). Again from the relation (14) we note that this is associated with torsion.

It may be observed that for the $SL(2,C)$ gauge theoretical extension corresponding to thisnoncommutative space we can take for the Hermitian representation the group manifold corresponding to the gauge field $A_\mu=SU(\in)$. Taking ∇_μ as the covariant derivative $\nabla_\mu=\partial_\mu-iA_\mu$ we have the commutator

$$(\nabla_\mu,\nabla_\nu)=\Gamma_{\mu\nu}^\lambda\nabla_\lambda \quad (28)$$

where $\Gamma_{\mu\nu}^\lambda$ is the structure constant. Noting that the commutator $[\nabla_\mu,\nabla_\nu]$ effectively coresponds to the field strength given by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] \quad (29)$$

we can relate the structure constant as the axial vector torsion.

Cartan's structural equation in Riemann-Cartan space-time U4 is given by [10]

$$T^\alpha=de^\alpha+\Gamma_\beta^\alpha\wedge e^\beta$$

$$R_\beta^\alpha=d\Gamma_\beta^\alpha\wedge\Gamma_\beta^\alpha\wedge\Gamma_\alpha^\beta \quad (30)$$

where $\Gamma_{\alpha\beta}$ and e^α represent spin connection and the local frames respectivley. The minimalLagrangian density of a spin $\frac{1}{2}$ field ψ with an external gravitational field with torsion is givenby [11]

$$L_D=\frac{i}{2}[\overline{\psi}^*\gamma\wedge D\psi+D\overline{\psi}\wedge^*\gamma\psi+m\psi\overline{\psi}-\frac{1}{4}A\wedge\overline{\psi}\gamma_5^*\gamma\psi] \quad (31)$$

where the exterior covariant derivative D is torsion-free, A is the axial vector part of the torsion one-form and $*$is the Hodge duality operator. It is noted that the axial vector torsion one-form is given by $A=*(e^\alpha\wedge T_\alpha)$. Here $\gamma=\gamma^\alpha e_\alpha$. In Riemannian Cartan space-time U4 there arisetwo boundary terms BT and BL given by [12].

$$2B_T = T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge e^\alpha \wedge e^\beta = d(e^\alpha \wedge T_\alpha) \quad (32)$$

$$B_L = -\frac{1}{2} R_\alpha^\beta \wedge R_\beta^\alpha \equiv dC_L$$

Where

$$C_L = -\frac{1}{2}[\Gamma_\alpha^\beta \wedge R_\beta^\alpha + \frac{1}{3}(\Gamma_\alpha^\beta \wedge \Gamma_\beta^\gamma \wedge \Gamma_\alpha^\gamma)]$$

is the Chern-Simons term for the curvature. The translation Chern-Simons term is given by

$$C_T = \frac{1}{2l^2}(e^\alpha \wedge T_\alpha) \quad (33)$$

Chandia and Zanelli [13] have observed that the quantum violation of the chiral current conservation in a Riemannian background with torsion is given by

$$\partial_\mu \langle j_\mu^5 \rangle = \frac{1}{8\pi^2}[R^{\alpha\beta} \wedge R_{\alpha\beta} + \frac{2}{l^2}(T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge e^\alpha \wedge e^\beta)] \quad (34)$$

where the first term on the R. H. S. is the well known Pontryagin density and represents theanomaly in Riemannain space-time without torsion. The second term on the R. H. S. is associated with torsion and corresponds to the Nieh-Yan density given by [14]

$$N = 2l^2 dC_T = d(e^\alpha \wedge T_\alpha)$$

$$= (T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge e^\alpha \wedge e^\beta) \quad (35)$$

It has been pointed out by these authors that if we combine the spin connection and vierbeintogether in a connection for $SO(5)$ in a tangent space in the form

$$W^{AB} = \begin{bmatrix} \Gamma^{\alpha\beta} & \frac{1}{l}e^\alpha \\ -\frac{1}{l}e^\alpha & 0 \end{bmatrix} \quad (36)$$

whereα, β=1, 2, 3, 4 and A, B=1, 2, 5, then we obtain the $SO(5)$ curvature two-form

$$F^{AB} = dW^{AB} + W^{AC} \wedge F^{CB} \quad (37)$$

from which we obtain $SO(5)$ Pontryagin density

$$F^{AB} \wedge F_{AB} = R^{\alpha\beta} \wedge R_{\alpha\beta} + \frac{2}{l^2}(T^\alpha \wedge T_\alpha - R^{\alpha\beta} \wedge e_\alpha \wedge e_\beta) \quad (38)$$

From this we observe that the N - Y term is related to the difference between the $SO(5)$ and $SO(4)$ Pontryagin densities

$$\frac{2}{l^2}\int_{M_4} N = P(SO(5)) - P(SO(4)) \quad (39)$$

In view of this, the N -Y density is also found to be a topological invariant. As we have pointedout that the introduction of the internal variable ξ_μ attached to the space-time point leads the deSitter space $M^{4,1}$ and the N-Y density N is given by the difference between the SO(5) and SO(4) Pontryagin density, the topological invariant N effectively corresponds to the contribution of the direction vector ξ_μ. In case of axial vector torsion $T^a \wedge T_a = 0$ in eqn. (38) and the SO(5) Pontryagin density reduces to SO(4) Pontryagin density when the torsion contributing density $R_{ab} \wedge e^a \wedge e^b = 0$. Indeed in the tangent space only SO(3, 1) symmetry is preserved and SO(4, 1) symmetry is broken. This implies that the topological invariant N is in effect associated with the discrete space Z_2 in the manifold $M_4 \times Z_2$ with the broken Z_2 symmetry and thus maybe taken to be a contribution related to the noncommutativity of space. Kreimer and Mielke [15] have objected to the inclusion of the N- Y term in chiral anomaly on the ground that toobtain a finite quantity the tetrads have to be rescaled. Indeed to this end Chandia and Zanellihave prescribed the rescaling

$$e^\alpha \to e^{-\alpha} = \frac{1}{M}e^\alpha$$

when in the limit $M \to \infty$, Ml fixed, we have the expression for the anomaly

$$A = \frac{1}{8\pi^2}[R^{\alpha\beta} \wedge R_{\alpha\beta} + 2(T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge e^\alpha \wedge e^\beta)] \quad (40)$$

However, Kreimer and Mielke have observed that this rescaling is not merely a simple manipulation but it also requires the change in the wave function renormalization Z-factor which willultimately lead to the vanishing value of the N -Y density in the expression for chiral anomaly. The crux of the trouble lies in the incorporation of the length scale l in the expression for thecontribution of the N - Y density. However we may observe here that in noncommutativegeometry, there is an implicit length scale governed by the commutation relation

$$[x^\mu, x^\nu] = \theta^{\mu\nu} = K\varepsilon^{\mu\nu} \quad (41)$$

where K has the dimension of l^2 . So we can associate the length scale in equation (38) withthe measure of the noncommutativity of space coordinates. Indeed it is very natural here inthe sense that the noncommutative manifold $M_4 \times Z_2$ which induces chiral anomaly leads totorsion. However this anomaly induced torsion is the axial vector one as only the axial vectorpart of torsion can interact with a Dirac fermion. It has been pointed out in an earlier paper [16] that chiral anomaly induces the topological origin of mass in flat space. The mass-energy of matter is related to the curvature when chiral anomaly gives rise to torsion. In view of this we may take torsion as the fundamental entity and Einstein's general relativity may be associated with the teleparallel gravity. This suggests that the noncommutative geometry which induces chiral anomaly effectively gives rise to a torsioned space-time.

4. Discussion

Chamseddine, Felder and Fröhlich [1] have studied gravity in noncommutative geometry byintroducing a gravity action for a space-time which is the product of a four-dimensional manifoldby a two-point space. In the simplest case where the Riemannian metric is taken to be thesame on the two copies of the manifold, we have a model of a scalar field coupled to Einsteingravity. The field geometrically describes the distance between the two points in the internalspace. It has been pointed out that vacuum expectation value of the field ϕ determines theelectroweak scale and thus forms a connection between gravity and the standard model. In ourpresent formalism the noncommutative manifold has been taken to be $M_4 \times Z_2$ where Z_2 isnot a two-point space but corresponds to a 'direction-vector' attached to a space-time point. In this geometry we have found that this leads to the introduction of torsion and gives rise toa Einstein-Cartan space-time. This torsion is an axial vector one and is associated with thechiral anomaly giving rise to the topological origin of mass [16] which in turn is responsiblefor curvature. This gives an indication that torsion may be taken to be the fundamentalentity in gravitational interaction and general relativity and it can be effectively described bythe teleparallel gravity. Gravity is a teleparallel theory Poincare gauge theory, zero curvaturereduces as the show go to the Einstein - Cartan space-time nonvanishing torsion and bent bothto the connection with this that the zero torsion range in our Einstein's similar to a Riemannianstructure is general relativity A recent paper [4] It is mentioned that was the extent $M_4 \times Z_2$, where the isolated space in a direction - vector, Z_2 symmetry broken chiral contradiction tothe mass topological origin for the responsible and consistently electroweak theory describes it. Thus such a non - commutative manifold cases where Z_2 is a two - point in space where theconnection between gravity and electroweak theory of a link appear at the chiral contradictionsand torsion realization through the acquisition to have.

5. Conclusion

We have observed that the Weitzenböck geometry which induces the equivalence of the teleparallel gravity with Einstein general relativity has some salient features. Nester [17] has shown that it leads to a pure tensorial proof of the positivity of energy in general relativity. Mielke [12] has pointed out that it yields a natural introduction of Ashtekar variables. Besides, it indicates the reality of torsioned space-time whereby torsion may be taken to be the fundamental entity rather than curvature. In the noncommutative manifold $M_4 \times Z_2$ where Z_2 corresponds to a direction-vector as well as in noncommutative $U(1)$ Yang-Mills theory we have a natural choice of torsioned space-time. However, in noncommutative manifold $M_4 \times Z_2$ torsion is related to deformation of symplectic structure whereas noncommutative gauge theory is chracterised by area preserving diffeomorphism. Recently it has been pointed out that the N-Y density of U_4 space may be taken to be related to the gravitational constant [18, 19]. Consequently this method is very much akin to our present line if we reflect that torsion is linked to internal space which types the space-time noncommutative such that the gravitational constant shots the rule of noncommutativity.

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