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Comparison Between Adomain Decomposition Method and Numerical Solutions of Linear Volterra Integral Equations of the Second Kind by Using the Fifth Order of Non-Polynomial Spline Functions

Elgaili Abdalla Elhassan Ibrahim[1], Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman[2, *], Neama Yahia Mohammed[3], Nageeb Abdallah Hamed Haroun[2]

[1]Department of Mathematics, Faculty of Science, Sudan University for Science and Technology, Khartoum, Sudan

[2]Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman, Sudan

[3]Department of Mathematics, College of Science, Tabuk University, Tabuk, Saudi Arabia

Email address:
dibdelradi78@gmail.com (A. R. A. R. A. G. A. Rahman)

[*]Corresponding author

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Abstract: Volterra integral equations are a special type of integrative equations; they are divided into two categories referred to as the first and second type. This paper will deal with the second type which has wide range of the applications in physics and engineering problems. Spline functions are piece-wise polynomials of degree n joined together at the break points with n-1 continuous derivatives. The break points of splines are called Knot, spline function can be integrated and differentiated due to being piece wise polynomials and can easily store and implemented on digital computer, non-polynomial spline function apiece wise is a blend of trigonometric, as well as, polynomial basis function, which form a complete extended Chebyshev space. Matlab is a powerful computing system for handling the calculations involved scientific and engineering problems. The aim of this paper is to compare between Adomain decomposition method and numerical solution to solve Volterra Integral Equations of second kind using the fifth order non-polynomial Spline functions by Matlab. We followed the applied mathematical method numerically by Matlab. Numerical examples are presented to illustrate the applications of this methods and to compare the computed results with analytical solutions. Finally by comparison of numerical results, Simplicity and efficiency of this method be shown.

Keywords: Linear Volterra Integral Equations, Second Kind, Non-Polynomial Spline Functions, Fifth Order, Adomain Decomposition Method

1. Introduction

Recently there is increase in concern by integrated equations, particularly Volterra integral equations has wide range of the applications in physics and engineering such as potential theory, Dirichlet problems, electrostatics, the particle transport problems of astrophysics, reactor problems, contact problems, diffusion problems and heat transfer problems(for more details see [4]).

In the mid of 1960s the numerical solution of integral equations was begun when the Kernel is unwell

In the beginning of 1980s the numerical methods began to take more importance when there is no analytical solution.

In 1997 Diogo, T and Lima, B presented deductive method [6], in order to find a numerical solution for Voltera integral equations with weakly singular kernel.

In 2007, Danaf & Ramadan [11] presented applications of non-polynomial spline functions to solve Volterra equations.

In 2009 Haq [2] studied the numerical solution to the boundary value problem and the initial value problem using the Spline function.

In the year 2011, Zarebnia - Hoshyar - Sedahti [17] presented the numerical solution based on the Cube Spline function to find a solution to the boundary value problem.

In 2011 Majeed,S [8] presented a numerical solution to the Volterra differential integrative equation of the second kind using non-polynomial Spline functions.

In 2012 Hossinpour presented [5] a solution for differential integral equations by using non-polynomial spline functions.

In 2015, Harbis, S, Murad, M and Majed, S [3] presented a numerical solution for linear Voltera integral equation from second kind by using non-polynomial spline function from the third degree.

In the year 2016; Taqi. A, Jumma. B [15] Method of Non-Boundary Spline Functions to Solve the System of Two Nonlinear Volterra Integral Equations.

In 2017, Najwa, S and Mohammed, S [7] presented a numerical solution for linear Voltera integral equations with weakly singular kernel by using non-polynomial spline function from the fifth degree.

In this paper we introduced a numerical solution for linear Voltera integral equation from second kind with by using non-polynomial spline function from the fifth degree by Matlab.

2. Analytical Solution of Linear Volterra Integral Equations of the Second Kind: [1, 9, 16]

First we define Voltera integral equations of the second kind are given by:

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt \quad 0 \le x \le b \tag{1}$$

The unknown function $u(x)$ that determined, occurs inside and outside the integral sign, The Kernel $k(x,t)$ and the function $f(x)$ are given continues functions. [1, 9, 16]

There are many analytical methods available for solving Linear Volterra Integral Equations of second kind. We focus on the follow method: the Adomain decomposition method:

3. Adomain Decomposition Method: [1, 9, 16]

The Adomain decomposition method (ADM) was introduced and developed by George Adomain. The Adomain decomposition method consists of decomposition the unknown function $u(x)$ of any equation into a sum of infinite number of components defined by the decompositions series

$$u(x) = \sum_{n=0}^{n} u_n(x) \tag{2}$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{3}$$

Where the components $(u_n(x), n \ge 0$ are to be determined in a recursive manner. The decomposition method concernsitself with finding the components individually; we substitute (2) into the Voltera integral equation (1) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \int_a^x k(x,t)(\sum_{n=0}^{\infty} u_n(t))dt \tag{4}$$

The zeroth components $u_0(x)$ is identified by all items that are not included under the integral sign. Consequently, the components $u_j(x), j \ge 1$ of the unknown function $u(x)$ is completely determined by setting the recurrence relation:

$$u_0(x) = f(x_0) \tag{5}$$

$$u_{n+1}(x) = \int_a^x k(x,t)u_n(t)dt, n \ge 0 \tag{6}$$

Example (1):[1,9,16]

To solve the following Voltera integral equation:

$$u(x) = 1 - \int_0^x u(t)dt \tag{7}$$

Where$f(x) = 1 \; and \; k(x,t) = -1$,

Substituting decomposition series (2) into the both two sides of VIE's (12) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)dt \tag{8}$$

We identify the zeroth components by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1,$$

$$u_{n+1}(x) = -\int_0^x u_n(t)dt, k \ge 0 \tag{9}$$

So that $u_0(x) = 1$,

$$u_1(x) = -\int_0^x u_0(t)dt = -\int_0^x 1dt, = -x,$$

$$u_2(x) = -\int_0^x u_1(t)dt = -\int_0^x -tdt, = \frac{x^2}{2!},$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x \frac{t^2}{2}dt, = -\frac{x^3}{3!},$$

$$u_4(x) = -\int_0^x u_3(t)dt = -\int_0^x -\frac{t^3}{6}dt, = \frac{x^4}{4!}, \forall \; equations \tag{10}$$

And so on, gives the series solution

$$u(x) = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^{-x} \tag{11}$$

which is equation (11) the exact solution for equation (7)

4. A Numerical Solution of Linear VIE's of 2nd Kind Using the 5th Order of Non-Polynomial Spline Functions

The form of the five non-polynomial spline function is:

$$P_i(t) = a_i cosK(t - t_i) + b_i sinK(t - t_i) + c_i(t - t_i) + d_i(t - t_i)^2 + e_i(t - t_i)^3 + r_i(t - t_i)^4 + z_i(t - t_i)^5 + g_i \quad (12)$$

$where\ a_i, b_i, c_i, d_i, e_i, r_i, z_i, and\ g_i$are constants, to be determined. In order to obtain the values of $a_i, b_i, c_i, d_i, e_i, r_i, z_i, and\ g_i$, we differentiate equation (12) seven times with respect to t, and we get the following equations

$$\left.\begin{array}{c} P_i^{(1)}(t) = -Ka_i sinK(t - t_i) + Kb_i cosK(t - t_i) + c_i + 2d_i(t - t_i) + 3e_i(t - t_i)^2 \\ +4r_i(t - t_i)^3 + 5z_i(t - t_i)^4 \\ P_i^{(2)}(t) = -K^2 a_i cosK(t - t_i) - K^2 b_i sinK(t - t_i) + 2d_i + 6e_i(t - t_i) \\ +12r_i(t - t_i)^2 + 20z_i(t - t_i)^3 \\ P_i^{(3)}(t) = K^3 a_i sinK(t - t_i) - K^3 b_i cosK(t - t_i) + 6e_i + 24r_i(t - t_i) + 60z_i(t - t_i)^2 \\ P_i^{(4)}(t) = K^4 a_i cosK(t - t_i) + K^4 b_i sinK(t - t_i) + 24r_i + 120z_i(t - t_i) \\ P_i^{(5)}(t) = -K^5 a_i sinK(t - t_i) + K^5 b_i cosK(t - t_i) + 120z_i \\ P_i^{(6)}(t) = -K^6 a_i cosK(t - t_i) - K^6 b_i sinK(t - t_i) \\ P_i^{(7)}(t) = K^7 a_i sinK(t - t_i) - K^7 b_i cosK(t - t_i) \end{array}\right\} \quad (13)$$

Hence replace $t\ by\ t_i$ in the relation (12) and (13) yields:

$$P(t) = a_i + g_i$$

$$P_i^{(1)}(t) = b_i K + c_i$$

$$P_i^{(2)}(t) = -K^2 a_i + 2d_i$$

$$P_i^{(3)}(t) = -K^3 b_i + 6e_i$$

$$P_i^{(4)}(t) = K^4 a_i + 24r_i$$

$$P_i^{(5)}(t) = K^5 a_i + 120z_i$$

$$P_i^{(6)}(t) = -K^6 a_i$$

$$P_i^{(7)}(t) = -K^7 b_i$$

We obtained the values of$a_i, b_i, c_i, d_i, e_i, r_i, z_i, and\ g_i$ from the above relations as follows:

$$a_i = -\frac{1}{K^6} P_i^{(6)}(t) \quad (14)$$

$$b_i = \frac{1}{K^7} P_i^{(7)}(t) \quad (15)$$

$$c_i = P_i^{(1)}(t) - Kb_i \quad (16)$$

$$d_i = \frac{1}{2}[P_i^{(2)}(t) + K^2 a_i] \quad (17)$$

$$e_i = \frac{1}{6}[P_i^{(3)}(t) + K^3 b_i] \quad (18)$$

$$r_i = \frac{1}{24}[P_i^{(4)}(t) - K^4 a_i] \quad (19)$$

$$z_i = \frac{1}{120}[P_i^{(5)}(t) - K^5 b_i] \quad (20)$$

$$g_i = P_i(t) - a_i \quad (21)$$

$for\ i = 0,1, \dots, n$

5. Linear Volterra Integral Equations of the Second Kind [1, 9, 16]

First we define Volterra integral equations of the second kind are given by:

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt \quad 0 \le x \le b \quad (22)$$

To solve the equation (22). We need to differentiate equation (22) seven times with respect to x, by using Libenze formula we realize:

$$u^{(1)}(x) = f^{(1)}(x) + \int_a^x \frac{\partial k(x,t)}{\partial x} u(t)dt + k(x,x)u(x) \quad (23)$$

$$u^{(2)}(x) = f^{(2)}(x) + \int_a^x \frac{\partial^{(2)} k(x,t)}{\partial x^{(2)}} u(t)dt + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u(x) + \frac{dk(x,x)}{dx} u(x) + k(x,x)u^{(1)}(x) \quad (24)$$

$$u^{(3)}(x) = f^{(3)}(x) + \int_a^x \frac{\partial^{(3)} k(x,t)}{\partial x^{(3)}} u(t)dt + \left(\frac{\partial^{(2)} k(x,t)}{\partial x^{(2)}}\right)_{t=x} u(x) + \frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u(x) + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(1)}(x) + \frac{d^{(2)} k(x,x)}{dx^{(2)}} u(x) + 2\frac{dk(x,x)}{dx} u^{(1)}(x) + k(x,x)u^{(2)}(x) \quad (25)$$

$$u^{(4)}(x) = f^{(4)}(x) + \int_a^x \frac{\partial^{(4)} k(x,t)}{\partial x^{(4)}} u(t)dt + \left(\frac{\partial^{(3)} k(x,t)}{\partial x^{(3)}}\right)_{t=x} u(x) + \frac{d}{dx}\left(\frac{\partial^{(2)} k(x,t)}{\partial x^{(2)}}\right)_{t=x} u(x) + \left(\frac{\partial^{(2)} k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(1)}(x) + \frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u(x) + 2\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(1)}(x) + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(2)}(x) + \frac{d^{(3)} k(x,x)}{dx^{(3)}} u(x) + 3\frac{d^{(2)} k(x,x)}{dx^{(2)}} u^{(1)}(x) + 3\frac{dk(x,x)}{dx} u^{(2)}(x) + k(x,x)u^{(3)}(x) \quad (26)$$

$$u^{(5)}(x) = f^{(5)}(x) + \int_a^x \frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}u(t)dt + \left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u(x) + \frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u(x) + \left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(1)}(x) + \frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u(x) + 2\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(1)}(x) + \left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(2)}(x) + \frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u(x) + 3\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(1)}(x) + 3\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(2)}(x) + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(3)}(x) + \frac{d^{(4)}k(x,x)}{dx^{(4)}}u(x) + 4\frac{d^{(3)}k(x,x)}{dx^{(3)}}u^{(1)}(x) + 6\frac{d^{(2)}k(x,x)}{dx^{(2)}}u^{(2)}(x) + 4\frac{dk(x,x)}{dx}u^{(3)}(x) + k(x,x)u^{(4)}(x) \tag{27}$$

$$u^{(6)}(x) = f^{(6)}(x) + \int_a^x \frac{\partial^{(6)}k(x,t)}{\partial x^{(6)}}u(t)dt + \left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x} u(x) + \frac{d}{dx}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u(x) + \left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u^{(1)}(x) + \frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{1}(x) + 2\frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(1)}(x) + \left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(2)}(x) + \frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u(x) + 3\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(1)}(x) + 3\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(2)}(x) + \left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(3)}(x) + \frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u(x) + 4\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(1)}(x) + 6\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(2)}(x) + 4\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(3)}(x) + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(4)}(x) + \frac{d^{5}k(x,x)}{dx^{(5)}}u(x) + 5\frac{d^{(4)}k(x,x)}{dx^{(4)}}u^{(1)}(x) + 10\frac{d^{(3)}k(x,x)}{dx^{(3)}}u^{(2)}(x) + 10\frac{d^{(2)}k(x,x)}{dx^{(2)}}u^{(3)}(x) + 5\frac{dk(x,x)}{dx}u^{(4)}(x) + k(x,x)u^{(5)}(x) \tag{28}$$

$$u^{(7)}(x) = f^{(7)}(x) + \int_a^x \frac{\partial^{(7)}k(x,t)}{\partial x^{(7)}}u(t)dt + \left(\frac{\partial^{(6)}k(x,t)}{\partial x^{(6)}}\right)_{t=x} u(x) + \frac{d}{dx}\left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x} u(x) + \left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x} u^{(1)}(x) + \frac{d^{2}}{dx^{2}}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u(x) + 2\frac{d}{dx}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u^{(1)}(x) + \left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x} u^{(2)}(x) + \frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u(x) + 3\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(1)}(x) + 3\frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(2)}(x) + \frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u(x) + 4\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(1)}(x) + 6\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(2)}(x) + 4\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(3)}(x) + \left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x} u^{(3)}(x) + \left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x} u^{(4)}(x) + \left(\frac{d^{(5)}}{dx^{(5)}}\left(\frac{\partial k(x,t)}{\partial x}\right)\right) u(x) + 5\frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(1)}(x) + 10\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(2)}(x) + 10\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(3)}(x) + 5\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(4)}(x) + \left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x} u^{(5)}(x) + \frac{d^{(6)}k(x,x)}{dx^{(6)}}u(x) + 6\frac{d^{(5)}k(x,x)}{dx^{(5)}}u^{(1)}(x) + 15\frac{d^{(4)}k(x,x)}{dx^{(4)}}u^{(2)}(x) + 20\frac{d^{(3)}k(x,x)}{dx^{(3)}}u^{(3)}(x) + 15\frac{d^{(2)}k(x,x)}{dx^{(2)}}u^{(4)}(x) + 6\frac{dk(x,x)}{dx}u^{(5)}(x) + k(x,x)u^{(6)}(x) \tag{29}$$

To complete our procedure for solving VIE's we substitute $x = a$ in equations (22)-(29) then we get:

$$u_0 = u(a) = f(a) \tag{30}$$

$$u_0^{(1)} = u^{(1)}(a) = f^{(1)}(a) + k(a,a).u(a) \tag{31}$$

$$u_0^{(2)} = u^{(2)}(a) = f^{(2)}(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{dk(x,x)}{dx}\right)_{x=a} u(a) + k(a,a)u^{(1)}(a) \tag{32}$$

$$u_0^{(3)} = u^{(3)}(a) = f^{(3)}(a) + \left(\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\frac{d^{(2)}k(x,x)}{dx^{(2)}}\right)_{x=a} u(a) + 2\left(\frac{dk(x,x)}{dx}\right)_{x=a} u^{(1)}(a) + k(x,x)u^{(2)}(a) \tag{33}$$

$$u_0^{(4)} = u^{(4)}(a) = f^{(4)}(a) + \left(\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + 2\left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\frac{d^{(3)}k(x,x)}{dx^{(3)}}\right)_{x=a} u(a) + 3\left(\frac{d^{(2)}k(x,x)}{dx^{(2)}}\right)_{x=a} u^{(1)}(a) + 3\left(\frac{dk(x,x)}{dx}\right)_{x=a} u^{(2)}(a) + k(x,x)u^{(3)}(a) \tag{34}$$

$$u_0^{(5)} = u^{(5)}(a) = f^{(5)}(a) + \left(\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u(a) + 2\left(\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + 3\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 3\left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + \left(\frac{d^{(4)}k(x,x)}{dx^{(4)}}\right)_{x=a} u(a) + 4\left(\frac{d^{(3)}k(x,x)}{dx^{(3)}}\right)_{x=a} u^{(1)}(a) + 6\left(\frac{d^{(2)}k(x,x)}{dx^{(2)}}\right)_{x=a} u^{(2)}(a) +$$

$$4\left(\frac{dk(x,x)}{dx}\right)_{x=a} u^{(3)}(a) + k(x,x)u^{(4)}(a) \tag{35}$$

$$u_0^{(6)} = u^{(6)}(a) = f^{(6)}(a) + \left(\left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u(a) + 2\left(\frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u(a) + 3\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 3\left(\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + \left(\frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + 4\left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 6\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(2)}(x) + 4\left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(4)}(a) + \left(\frac{d^{(5)}k(x,x)}{dx^{(5)}}\right)_{x=a} u(a) + 5\left(\frac{d^{(4)}k(x,x)}{dx^{(4)}}\right)_{x=a} u^{(1)}(a) + 10\left(\frac{d^{(3)}k(x,x)}{dx^{(3)}}\right)_{x=a} u^{(2)}(a) + 10\left(\frac{d^{(2)}k(x,x)}{dx^{(2)}}\right)_{x=a} u^{(3)}(a) + 5\left(\frac{dk(x,x)}{dx}\right)_{x=a} u^{(4)}(a) + k(x,x)\,u^{(5)}(a) \tag{36}$$

$$u_0^{(7)} = u^{(7)}(a) = f^{(7)}(a) = \left(\left(\frac{\partial^{(6)}k(x,t)}{\partial x^{(6)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\frac{d}{dx}\left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x}\right)_{x=a} u(a) + \left(\left(\frac{\partial^{(5)}k(x,t)}{\partial x^{(5)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u(a) + 2\left(\frac{d}{dx}\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + \left(\left(\frac{\partial^{(4)}k(x,t)}{\partial x^{(4)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u(a) + 3\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 3\left(\frac{d}{dx}\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + \left(\frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u(a) + 4\left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 6\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + 4\left(\frac{d}{dx}\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + \left(\left(\frac{\partial^{(3)}k(x,t)}{\partial x^{(3)}}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + \left(\left(\frac{\partial^{(2)}k(x,t)}{\partial x^{(2)}}\right)_{t=x}\right)_{x=a} u^{(4)}(a) + \left(\frac{d^{(5)}}{dx^{(5)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u(a) + 5\left(\frac{d^{(4)}}{dx^{(4)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(1)}(a) + 10\left(\frac{d^{(3)}}{dx^{(3)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(2)}(a) + 10\left(\frac{d^{(2)}}{dx^{(2)}}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(3)}(a) + 5\left(\frac{d}{dx}\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(4)}(a) + \left(\left(\frac{\partial k(x,t)}{\partial x}\right)_{t=x}\right)_{x=a} u^{(5)}(a) + \left(\frac{d^{(6)}k(x,x)}{dx^{(6)}}\right)_{x=a} u(a) + 6\left(\frac{d^{(5)}k(x,x)}{dx^{(5)}}\right)_{x=a} u^{(1)}(a) + 15\left(\frac{d^{(4)}k(x,x)}{dx^{(4)}}\right)_{x=a} u^{(2)}(a) + 20\left(\frac{d^{(3)}k(x,x)}{dx^{(3)}}\right)_{x=a} u^{(3)}(a) + 15\left(\frac{d^{(2)}k(x,x)}{dx^{(2)}}\right)_{x=a} u^{(4)}(a) + 6\left(\frac{dk(x,x)}{dx}\right)_{x=a} u^{(5)}(a) + k(x,x)u^{(6)}(a) \tag{37}$$

Now, we try to solve equation (22) using fifth order non-polynomial Spline functions.

6. Using Fifth Order Non-Polynomial Spline Function

In order to approximate the solution of linear VIE's of second kind (22) by using fifth order non-polynomial spline function (1). we present a method of solution in algorithm (VIE2NPS5).

The algorithm (VIE2NPS5):

Step 1: set $h = \frac{b-a}{n}, t_i = t_0 + ih, i = 0,1,\ldots.,n$ $where\ t_0 = a, t_n = b\ and\ u_0 = f(a)$.

Step 2: Evaluate $a_0, b_0, c_0, d_0, e_0, r_0 z_0\ and\ g_0$ by substituting (30)-(37) In equations (13) - (21).

Step 3: Calculate $p_0(t)$using step 2 and equation (12).

Step 4: Approximate $u_1 = p_0(t_i)$.

Step 5: For $i = 1\ to\ n-1$ do following steps:

Step 6: Evaluate $a_i, b_i, c_i, d_i, e_i, r_i, z_i\ and\ g_i$ by using equations (13)-(21) and replacing $u(t_i), u^1(t_i), u^2(t_i), u^3(t_i), u^4(t_i), u^5(t_i), u^6(t_i)\ and\ u^7(t_i)$ $by\ p_i(t_i), {p_i}^1(t_i), {p_i}^2(t_i), {p_i}^3(t_i), {p_i}^4(t_i), {p_i}^5(t_i), {p_i}^6(t_i)\ and\ {p_i}^7(t_i)$

Step 7: Calculate $p_i(t)$using step 6 and equation (12).

Step 8: Approximate $u_{i+1} = p_i(t_{i+1})$.

7. Programming

Program 1: linear non-polynomial spline function for solving VIE's of the second kind.

```
function [x, y, err, u] = volnonpolyspline1st(ker,f, ex, a, b, n)
syms X T s
h   = (b-a)/(n-1);
u   = subs(f, X, a)
ifisempty(diff(f,1))==1
   z1=0;
else
   z1=diff(f,1)
end
du = subs(z1,a) + subs(ker,{X,T},{a,a})*u(1);
ifisempty(diff(f,2))==1
   z2=0;
else
   z2=diff(f,2);
end
ifisempty(diff(subs(ker,{T},{X},'X')))==1
   z3=0;
else
```

```
    z3=diff(subs(ker,{T},{X}),'X');
  end
  ifisempty(diff(ker,'X'))==1
    zz3=0;
  else
    zz3=diff(ker,'X');
  end
  d2u(1)=subs(z2,a)+subs(zz3,{X,T},{a,a})*u(1)+subs(z3,a)
*u(1)+subs(ker,{X,T},{a,a})*du(1);
  ifisempty(subs(diff(subs(diff(ker,X), T, X), X), X, a))==1;
    s=0;
  else
    s=subs(diff(subs(diff(ker,X), T, X), X), X, a);
  end
  ifisempty(diff(f,3))==1
    z4=0;
  else
    z4=diff(f,3);
  end
  ifisempty(diff(ker,2,'X'))==1
    z5=0;
  else
    z5=diff(ker,2,'X');
  end
  ifisempty(subs(diff(subs(ker, T, X), X, 2), X, a))==1
    z6=0;
  else
    z6=subs(diff(subs(ker, T, X), X, 2), X, a);
  end
  ifisempty(subs(diff(subs(ker, T, X), X), X, a))==1
    z7=0;
  else
    z7=subs(diff(subs(ker, T, X), X), X, a);
  end
  d3u(1)=subs(z4, a)+subs(subs(z5, T, X), X, a)*u(1)+...
  s*u(1)+subs(subs(diff(ker, X), T, X), X,a)*du(1)+...
    z6*u(1)+2*z7*du(1)+subs(ker,{X,T},{a,a})*d2u(1);
  a(1)=-d2u(1);
  b(1)=-d3u(1);
  c(1)=du(1)+d3u(1);
  d(1)=u(1)+d2u(1);
  for i=1:n-1
     u(i+1)=a(i)*cos(h)+b(i)*sin(h)+h*c(i)+d(i);
     du(i+1)=-a(i)*sin(h)+b(i)*cos(h)+c(i);
  d2u(i+1)=-a(i)*cos(h)-b(i)*sin(h);
  d3u(i+1)=a(i)*sin(h)-b(i)*cos(h) ;
  d4u(i+1)=a(i)*cos(h)+b(i)*sin(h) ;
  a(i+1)=-d2u(i+1);
  b(i+1)=-d3u(1);
  c(i+1)=du(i+1)+d3u(i+1);
  d(i+1)=u(i+1)+d2u(i+1);
  end
   x = a:h:b;
   y = subs(ex, x);
  for i = 1:n
  err(i)=abs(u(i)-subs(ex,x(i)));
  end
  format long
  disp('   i          X          Y          u          err')
  [1:n; x; y; u; err]'
  figure(1)
  plot(x, y, 'b --', 'linewidth', 3)
  hold on
  plot(x, u, 'r.-', 'linewidth', 1)
  grid on
  xlabel('X')
  ylabel('Y')
  title('exact vs. approx.')
  legend('exact', 'approx.')
```

Program 2: sixth no –polynomial spline function for solving VIE's of the second kind with weakly singular kernel.

```
  function [X, Y, err, u] = volnonpolyspline5st(ker,f, ex, a, b, n)
  syms X T
  h   = (b-a)/(n-1);
  u   = subs(f, X, a)
   x = a:h:b;
   y = subs(ex, x);
  ifisempty(diff(f,1))==1
    z1=0;
  else
    z1=diff(f,1)
  end
  du = subs(z1,a) + subs(ker,{X,T},{a,a})*u(1);
  ifisempty(diff(f,2))==1
    z2=0;
  else
    z2=diff(f,2);
  end
  ifisempty(diff(subs(ker,{T},{X},'X')))==1
    z3=0;
  else
    z3=diff(subs(ker,{T},{X}),'X');
  end
  ifisempty(diff(ker,'X'))==1
    zz3=0;
  else
    zz3=diff(ker,'X');
  end
  d2u(1)=subs(z2,a)+subs(zz3,{X,T},{a,a})*u(1)+subs(z3,a)
*u(1)+subs(ker,{X,T},{a,a})*du(1);
  ifisempty(subs(diff(subs(diff(ker,X), T, X), X), X, a))==1;
    s=0;
  else
    s=subs(diff(subs(diff(ker,X), T, X), X), X, a);
  end
  ifisempty(diff(f,3))==1
    z4=0;
  else
    z4=diff(f,3);
  end
  ifisempty(diff(ker,2,'X'))==1
    z5=0;
  else
    z5=diff(ker,2,'X');
```

```
end
ifisempty(subs(diff(subs(ker, T, X), 2, X), X, a))==1
   z6=0;
else
   z6=subs(diff(subs(ker, T, X), 2, X), X, a);
end
ifisempty(subs(diff(subs(ker, T, X), X), X, a))==1
   z7=0;
else
   z7=subs(diff(subs(ker, T, X), X), X, a);
end
d3u(1)=subs(z4, a)+subs(subs(z5, T, X), X, a)*u(1)+...
   s*u(1)+subs(diff(subs(diff(ker, X), T, X), X,a))*du(1)+...
   z6*u(1)+2*z7*du(1)+subs(ker,{X,T},{a,a})*d2u(1);
ifisempty(diff(f,4))==1
   z8=0;
else
   z8=diff(f,4);
end
ifisempty(diff(ker,3,'X'))==1
   z9=0;
else
   z9=diff(ker,3,'X');
end
ifisempty(subs(diff(ker,X,2),T,X))==1
   z10=0;
else
   z10=subs(diff(ker, X, 2),T,X);
end
ifisempty(subs(diff(z10,X),X,a))==1
   z1010=0;
else
   z1010=subs(diff(z10,X),X, a);
end
ifisempty(diff(ker,2,'X'))==1
   z11=0;
else
   z11=diff(ker,2,'X');
end
ifisempty(subs(diff(ker,'X'),T,X))==1
   z12=0;
else
   z12=subs(diff(ker,'X'),T,X);
end
ifisempty(subs(diff(z12,X,2),X,a))==1
 z1212=0;
else
 z1212= subs(diff(z12,X,2),X,a);
end
ifisempty(subs(diff(subs(ker, T, X), X, 3), X, a))==1
   z14=0;
else
   z14=subs(diff(subs(ker, T, X), X, 3), X, a);
end
ifisempty(subs(diff(subs(ker, T, X), X, 2), X, a))==1
   z15=0;
else
   z15=subs(diff(subs(ker, T, X), X, 2), X, a);
end
ifisempty(subs(diff(subs(ker, T, X), X), X, a))==1
   z16=0;
else
   z16=subs(diff(subs(ker, T, X), X), X, a);
end
d4u(1)=subs(z8, a)+subs(subs(z9, T, X), X, a)*u(1)+...
   z1010*u(1)+subs(subs(z11,  T, X), X,a)*du(1)+...
   z1212*u(1)+2*s*du(1)+subs(subs(diff(ker,X),T,X),
X,a)*d2u(1)+...

z14*u(1)+3*z15*u(1)+3*z16*d2u(1)+subs(ker,{X,T},{a,a})
*d3u(1);
   ifisempty(diff(f,5))==1
   z17=0;
else
   z17=diff(f,5);
end
ifisempty(diff(ker,4,'X'))==1
   z18=0;
else
   z18=diff(ker,4,'X');
end
ifisempty(subs(diff(ker, X,3),T,X))==1
   z19=0;
else
   z19=subs(diff(ker,X,3),T, X);
end
ifisempty(subs(diff(z19,X), X,a))==1
   z1919=0;
else
   z1919=subs(diff(z19,X), X, a);
end
ifisempty(diff(ker,3,'X'))==1
   z20=0;
else
   z20=diff(ker,3,'X');
end
ifisempty(subs(diff(ker,2,'X'),T,X))==1
   z21=0;
else
   z21=subs(diff(ker,2,'X'),T,X );
end
ifisempty(subs(diff(z21,X,2),X,a))==1
  z2121 =0;
else
   z2121=subs(diff(z21,X,2),X,a);
end
ifisempty(subs(diff(ker,X, 2),T,X))==1
   z22=0;
else
   z22=subs(diff(ker, X,2),T, X);
end
ifisempty(subs(diff(z22,X), X, a))==1
   z2222=0;
else
```

```
    z2222=subs(diff(z22,X ), X, a);
  end
  ifisempty(subs(diff(ker,'X'),T,X))==1
    z24=0;
  else
    z24=subs(diff(ker,'X'),T,X);
  end
  ifisempty(subs(diff(z24,X,3),X,a))==1
   z2424 =0;
  else
    z2424=subs(diff(z24,X,3),X,a);
  end
  ifisempty(subs(diff(subs(ker, T, X), X, 4), X, a))==1
    z28=0;
  else
    z28=subs(diff(subs(ker, T, X), X, 4), X, a);
  end
  d5u(1)=subs(z17,a)+subs(subs(z18, T, X), X, a)*u(1)+...
    z1919*u(1)+subs(subs(z20,          T,          X),          X,
a)*du(1)+z2121*u(1)+...

z2121*d2u(1)+2*z2222*du(1)+subs(subs(z11,T,X),X,a)*d2u
(1)+...
    z2424*u(1)+3*z1212*du(1)+3*s*d2u(1)+...
  subs(subs(diff(ker, X),T,X), X, a)*d3u(1)+...
    z28*u(1)+4*z14*du(1)+6*z15*d2u(1)+...
   4*z16*d3u(1)+ subs(ker,{X,T},{a,a})*d4u(1);
  ifisempty(diff(f,6))==1
    z31=0;
  else
    z31=diff(f,6);
  end
  ifisempty(diff(ker,5,'X'))==1
    z32=0;
  else
    z32=diff(ker,5,'X');
  end
  ifisempty(subs(diff(ker, X,4),T,X))==1
    z33=0;
  else
    z33=subs(diff(ker,X,4),T, X);
  end
  ifisempty(subs(diff(z33,X), X,a))==1
    z3333=0;
  else
    z3333=subs(diff(z33,X), X, a);
  end
  ifisempty(diff(ker,4,'X'))==1
    z34=0;
  else
    z34=diff(ker,4,'X');
  end
  ifisempty(subs(diff(ker,3,'X'),T,X))==1
    z35=0;
  else
    z35=subs(diff(ker,3,'X'),T,X );
  end
  ifisempty(subs(diff(z35,X,2),X,a))==1
   z3535 =0;
  else
    z3535=subs(diff(z35,X,2),X,a);
  end
  ifisempty(subs(diff(ker,X, 2),T,X))==1
    z36=0;
  else
    z36=subs(diff(ker, X,2),T, X);
  end
  ifisempty(subs(diff(z36,X,3), X, a))==1
    z3636=0;
  else
    z3636=subs(diff(z36,X,3), X, a);
  end
  ifisempty(subs(diff(ker,'X'),T,X))==1
    z37=0;
  else
    z37=subs(diff(ker,'X'),T,X);
  end
  ifisempty(subs(diff(z37,X,3),X,a))==1
   z3737 =0;
  else
    z3737=subs(diff(z37,X,3),X,a);
  end
  ifisempty(subs(diff(subs(ker, T, X), X, 5), X, a))==1
    z38=0;
  else
    z38=subs(diff(subs(ker, T, X), X, 5), X, a);
  end
  d6u(1)=subs(z31,a)+subs(subs(z32, T, X), X, a)*u(1)+...
    z3333*u(1)+subs(subs(z34,          T,          X),          X,
a)*du(1)+z3535*u(1)+...

2*z1919*du(1)+z3636*u(1)+3*z2121*du(1)+3*z2222*d2u(
1)+...
    subs(subs(z11,T,X),X,a)*d3u(1)+z3737*u(1)+...
    4*z2424*du(1)+6*z1212*d2u(1)+4*s*d3u(1)+...
  subs(subs(diff(ker, X),T,X), X, a)*d4u(1)+...

z38*u(1)+5*z28*du(1)+10*z14*d2u(1)+10*z15*d3u(1)+...
   5*z16*d4u(1)+ subs(ker,{X,T},{a,a})*d5u(1);
  ifisempty(diff(f,7))==1
    z39=0;
  else
    z39=diff(f,7);
  end
  ifisempty(diff(ker,6,'X'))==1
    z40=0;
  else
    z40=diff(ker,6,'X');
  end
  ifisempty(subs(diff(ker, X,5),T,X))==1
    z41=0;
  else
    z41=subs(diff(ker,X,5),T, X);
  end
```

```
ifisempty(subs(diff(z41,X), X,a))==1
   z4141=0;
else
   z4141=subs(diff(z41,X), X, a);
end
ifisempty(diff(ker,5,'X'))==1
   z42=0;
else
   z42=diff(ker,5,'X');
end
ifisempty(subs(diff(ker,4,'X'),T,X))==1
   z43=0;
else
   z43=subs(diff(ker,4,'X'),T,X );
end
ifisempty(subs(diff(z43,X,2),X,a))==1
  z4343 =0;
else
   z4343=subs(diff(z43,X,2),X,a);
end
ifisempty(subs(diff(ker,X, 3),T,X))==1
   z44=0;
else
   z44=subs(diff(ker, X,3),T, X);
end
ifisempty(subs(diff(z44,X,3), X, a))==1
   z4444=0;
else
   z4444=subs(diff(z44,X,3), X, a);
end
ifisempty(subs(diff(ker,X, 2),T,X))==1
   z45=0;
else
   z45=subs(diff(ker, X,2),T, X);
end
ifisempty(subs(diff(z45,X,3), X, a))==1
   z4545=0;
else
   z4545=subs(diff(z45,X,3), X, a);
end
ifisempty(subs(diff(ker,'X'),T,X))==1
   z46=0;
else
   z46=subs(diff(ker,'X'),T,X);
end
ifisempty(subs(diff(z46,X,5),X,a))==1
  z4646 =0;
else
   z4646=subs(diff(z46,X,5),X,a);
end
ifisempty(subs(diff(subs(ker, T, X), X, 6), X, a))==1
   z47=0;
else
   z47=subs(diff(subs(ker, T, X), X, 6), X, a);
end
d7u(1)=subs(z39,a)+subs(subs(z40, T, X), X, a)*u(1)+...
   z4141*u(1)+subs(subs(z42, T, X), X,
a)*du(1)+z4343*u(1)+...
      2*z3333*du(1)+subs(subs(z34,T,X),X,a)*d2u(1)+...
      z4444*u(1)+3*z3535*du(1)+3*z1919*d2u(1)+...
      z4545*u(1)+4*z3636*du(1)+6*z2121*d2u(1)+...
      4*z2222*d3u(1)+ subs(subs(z20,T,X),X,a)*d3u(1)+...

subs(subs(z11,T,X),X,a)*d4u(1)+z4646*u(1)+5*z3737*du(1
)+...
      10*z2424*d2u(1)+10*z1212*d3u(1)+5*s*d4u(1)+...
   subs(subs(diff(ker, X),T,X), X, a)*d5u(1)+...

z47*u(1)+6*z38*du(1)+15*z28*d2u(1)+20*z14*d3u(1)+...

15*z15*d4u(1)+6*z16*d5u(1)+subs(ker,{X,T},{a,a})*d6u(1);
   a(1)=-d6u(1);
   b(1)= d7u(1);
   c(1)=du(1)-d7u(1);
   d(1)=(1/2)*(d2u(1)-d6u(1));
   e(1)=(1/6)*(d3u(1)+d7u(1));
   r(1)=(1/24)*(d4u(1)+d6u(1));
   z(1)=(1/120)*(d5u(1)-d7u(1));
   g(1)=u(1)+d6u(1);
   for i=1:n-1
      u(i+1) = a(i)*cos(h) + b(i)*sin(h) + h*c(i) + d(i)*h^2 +
e(i)*h^3 + r(i)*h^4 + z(i)*h^5 + g(i);
      du(i+1)  = -a(i)*sin(h) + b(i)*cos(h) + c(i) + 2*d(i)*h +
3*e(i)*h^2 + 4*r(i)*h^3 + 5*z(i)*h^4 ;
      d2u(i+1) = -a(i)*cos(h) - b(i)*sin(h) + 2*d(i) + 6*e(i)*h
+ 12*r(i)*h^2 + 20*z(i)*h^3 ;
      d3u(i+1) = a(i)*sin(h) - b(i)*cos(h) + 6*e(i) + 24*r(i)*h
+ 60*z(i)*h^2 ;
      d4u(i+1)  =  a(i)*cos(h)  +  b(i)*sin(h)  +  24*r(i)  +
120*z(i)*h ;
   d5u(i+1) = -a(i)*sin(h) + b(i)*cos(h) + 120*z(i) ;
   d6u(i+1) = -a(i)*cos(h) - b(i)*sin(h) ;
   d7u(i+1) =  a(i)*sin(h) - b(i)*cos(h);
   a(i+1)=-d6u(i+1);
   b(i+1)= d7u(i+1);
   c(i+1)= du(i+1)-d7u(i+1);
   d(i+1)=(1/2)*(d2u(i+1)-d6u(i+1));
   e(i+1)=(1/6)*(d3u(i+1)+d7u(i+1));
   r(i+1)=(1/24)*(d4u(i+1)+d6u(i+1));
   z(i+1)=(1/120)*(d5u(i+1)-d7u(i+1));
   g(i+1)=u(i+1)+d6u(i+1);
   end
   err = zeros(1,n);
   disp(h)
   disp('length of x')
   disp(length(x))
   disp('length of y')
   disp(length(y))
   for k = 1:n
   err(k) = abs(u(k) - y(k));
   end
   format long
   disp('    i          X          Y          u          err')
   [1:n; x; y; u; err]'
```

```
figure(1)
plot(x, y, 'b --', 'linewidth', 3)
hold on
plot(x, u, 'r.-', 'linewidth', 1)
grid on
xlabel('X')
ylabel('Y')
title('exact vs. approx.')
legend('exact', 'approx.')
```

8. Numerical Examples and Figures

Example (2): Consider the VIE of second kind [10]:

$$u(x) = x + \int_0^x (t-x)u(t)dt \quad 0 \le x \le 1$$

Where $f(x) = x \ and \ kernel = t - x, with \ y = sinx$

Table 1 present a comparison between the exact and numerical solution using linear [13] and fifth order non-polynomial spline functions, where $u_i(x)$ denotes the approximate solution using non- polynomial spline functions, with $h = 0.1$.

Table 1. *Analytical and numerical solution of test example(2).*

x	Exact solution	$u_i(x)$	
		linear	Five order
0	0.00000000000000	0.00000000000000	0.00000000000000
0.1	0.09983341664683	0.09983341664683	0.09983341668651
0.2	0.19866933079506	0.19916808204436	0.19866933583213
0.3	0.29552020666134	0.29800648787140	0.29552028832715
0.4	0.38941834230865	0.39635111335875	0.38941890981745
0.5	0.47942553860420	0.49420442535136	0.47942806425312
0.6	0.56464247339504	0.59156887837027	0.56465102008993
0.7	0.64421768723769	0.68844691467410	0.64424167375464
0.8	0.71735609089952	0.78484096432036	0.71741482576318
0.9	0.78332690962748	0.88075344522641	0.78345650410261
1	0.84147098480790	0.97618676323008	0.84173434026604

Table 2: present a comparison between the error in our methods and other method] where error=|exact value - numerical

Table 2. *Comparison between the error with.*

x	Error in linear	Error in five order
0	0.00000000000000	0.00000000000000
0.1	0.00000000000000	0.00000000003968
0.2	0.00049875124930	0.00000000503707
0.3	0.00248628121006	0.00000008166582
0.4	0.00693277105010	0.00000056750880
0.5	0.01477888674716	0.00000252564892
0.6	0.02692640497523	0.00000854669489
0.7	0.04422922743640	0.00002398651695
0.8	0.06748487342084	0.00005873486365
0.9	0.09742653559892	0.00012959447513
1	0.13471577842218	0.00026335545814

Example (3): Consider the VIE of second kind [14]:

$$u(x) = 1 + \int_0^x (t-x)u(t)dt \quad 0 \le x \le 1$$

Where $f(x) = 1 \ and \ kernel = t - x, with \ y = cosx$

Table 3 present a comparison between the exact and numerical solution using linear [13] and five order non-polynomial spline functions, where $u_i(x)$ denotes the approximate solution using non- polynomial spline functions, with $h = 0.1$.

Table 3. *Exact and numerical solution of test example(3).*

x	Exact solution	$u_i(x)$	
		linear	Five order
0	1.00000000000000	1.00000000000000	1.00000000000000
0.1	0.99500416527803	0.99500416527803	0.99500416527803
0.2	0.98006657784124	0.99003328892062	0.98006657783728
0.3	0.95533648912561	0.98508724623992	0.95533648861880
0.4	0.92106099400289	0.98016591317098	0.92106098534563
0.5	0.87758256189037	0.97526916626867	0.87758249662003
0.6	0.82533561490968	0.97039688270455	0.82533529782125
0.7	0.76484218728449	0.96554894026384	0.76484101853443
0.8	0.69670670934717	0.96072521734230	0.69670315178005
0.9	0.62160996827066	0.95592559294323	0.62160056477445
1	0.54030230586814	0.95114994667438	0.54028001149101

Table 4: present a comparison between the error in our methods and other method in [10] where error=|exact value - numerical

Table 4. *Comparison between the error with.*

x	Error in linear	Error in five order
0	0.00000000000000	0.00000000000000
0.1	0.00000000000000	0.00000000000000
0.2	0.00996671107938	0.00000000000396
0.3	0.02975075711431	0.00000000050681
0.4	0.05910491916810	0.00000000865725
0.5	0.09768660437829	0.00000006527035
0.6	0.14506126779487	0.00000031708843
0.7	0.20070675297935	0.00000116875006
0.8	0.26401850799514	0.00000355756712
0.9	0.33431562467256	0.00000940349622
1	0.41084764080624	0.00002229437713

The Figures show the comparison between the analytical and the approximate solution using linear and fifth order non-polynomial spline functions for the examples.

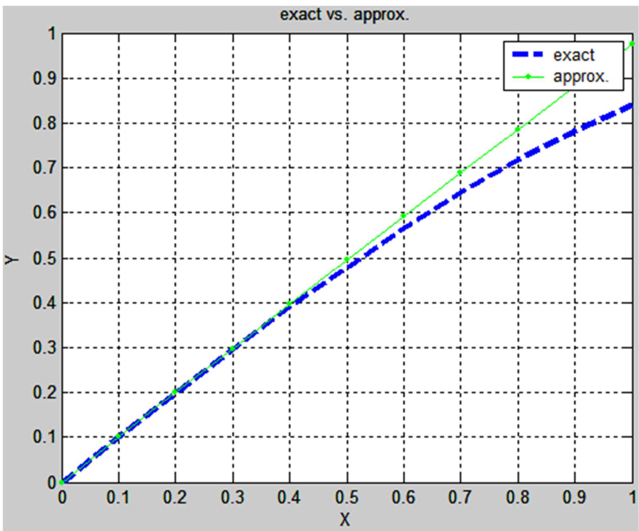

Figure 1. *Comparison between the analytical and the approximate solution using linear non-polynomial spline functions for example (2).*

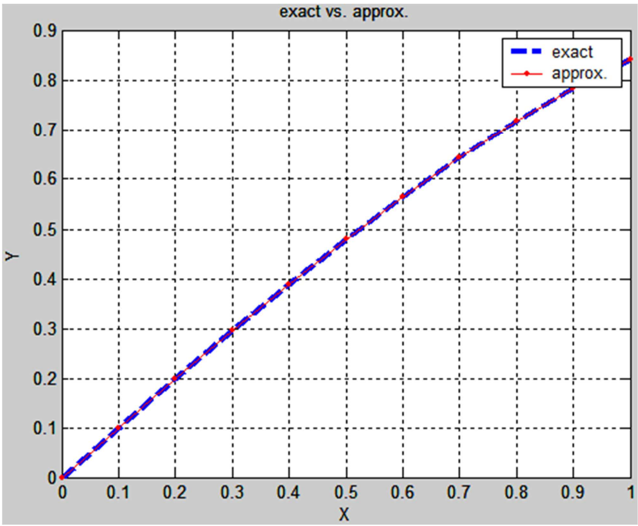

Figure 2. *Comparison between the analytical and the approximate solution using fifth order non-polynomial spline functions for example (2).*

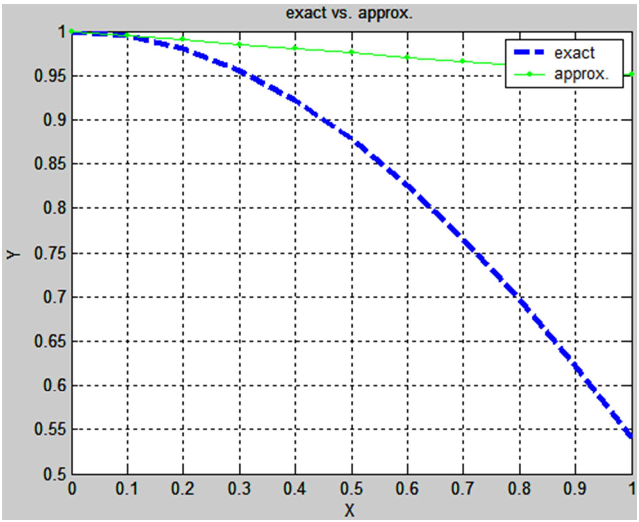

Figure 3. *Comparison between the analytical and the approximate solution using linear non-polynomial spline functions for example (3).*

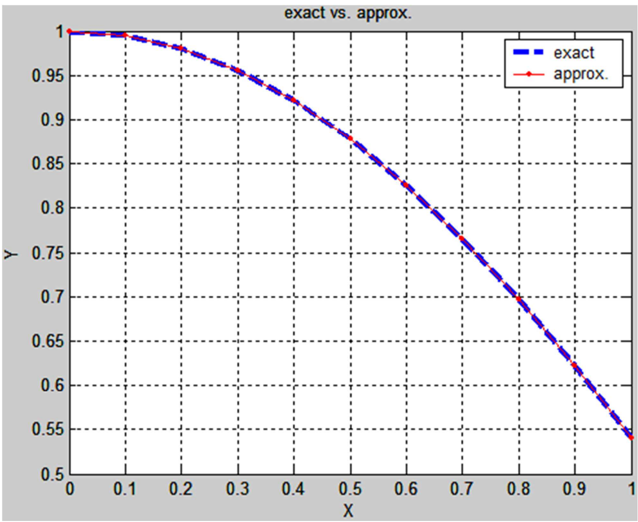

Figure 4. *Comparison between the analytical and the approximate solution using fifth order non-polynomial spline functions for example (3).*

9. Results

In general the methods which are used in this paper proved their effectiveness in solve linear Volterra Integral Equations of second kind numerically and finding an accurate results. The Figures show a comparison between the Adomain Decomposition and numerical solution which was presented in example (2) and example (3), the results show that the fifth order non-polynomial Spline functions gives the best approximation than the linear non-polynomial Spline functions.

10. Conclusion

Finally we found that the fifth order non- polynomial Spline functions characterized with an easy, fast and accuracy is high comparing with the less fifth order non-polynomial Spline functions and the results that we obtained show that the fifth order non-polynomial Spline functions gives the best approximation to solve the linear non-polynomial Spline functions. The proposed scheme is simple and computationally attractive and its accuracy is high and we can simply execute this method by Matlab.

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