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Blow-up of Solutions for Semilinear Timoshenko System with Damping and Source Terms

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Abstract: In this paper, we are concerned with one-dimensional Timoshenko model for a beam with nonlinear damping and source terms. We establish a blow-up result when the initial energy is positive and the initial data is not in a potential well. Under arbitrary positive initial energy, we also prove a finite-time blow-up result for a special case.

Keywords: Timoshenko System, Source Term, Damping Term, Blow-up

1. Introduction

In this paper, we study the semilinear Timoshenko system

$$u_{tt} - u_{xx} + k(u + v_x) + |u_t|^{p-1} u_t = f_1(u, v), \tag{1}$$

$$v_{tt} - k(u + v_x)_x + |v_t|^{q-1} v_t = f_2(u, v), \tag{2}$$

in $(0,1)\times(0,\infty)$, under the following boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \ge 0, \tag{3}$$

$$v(0,t) = v(1,t) = 0, \quad t \ge 0, \tag{4}$$

and initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,1), \tag{5}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x), \quad in \quad (0,1), \tag{6}$$

The function u is the rotation angle of a filament of the beam and v is the transverse displacement of the beam, k is a strict positive constant. The nonlinear function $f_1(u,v)$ and $f_2(u,v)$ act as strong source terms, $p,q \ge 1$. For the corresponding linearized system of (1)-(2)

$$u_{tt} - u_{xx} + k(u + v_x) = 0, \tag{7}$$

$$v_{tt} - k(u + v_x)_x = 0, \tag{8}$$

which is given by Timoshenko [1] as a simple model describing vibration of a beam, this model for Timoshenko beams have attracted vast interest during the last thirty years. Systems (7)-(8) has been studied by many authors and results concerning existence and asymptotic behavior have been established. The stabilization of the Timoshenko system has been studied with different type of damping, we refer the reader to [2, 3, 4, 5, 6] and their references.

Let us mention some known results for semilinear Timoshenko system. Parente et al [7] treated the existence and uniqueness for the problem

$$u_{tt} - u_{xx} + k(u + v_x) + f(u) = 0, \tag{9}$$

$$v_{tt} - k(u + v_x)_x + g(v) = 0, \tag{10}$$

with differential boundary conditions, where f, g are Lipschitz continuous functions. Araruna et al [8] investigated the existence and uniqueness of strong and weak solution of the one-dimensional Timoshenko model (9)-(10) for beams with a nonlinear external forces and a boundary damping mechanics. They also proved that the energy of solution decays exponentially. Chueshov and Lasiecka [9] studied the existence of a compact global attractor for (9)-(10) with nonlinearities of f and g being locally Lipschitz in the 2-dimensional case. Gorgi and Vegni [10] gave the uniform energy estimate and the estimate of an absorbing set for the Timoshenko beam with memory and Dirichlet boundary condition. Messaoudi and Soufyane [6, 11] established a general decay result for a nonlinear Timoshenko system with a boundary control of memory type

$$\rho_1(x)u_{tt} = \Delta u + \alpha\sum_{i=1}^{n}\frac{\partial v}{\partial x_i} - \beta u - \alpha(x)f_1(u,v), \tag{11}$$

$$\rho_2(x)v_{tt} = \Delta v + \alpha\sum_{i=1}^{n}\frac{\partial u}{\partial x_i} - \alpha(x)f_2(u,v), \tag{12}$$

However, there has been less focus on the Timoshenko system with nonlinear source terms. Recently, Pei et al [12, 13] studied the global well-posedness and long-term behavior of the Reissner Mindlin-Timoshenko plate systems, focusing on the interplay between nonlinear viscous damping and source terms, by the potential well framework [14, 15]. To the best of our knowledge, the system of nonlinear Timoshenko equation have not been well studied.

In this paper, we consider the blowup of the solutions of problem (1)-(6). We give an equivalent inequality between

$$\| u_x \|^2 + k\| u + v_x \|^2$$

and the standard norm on the function space $H_0^1 \times H_0^1$, then we obtain local existence of solution of problem (1)-(6) following very carefully the techniques used in [16]. We prove that the solutions blow up in finite time if when the initial energy is positive and the initial data is not in a potential well. The main tool of the proof is a technique introduced by paper [17] and some estimates used firstly by Vitillaro [18], in studying a class of a single wave equation. This tool has been used by many paper to deal with the global existence and blow-up of solutions to some nonlinear hyperbolic systems with damping and source terms in a bounded domain, for example see [19, 20]. Secondly, we extend the result of [21], established for the Klein-Gordon equation, to our problem, and prove a finite-time blow-up result for $p = q = 1$ for problem (1)-(6) under arbitrary positive initial energy.

2. Preliminaries

Throughout this paper, we denote $L^p(0,1)$ and $H_0^1(0,1)$ by L^p and H_0^1, respectively. $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual L^2 norm and L^p norm, respectively. And let us define

$$(\varphi,\psi) = \int_0^1 \varphi(x)\psi(x)dx$$

as the usual L^2 inner product. The standard duality between $(H^1)'$ and H^1 will be denote also by $(\cdot,\cdot)$. For $\varphi \in H_0^1$, sometimes we write equivalent norm $\|\varphi_x\|$ instead of H_0^1 norm

$$\| \varphi \|_{H_0^1}^2 = \| \varphi_x \|^2 + \| \varphi_x \|^2$$

Let V denotes the following Hilbert space $V = H_0^1 \times H_0^1$, and endowed with the following norm $\| U \|_V^2 = \| u_x \|^2 + \| v_x \|^2$, for $U = (u,v) \in V$. Throughout this paper, $C, C_1, C_2, \ldots$ are positive generic constants, which may be different in various occurrences. In addition, we denote C_* is the Poincare constants, that is, for $u \in H_0^1$

$$\| u \|_s \le C_* \| u_x \|, 2 \le s \le +\infty. \tag{13}$$

Concerning the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$, we assume that

$$f_1(u,v) = a\,|u+v|^{2(\rho+1)}(u+v) + b\,|u|^{\rho}\,u\,|v|^{\rho+2}, \tag{14}$$

$$f_2(u,v) = a\,|u+v|^{2(\rho+1)}(u+v) + b\,|v|^{\rho}\,v\,|u|^{\rho+2}, \tag{15}$$

where $a,b > 0$ and $\rho \ge 0$ are constants. It is easy to see that

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v), \tag{16}$$

for any $(u,v) \in R^2$, where $F(u,v) = \frac{1}{2(\rho+2)}[a\,|u+v|^{2(\rho+2)} + b\,|uv|^{\rho+2}]$. Moreover, a quick computation will show that there exist two positive constants C_0 and C_1 such that the following inequality holds (see [19, 20])

$$C_0(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}) \le 2(\rho+2)F(u,v) \le C_1(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}). \tag{17}$$

Now, we give the following lemmas which will be used later. By a simple computation, we have the following result:

Lemma 2.1 For $(u,v) \in H_0^1$, there exist positive constants $\alpha_1 > 0, \alpha_2 > 0$ such that the inequality holds

$$\alpha_1(\| u_x \|^2 + k\| u + v_x \|^2) \le \| u_x \|^2 + \| v_x \|^2 \le \alpha_2(\|u_x\|^2 + k\|u + v_x\|^2), \tag{18}$$

where

$$\alpha_2 = max\{1 + 2C_*), \tfrac{2}{k}\}, \tfrac{1}{\alpha_1} = max\{(1 + 2kC_*), 2k\}.$$

Now, we state the local existence of the problem (1)-(6) and the proof follows very carefully the techniques used in [16].

Lemma 2.2 (Local existence) Assume that the assumptions (14)-(18) hold. Suppose further that $p,q > 1, \rho > 0$, then for any initial data $u_0, v_0 \in H_0^1$ and $u_1, v_1 \in L^2$, there exists a local weak solution (u,v) of problem (1)-(6) defined in

$[0,T_0]$ for some $T_0>0$ and satisfies the energy identity

$$E(t)+\int_0^t (\| u_s(s)\|_{p+1}^{p+1}+\| v_s(s)\|_{q+1}^{q+1})ds = E(0), \quad (19)$$

where $E(t)$ is defined by

$$E(t)=\tfrac{1}{2}(\| u_t\|^2+\| v_t\|^2)+\tfrac{1}{2}\| u_x\|^2$$

$$+\tfrac{k}{2}\| u+v_x\|^2-\int_0^1 F(u,v)dx, \quad (20)$$

And

$$E(0)=\tfrac{1}{2}(\| u_1\|^2+\| v_1\|^2)+\tfrac{1}{2}\| u_{0x}\|^2$$

$$+\tfrac{k}{2}\| u_0+v_{0x}\|^2-\int_0^1 F(u_0,v_0)dx. \quad (21)$$

It follows from Theorem 2.2 that

$$\tfrac{d}{dt}E(t)=-\| u_t\|_{p+1}^{p+1}-\| v_t\|_{q+1}^{q+1}, \quad (22)$$

then the energy function $E(t)$ is a nonincreasing function.

Lemma 2.3 There exist $\eta>0$ such that for any $(u,v)\in H_0^1\times H_0^1$, the following inequality holds

$$\| u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\| uv\|_{2(\rho+2)}^{2(\rho+2)}\le \eta(\| u_x\|^2+\| v_x\|^2)^{\rho+2}$$

$$\le \eta\alpha_2^{\rho+2}(\| u_x\|^2+k\| u+v_x\|^2)^{\rho+2}. \quad (23)$$

Proof A combination of the following inequality [17]

$$\| u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\| uv\|_{2(\rho+2)}^{2(\rho+2)}\le \eta(\| u_x\|^2+\| v_x\|^2)^{\rho+2}$$

with Lemma 2.1 yields (23).

3. Blow-up of Solution

In this section, we deal with the blow-up of the solution of problem (1)-(6). Firstly, based on the technique introduces by paper [17] and some estimates used firstly by Vitillaro [18], in studying a class of single wave equation, we establish a blow-up result when the initial energy is positive and the initial data is not in a potential well. Secondly, we extend the result of [21], established for the Klein-Gordon equation, to our problem. Under arbitrary positive initial energy, we prove a finite-time blow-up result for $p=q=1$ for problem (1)-(6).

For the sake of simplicity, we set $a=b=1$. Let

$$B=(\eta\alpha_2^{\rho+2})^{\frac{1}{2(\rho+2)}},\quad \beta_1=B^{\frac{\rho+2}{\rho+1}},$$

$$E_1=(\tfrac{1}{2}-\tfrac{1}{2(\rho+2)})\beta_1^2. \quad (24)$$

where η is the optimal constant in (23).

Lemma 3.1 Let (u,v) be a solution of problem (1)-(6), $E(0)<E_1$, and

$$\| u_{0x}\|^2+k\| u_0+v_{0x}\|^2\ge \beta_1^2, \quad (25)$$

then there exist a constant $\beta_2\ge\beta_1$, such that

$$\| u_x\|^2+k\| u+v_x\|^2\ge \beta_2^2, \quad (26)$$

$$\| u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\| uv\|_{2(\rho+2)}^{2(\rho+2)}\le (B\beta_2)^{2(\rho+2)}. \quad (27)$$

Proof. By the definition of $E(t)$, (14)-(18), and the definition of B, we have

$$E(t)\ge \tfrac{1}{2}(\| u_x\|^2+k\| u+v_x\|^2)$$

$$-\tfrac{1}{2(\rho+2)}(\| u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\| uv\|_{2(\rho+2)}^{2(\rho+2)})$$

$$\ge \tfrac{1}{2}(\| u_x\|^2+k\| u+v_x\|^2)$$

$$-\tfrac{1}{2(\rho+2)}\eta\alpha_2^{\rho+2}(\| u_x\|^2+k\| u+v_x\|^2)^{\rho+2}$$

$$=g(\sqrt{\| u_x\|^2+k\| u+v_x\|^2}), \quad (28)$$

where $g(s)=\tfrac{1}{2}s^2-\tfrac{1}{2(\rho+2)}\eta\alpha_2^{\rho+2}s^{2(\rho+2)}$. Then we can get the result by using the proof of Lemma 3.3 in [17] word for word.

Now, we state the result of blow-up of solution.

Theorem 3.2 Assume that

$$2(\rho+2)>\max\{p,q\}+1,$$

then any solution of problem (1)-(6) with initial data satisfying

$$\| u_{0x}\|^2+k\| u_0+v_{0x}\|^2\ge \beta_1^2$$

and $E(0)<E_1$, cannot exist for all time.

The proof is similar to that in [17], so we give only the result here.

Now, we prove a finite-time blow-up result for $p=q=1$ for problem (1)-(6) under arbitrary positive initial energy.

Lemma 3.3 [21] Suppose that $\Phi(t)\in C^2([0,T))$ satisfies

$$\Phi\Phi_{tt}-\alpha\Phi_t^2+r\Phi\Phi_t+\beta\Phi\ge 0, \quad (29)$$

and

$$\Phi(t)\ge 0,\Phi(0)>0,\Phi_t(0)>\tfrac{r}{\alpha-1}\Phi(0),$$

$$(\Phi_t(0)-\tfrac{r}{\alpha-1}\Phi(0))^2>\tfrac{2\beta}{2\alpha-1}\Phi(0), \quad (30)$$

Then, there exist $T_0>0$ such that

$$\limsup_{t\to T_0}\Phi(t)=+\infty$$

and the estimate of the life span of the solution is given by $T_0\le\Phi^{1-\alpha}(0)A^{-1}$, where

$$A^2=(\alpha-1)^2\Phi^{-2\alpha}(0)\times[(\Phi_t(0)-\tfrac{r}{\alpha-1}\Phi(0))^2-\tfrac{2\beta}{2\alpha-1}\Phi(0)], \quad (31)$$

Theorem 3.4 Assume that $p=q=1$ in equations (1) and (2). If

$$E(0)>0, \Phi_t(0)>\tfrac{2}{\rho+1}\Phi(0)>0,$$

$$[\Phi_t(0)-\tfrac{2}{\rho+1}\Phi(0)]^2>4E(0)\Phi(0),$$

where

$$\Phi(t)=\|u\|^2+\|v\|^2.$$

Then there exist $T_0\le\Phi^{\frac{-(\rho+1)}{2}}(0)A^{-1}$, such that $\limsup_{t\to T_0}\Phi(t)=+\infty$, where A is defined in (31).

Proof We first multiply both sides of equation (1) and (2) by u and v, respectively, and integrate over $(0,1)$, then summing up and integrating by parts, we obtain the following equality

$$\tfrac{1}{2}\Phi_{tt}(t)-G(t)+\|u_x\|^2+k\|u+v_x\|^2$$

$$=\int_0^1[uf_1(u,v)+vf_2(u,v)]dx, \quad (32)$$

where $G(t)=\|u_t\|^2+\|v_t\|^2$.

Similarly, multiplying both sides of equation (1) and (2) by u_t and v_t, respectively, summing up and integrating by parts, we obtain the following equality

$$\tfrac{1}{2}\tfrac{d}{dt}(G(t)+\|u_x\|^2+k\|u+v_x\|^2)+G(t)$$

$$=\int_0^1[u_tf_1(u,v)+v_tf_2(u,v)]dx, \quad (33)$$

Integrating (33) over $(0,t)$ and noting $G(t)>0$, we arrive that

$$\tfrac{1}{2}(G(t)+\|u_x\|^2+k\|u+v_x\|^2)$$

$$\le\int_0^1F(u,v)dx-\int_0^1F(u_0,v_0)dx,$$

and then, combining the expression of $E(0)$, we have

$$\tfrac{1}{2}(G(t)+\|u_x\|^2+k\|u+v_x\|^2)-E(0)\le\int_0^1F(u,v)dx. \quad (34)$$

By (17), combining (32) and (34), we get

$$(\rho+2)G(t)+(\rho+2)(\|u_x\|^2+k\|u+v_x\|^2)-2(\rho+2)E(0)$$

$$\le\tfrac{1}{2}\Phi_{tt}-G+\tfrac{1}{2}\Phi_t+\|u_x\|^2+k\|u+v_x\|^2. \quad (35)$$

Thus (35) yields the following differential inequality

$$\tfrac{1}{2}\Phi_{tt}-G(t)+\tfrac{1}{2}\Phi_t+2(\rho+2)E(0)\ge(\rho+3)G(t). \quad (36)$$

From Cauchy-Schwartz inequality, we have

$$(\Phi_t)^2\le4G(t)\Phi_t. \quad (37)$$

Multiplying (36) by $\Phi(t)$ (t) and using (37), we obtain

$$\Phi\Phi_{tt}-\tfrac{\rho+3}{2}(\Phi_t)^2+\Phi\Phi_t+2(\rho+2)E(0)\Phi\ge0. \quad (38)$$

Comparing this differential inequality with (29), we easily see that for

$$\alpha=\tfrac{\rho+3}{2}>1, r=1, \beta=2(\rho+2)E(0),$$

$$\tfrac{2\beta}{2\alpha-1}=4E(0), \tfrac{r}{\alpha-1}=\tfrac{2}{\rho+1}$$

as in (30), the time $T>0$ is bounded above by

$$T_0\le\Phi^{\frac{-(\rho+1)}{2}}(0)A^{-1}, \quad A^2=(\tfrac{\rho+1}{2})^2\Phi^{-(\rho+3)}(0)\times$$

$$[(\Phi_t(0)-\tfrac{2}{\rho+1}\Phi(0))^2-4E(0)\Phi(0)] \quad (39)$$

For

$$\Phi_t(0)=2\int_0^1(u_0u_1+v_0v_1)dx,$$

$$\Phi(0)=\|u_0\|^2+\|v_0\|^2.$$

Thus we have the desired results.

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