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A New Exact Solution of One Dimensional Steady Gradually Varied Flow in Open Channels

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Abstract: One dimensional steady gradually varied flow in open channels is of academic and practical importance. Ita been studied for various applications and in various contexts since the 19th Century. There several classes of gradually varied flow; i.e., one or more dimensions, steady and transient flows. Gradually varied flow may occur in several channel geometries comprising rectangular, trapezoidal, parabolic bottom surfaces and diverse configurations: simple channels, compound channels, and channel networks. The wide rectangular channel case is of particular interest in its own right, as well as serving as a validation benchmark for transient, and multiple dimensional gradually varied flow, the latter normally solved by numerical techniques and therefore requiring calibration. In this paper, a new exact analytical and easy to compute solution is developed. It is shown that this solution possesses the ease of computation as an advantage in comparison with existent exact solutions reported in the literature. As this solution involves a multiple valued function, it is consistent with the nonuniqueness propert of the intial value problem of one dimensional steady gradually varied flow.

Keywords: Gradually Varied Flow, Open Channels, Steady One Dimensional, Exact Solution

1. Introduction

Gradually varied flow (GVF) in open channels has been studied since the 19th Centuty, both for its academic and practical importance. GVF constitutes one of the the major types of fluid flow in open channels; cf. [1-3]. An extensive literature exists on various classes of GVF; e.g., transient, steady, one dimensional, two dimensional, three dimensional, and combinations thereof, as well sediment transport effects and fluid kinetic conversion to electrical power; cf. [4-6] for recent work on open channel work.

This paper is concerned with one class of GVF; i.e., one dimensional steady GVF. This class of GVF is important both in its own right in that it provides a good representation of some open channel flows in practice, as well as constituting a benchmark for the validation of models of more complex GVF classes involving transient and multiple dimensional characteristics, where numerical methods are required [7]. The contribution of the work reported in this paper is that a new exact analytical and easy to compute solution is developed for one dimensional steady GVF, where channel wall and bottom friction is defined by the Manning equation [8].

The paper is organised as follows. After the Introduction Section, the differential equation of one dimensional steady GVF is presented in Section 2, along with a review of existent exact analytical solutions. This is followed, in Section 3, with the development of the new exact analytical solution which constitutes the contribution of this work. The paper is concluded in Section 4 with a comparison of the solution presented in this paper with existent exact solutions reported in the literature, as well as providing suggestions for future research in this area.

2. One Dimensional Steady State GVF in Wide Open Channels

One dimensional steady GVF may be described by the ordinary differential equation as [1-3]

$$dh/dx = (I_S – I_E)/(1 – F_r^2) \quad (1)$$

notation employed in Eq. (1) and the rest of the paper being provided in Appendix 1. The friction slope, I_E, depends on the channel geometry and the physical characteristics of the channel side walls and bottom surface. There are two major constitutive equations for I_E [9]: the Chezy and the Manning equations, whose respective applicability in practice is compared in [9]. In this work, the following assumptions are made:

1) Constant fluid volumetric flow rate;

2) The Manning equation is assumed for the channel friction, in view of the general consensus in research work that the Manning equation provides better agreement with experiment than the Chezy equation;

3) A wide open channel is considered; i.e., fluid level is very small in comparison with channel width.

Prior to development of the new exact solution, the one dimensional steady equation of gradually varied flow in an open channel is put into adequate form for solution development. As stated in the last paragraph, the one dimensional steady GVF equation in an open channel may be written as

$$dh/dx = (I_S – I_E)/(1 – Fr^2) \quad (1)$$

For a wide rectangular channel of uniform cross section, b >> h. Then, the hydraulic radius for a wide channel may be written as

$$Rh_y = bh/(b + 2h) \cong h \quad (2)$$

By definition of the Froude number, may be written as

$$Fr^2 = v^2/(gh) \quad (3)$$

The mass balance may be written as

$$v = q/h \quad (4)$$

Using Eq. (4), then Eq. (3) may be rewritten as

$$Fr^2 = q^2/(gh^3) \quad (5)$$

The Manning equation for the friction slope may be written as [1]

$$I_E = n^2 Q^2 P^{4/3}/A^{10/3} \quad (6)$$

and as A = b h, Eq.(6) may be rewritten as

$$I_E = n^2 Q^2 P^{4/3}/(h^{10/3} b^{10/3}) \quad (7)$$

which may be conveniently rewritten as

$$I_E = C_1 h^{-10/3} \quad (8)$$

where

$$C_1 = n^2 Q^2 P^{4/3}/b^{10/3} \quad (9)$$

Using Eqs. (5) and (7), Eq.(1), may be rewritten as

$$dh/dx=(I_S – C_1 h^{-10/3}/(1–C_2 h^{-3}) \quad (10)$$

where

$$C_2 = q^2/g \quad (11)$$

Eq. (10) is the starting point for the development of the new exact solution.

3. New Exact Solution Development

In the development of the exact solution of Eq. (10) to be presented in this subsection, either x or h may considered as the dependent variable, and the other as the independent variable. Since the right side of Eq. (10) is already written as a function of h, x is chosen as the dependent variable. Thus, Equation (10) may be rewritten as

$$dx = ((1 – C_2 h^{-3})/(I_S – C_1 – h^{-10/3})) dh \quad (12)$$

and separating the two terms on the right side of Eq. (12), we obtain

$$dx=dh/(I_S–C_1h^{-10/3})–C_2 h^{-3} dh/(I^S h^3 – C_1) \quad (13)$$

Integration of Eq. (13) yields

$$\int dx=\int dh/(I_S – C_1h^{-10/3}) –\int C_2 h^{-3} dh/(I_S h^3 – C_1) + C_0 \quad (14)$$

where in Equation (14), there are three indefinite integrals and C_0 is a constant of integration. For convenience, Eq. (14) may be rewitten as

$$\int dx = C_1\text{-}1 \int dh/(K_1 – h^{-10/3}) – K_2 \int h^{-3} dh/(K_1 – h^{-10/3}) + C_0 \quad (15)$$

The indefinite integrals on the right side of Eq. (15) may be evaluated as described in the next two subsections.

3.1. First Definite Integral Evaluation

Recognising the equivalence of $C_1\text{-}1\int dh/(K_1 – h^{-10/3})$ in Eq. (15) and $\int dy/(a^3 – y^3)$ in Eq. (A.2.1), given in Appendix 2, which is based on [10], and putting $a = K_1^{1/3}$ and $y = h^{-10/9}$, Eq. (15) is obtained

$$C_1\text{-}1 \int dh/((K_1^{1/3})^3 – (h^{10/9})^3) = C_1\text{-}1/(6K_1^{2/3})$$

$$\ln ((K12/3+K11/3h\text{-}10/9+h\text{-}20/9)/$$

$$(K_1^{1/3}–h^{-10/9})^2)+$$

$$((C_1/K_1^{2/3} \sqrt{3})) \arctan$$

$$((2 h^{-10/9} + K_1^{1/3})/$$

$$(K_1^{1/3} \sqrt{3})) \quad (16)$$

3.2. Second Definite Integral Evaluation

Multiplying the numerator and the denominator by $-h^{10/3}$, the second definite integral in Eq. (15) may be rewritten as

$$K_2 \int h^{1/3} dh/(K_1 h^{10/3} – 1) \quad (17)$$

and recognising the equivalence of the indefinite integral (17) and $\int y^m (ay^n + b)^p$ in Eq.(A2.2), given in Appendix 2, which is based on [10], and putting y = h, a = K_1, b = -1, p = -1, m = 1/3, n = 10/3, w = ay^n + b, indefinite integral (17) may be rewitten as

$$K_2 \int h^{1/3}\, dh/(K_1 h^{10/3} - 1) = K_2/K_1 (1 - 2 h^2) \int dh/(K_1 h^{19/3} - h^3) \tag{18}$$

Recognising the equivalence of $\int dh/(K_1 h^{19/3} - h^3)$ and $\int dy/(y(ay^n + b)$, and using Eq. (A2.3) with y = h^3, b = -1, n = 10/3, a = K_1, we may write

$$\int dh/(h^3 (K_1 h^{10/3} - 1) = -3/10 \ln| h^{10}/(K_1 h^{10} - 1)|,$$

and the right side of Eq. (18) may be rewritten as

$$K_2/K_1 (1- 2h^2) \int dh/K_1 h^{10/3} – h^3) = K_2/K_1(1–2h^2)– 3/10\ln|h10/(K_1h^{10}–1)| \tag{19}$$

3.3. Final Form of New Exact Solution

Using Eq. (16) and (19) in Eq. (15), the final form of the new exact solution may be writtten a

$$x = f_1(h) + f_2(h) + C_0 \tag{20}$$

where $f_1(h)$ and $f_2(h)$ are given by Eqs. (16) and (19), respectively.

4. Conclusion

It may be observed from Eq. (20), along with Eqs. (16) and (19) that the exact solution developed in this work is easy to compute. The only existent exact solutions in the literature for one dimensional steady GVF in wide open channels are those developed in [8, 9]. In order to facilitate comparison with the exact solution developed in this paper (Eqs. (15), (16), and (19)), the aforesaid solutions are transcribed in full in Appendices 3 and 4. There are two points which distinguish the exact solution presented in this paper from the exact solutions presented in [11] and [12]. The exact solution presented in this paper is the sum of four explicit algebraic terms (Eqs. (16) and (19)) which are easy to compute. The exact solutions presented in [8] and [9] involve the sum of a significantly larger number of explicit algebraic terms; cf. Eqs. (A3.1) - (A3.2) in Appendix 3 and Equation (A4.2) in Appendix 4. This distinguishing feature means that the determination of the constant parameter in each of the three exact solutions by applying initial conditions may be much more complicated in the solutions presented in [11] and [12] in comparison with that presented in this paper.

All three exact solutions, those presented in [11] and [12] and that presented in this paper, possess terms which include the function arctan, which is a multiple valued function. This is in full accordance with the nonuniqueness of the solution of the initial value problem of the one dimensional steady GVF equation, in view of the fact that the aforesaid problem does not satisfy the Lipschitz condition; cf. [13].

In practice, as is the case with natural rivers and estuaries, channel width varies along its length. A natural extension of the work reported in this paper is the search for exact solutions to the one dimensional steady GVF equation for a rectangular open channel with varying width with length along the channel. Another research direction of importance in practice is the the consideration of one dimensional steady GVF in recatngular open channels with varying volumetric flowrate due to rainfall and seepage.

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Appendix

Appendix 1. Notation

A – channel cross sectional area
b – channel width
C_0, C_1, C_2, K_1, K_2 – constants
f_1, f_2 – functions
Fr – Froude number
h – fluid level
g – gravitational acceleration
I_E – friction slope
I_S – channel slope
n – Manning friction coefficient
P - channel wetted perimeter
q – fluid volumetric flowrate per unit channel width
Q – fluid volumetric flowrate
R_{hy} – channel hydraulic radius
v – fluid velocity
x – longitudenal distance

Appendix 2. Indefinite Integral Formulae

Indefinite integral formulae are presented, employing [10] as a basis.

1. From [7], Section T2.1.1-6, p.1131, Eq. (1) may be written as

$$\int dx/(a^3 – y^3) = 1/(6a^2) \ln((a^2 + ay + y^2) + 1/(a^2\sqrt{3}) \arctan ((2y + a)/(a\sqrt{3})). \tag{A2.1}$$

2. From [7], section T2.1.2-6, p.1136, Eq. (8) may be written as

$$\int x^m (ay^n + c)^p\, dy = 1/(a(m +np + 1))$$

$$(y^{m-n}+1\, w^{p+1} - c(m-n+1)\int y^{m-n}\, w^p\, dy), \qquad (A2.2)$$

where $w = ay^n + c$.

3. From [7], section T2, 1.2-6, p.1136, Eq. (1) may be written as

$$\int dx/(x(ax^n + c)) = (1/(c^n))\ \ln|y/(ay^n + c)|. \qquad (A2.3)$$

Appendix 3. Venutelli [8] Exact Solution

The exact solution developed in [7] is presented in the notation employed therein.

$$x=yn/S0\ (\eta - 3F + 3\eta c3G) + \text{constant}, \qquad (A3.1)$$

where

$$F=\alpha_1(\arctan f_1 + \arctan f_4)-\alpha 2(\arctan f_2 + \arctan f_3) + (1/40)\ (\beta_1(\ln|z_1|-\ln|z_4|-\beta 2(\ln|z_2|-\ln|z_3|))-(1/10)\ (\ln|\eta^{1/3} - 1| - \ln|\eta^{1/3} + 1|), \qquad (A3.2)$$

$$G=\alpha_2(\arctan f_1-\arctan f_4)-\alpha_1(\arctan f_2-\arctan f_3) + (1/40)(\beta_2(\ln|z_1|+\ln|z_4|)+\beta_1\ (\ln|z_2|+\ln|z_3|)) - (1/10)\ (\ln|\eta^{1/3} - 1| + \ln|\eta^{1/3} + 1|), \qquad (A3.3)$$

$$\alpha_{1,\,2} = (1/10)\ ((1/2)\ (5 +/-\sqrt{5}),$$

$$\beta_{1,\,2} = 1 -/+\sqrt{5},$$

$$f_{1,\,4} = 2(2/(5-\sqrt{5})\ (\eta 1/3-/+(1/4)\beta_2),$$

$$f_{2,\,3} = 2(2/(5-\sqrt{5})\ (\eta 1/3+/-(1/4)\beta_1),$$

$$z_{1,\,4} = 1 +/-(1/2)\beta_2\eta^{1/3} + \eta^{2/3},$$

$$z_{2,\,3} = 1 -/+(1/2)\beta_1\eta^{1/3} + \eta^{2/3}.$$

Appendix 4. Vatankhah [9] Exact Solution

The exact solution developed in [8] is presented in the notation employed therein.

$$- S_0\ x/y = I(\eta,\delta) = \text{constant}, \qquad (A4.1)$$

where

$I(\eta,\delta) = -1/\eta^3 + (3/10)\ (\delta-1)\ln|1-\eta| +$
$(3/10)\ (\delta+1)\ln(1+\eta)+(3/40)\ (A_+-\delta A_-)\ \ln(2\eta^2+\eta A_-+2) +$
$(3/40)\ (A_--\delta A_+)\ln(2\eta^2+\eta A_++2) -$
$(3/40)\ (A_++\delta A_-)\ln(2\eta^2-\eta A_-+2) -$
$(3/40)\ (A_-+\delta A_+)\ln(2\eta^2-\eta A_++2) +$
$(3/20)\ (B_--\delta B_+)\ \arctan((4\eta+A_-)/B_+) -$
$(3/20)\ (B_++\delta B_-)\ \arctan((4\eta+A_+)/B_-) +$
$(3/20)\ (B_-+\delta B_+)\ \arctan((4\eta-A_-)/B_+) -$
$(3/20)\ (B_+-\delta B_-)\ \arctan((4\eta-A_+)/B_-)$, (A4.2)

where

$A+ = 1+\sqrt{5}$,
$A_- = 1-\sqrt{5}$,
$B_+ = \sqrt{(10+2\sqrt{5})}$,
$B_- = \sqrt{(10-2\sqrt{5})}$

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