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A Lot of Examples of Generalized Weak Bi-Frobenius Algebras

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Abstract: In this paper, by considering the tensor product of a bi-Frobenius algebra and a weak Hopf algebra, a lot of examples of the generalized weak bi-Frobenius algebras are given, such as the 16-dimensional, 24-dimensional and 40-dimensional GWBF algebras. They provide a common generalization of weak Hopf algebras introduced by Böhm, Nill, Szlachányi, and of bi-Frobenius algebras introduced by Doi and Takeuchi.

Keywords: Examples, Bi-Frobenius Algebras, Generalized Weak Bi-Frobenius Algebras

1. Introduction

Doi and Takeuchi axiomatized these above facts to obtain the notion of bi-Frobenius algebras in [1-2], which generalizes the notion of finite dimensional Hopf algebras. A bi-Frobenius algebra is a Frobenius algebra and a Frobenius coalgebra with a pair of right integrals $(\varphi,\ t)$ and an antipode- like anti-automorphism S. Further research on bi-Frobenius algebras could be found in [3-7].

In 2013, Chen and Wang [3] introduced the notion of a generalized weak bi-Frobenius algebra, which is a generalization of weak Hopf algebras [8-10] and bi-Frobenius algebras [2]. They investigated some properties of various modular elements of generalized weak bi-Frobenius algebras [11-13]. They also provided some examples of generalized weak bi-Frobenius algebras. However, their examples are in fact face algebras or the direct products of a weak Hopf algebra and a bi-Frobenius algebra [13-15]. So there still lacks non-trivial examples. This is the motivation of this manuscript.

The general organization of this paper is as the following. In Section 1, some notions of generalized weak bi-Frobenius algebras are recalled. In Section 2, some non-trivial examples of generalized weak bi-Frobenius algebras are given.

2. Preliminaries

In this section the basic definitions related to the paper are recalled.

Let $(H,\ m,\ \eta)$ be a finite dimensional algebra with unit 1 and coalgebra with counit ε, $0 \neq \psi \in H^*$. There is a bijective map $S: H \to H$ satisfying, for all $h,\ g \in H$,

$$\psi\,(hg_1)\,S\,(g_2) = \psi\,(h_1 g)\,h_2$$

The data $(H,\ \psi,\ S)$ is called a generalized weak biFrobenius algebra (or GWBF-algebra) if the following conditions hold:

Unit 1 satisfies the following identities:

$$(\Delta \otimes id)\Delta\,(1) = (\Delta\,(1) \otimes 1)(1 \otimes \Delta\,(1)) = (1 \otimes \Delta\,(1))(\Delta\,(1) \otimes 1);$$

Counit ε satisfies the following identities:

$$\varepsilon\,(fgh) = \varepsilon\,(fg_1)\,\varepsilon\,(g_2 h) = \varepsilon\,(fg_2)\,\varepsilon\,(g_1 h);$$

The pair $(H,\ \psi)$ is a Frobenius algebra, i.e.,

$$\psi' H = H^*;$$

The map S is an anti-algebra map, i.e.,

$$S\,(hg) = S\,(g)S\,(h),\ \ S\,(1) = 1;$$

The map S is an anti-algebra map, i.e.,

$$\Delta\left(S(h)\right) = \sum S\,(h_2) \otimes S\,(h_1),\ \ \varepsilon\left(S\,(h)\right) = \varepsilon(h).$$

For all $f,\ g,\ h \in H$.

Let $(H,\ \psi,\ S)$ be a GWBF-algebra. By a *GWBF-subalgebra* it means that a subalgebra and subcoalgebra K with $S\,(K) \subset K$ and $(K,\ \psi',\ S|_K)$ becomes a GWBF-algebra for some $\psi' \in K^*$.

3. The Examples of GWBF Algebra

In this section some examples of generalized weak biFrobenius algebra are given.

3.1. The 16-Dimensional GWBF Algebra

Define A as the 16-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) X_{00} is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 1$ if $i = 1$ or $s = 1$; and $q = (j + t)$ mod8.

(3) The coproduct and counit on A are given by:

$$\Delta\,(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta\,(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta\,(X_{01}) = (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11};$$

$$\Delta\,(X_{11}) = X_{11} \otimes X_{10} + X_{10} \otimes X_{11};$$

$$\Delta\,(X_{02}) = (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12};$$

$$\Delta\,(X_{12}) = X_{12} \otimes X_{10} + X_{10} \otimes X_{12};$$

$$\Delta\,(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta\,(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta\,(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta\,(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta\,(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta\,(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta\,(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta\,(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta\,(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta\,(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\varepsilon\ (X_{00}) = 2, \varepsilon\,(X_{10}) = 1,\ \varepsilon\,(X_{ij}) = 0\ \text{(for } j \geq 1\text{)};$$

(4) $\psi\,(X_{07}) = \frac{3}{2},\ \psi l\,(X_{17}) = \frac{1}{2},\ \psi(X_{ij}) = 0$ (for $j \leq 6$);

(5) $S = id_A$;

Then $(A,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \psi,\ S)$ is a GWBF-algebra.

Let K be the subspace generated by $X_{00},\ X_{10},\ X_{02},\ X_{12},\ X_{04},\ X_{14},\ X_{06},\ X_{16}$. It is a subalgebra and subcoalgebra of A. Define $\phi \in K^*$ by $\phi(X_{06}) = \frac{3}{2},\ \phi(X_{16}) = \frac{1}{2},\ \phi(X_{ij}) = 0$(for $j = 0,\ 2,\ 4$). Then $(K,\ \phi,\ S|_K)$ is a GWBF-subalgebra of A.

3.2. The 24-Dimensional GWBF Algebra (Type 1)

Define B as the 24-dimensional associative algebra generated by a set $\{X_{ij} | i = 0, 1, 2, j = 0, 1, \cdots, 7\}$ with the following relations:

(1) X_{00} is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 2$ if $i = 2$ or $s = 2$, $p = 1$ for the other condition; and $q = (j + t)$ mod8.

(3) The coproduct and counit on A are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta(X_{20}) = (X_{10} - X_{20}) \otimes X_{20} + X_{20} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{01}) = (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11};$$

$$\Delta(X_{11}) = X_{11} \otimes X_{10} + X_{10} \otimes X_{11};$$

$$\Delta(X_{21}) = (X_{10} - X_{20}) \otimes X_{21} + (X_{11} - X_{21}) \otimes X_{20} + X_{20} \otimes (X_{11} - X_{21}) + X_{21} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{02}) = (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10} + (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12};$$

$$\Delta(X_{12}) = X_{12} \otimes X_{10} + X_{10} \otimes X_{12};$$

$$\Delta(X_{22}) = (X_{10} - X_{20}) \otimes X_{22} + (X_{12} - X_{22}) \otimes X_{20} + X_{20} \otimes (X_{12} - X_{22}) + X_{22} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta(X_{23}) = (X_{10} - X_{20}) \otimes X_{23} + X_{20} \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes X_{22} + X_{21} \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes X_{21} + X_{22} \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes X_{20} + X_{23} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta(X_{24}) = (X_{10} - X_{20}) \otimes X_{24} + (X_{14} - X_{24}) \otimes X_{20} + X_{20} \otimes (X_{14} - X_{24}) + X_{24} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta(X_{25}) = (X_{10} - X_{20}) \otimes X_{25} + X_{20} \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes X_{24} + X_{21} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{21} + X_{24} \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes X_{20} + X_{25} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta(X_{26}) = (X_{10} - X_{20}) \otimes X_{26} + X_{20} \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes X_{24} + X_{22} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{22} + X_{24} \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes X_{20} + X_{26} \otimes (X_{10} - X_{20});$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\Delta(X_{27}) = (X_{10} - X_{20}) \otimes X_{27} + X_{20} \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes X_{26} + X_{21} \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes X_{25} + X_{22} \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes X_{24} + X_{23} \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes X_{23} + X_{24} \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes X_{22} + X_{25} \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes X_{21} + X_{26} \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes X_{20} + X_{27} \otimes (X_{10} - X_{20});$$

$$\varepsilon\,(X_{00}) = 2,\ \varepsilon\,(X_{10}) = 1,\ \varepsilon\,(X_{ij}) = 0(\text{for } i = 2 \text{ or } j \geq 1);$$

(4) $\psi\,(X_{07}) = 3,\ \psi\,(X_{17}) = 2,\ \psi\,(X_{27}) = 1,\ \psi\,(X_{ij}) = 0(\text{for } j \leq 6);$

(5) $S = id_B$,

Then $(B,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \psi,\ S)$ is a GWBF-algebra. Let L be the subspace generated by X_{00}, X_{10}, X_{20}, X_{02}, X_{12}, X_{22}, X_{04}, X_{14}, X_{24}, X_{06}, X_{16} and X_{26}. It is a subalgebra and subcoalgebra of B.

Moreover, define $\rho \in L^*$ by

$$\rho\,(X_{06}) = 3,\ \rho\,(X_{16}) = 2,\ \rho\,(X_{26}) = 1,\ \rho\,(X_{ij}) = 0\ (\text{for } j = 0,\ 2,\ 4),$$

Then $(L,\ \rho,\ S|_L)$ is a GWBF-subalgebra of B.

3.3. The 24-Dimensional GWBF Algebra (Type 2)

Define B' as the 24-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ 2,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) X_{00} is the unit;

(2) $X_{ij}X_{st} = X_{pq}$, where $p = 0$ if $i = 0$ and $s = 0$, $p = 2$ if $i = 2$ or $s = 2$, $p = 1$ for the other condition; and $q = (j + t) \bmod 8$.

(3) The coproduct and counit on A are given by:

$$\Delta\,(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{20};$$

$$\Delta\,(X_{10}) = (X_{10} - X_{20}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{20};$$

$$\Delta\,(X_{20}) = X_{20} \otimes X_{20};$$

$$\Delta\,(X_{01}) = (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{11} - X_{21}) + (X_{11} - X_{21}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta\,(X_{11}) = (X_{10} - X_{20}) \otimes (X_{11} - X_{21}) + (X_{11} - X_{21}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta\,(X_{21}) = X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta\,(X_{02}) = (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + (X_{02} - X_{12}) \otimes (X_{00} - X_{10})) + (X_{10} - X_{20}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{10} - X_{20}) + X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta\,(X_{12}) = (X_{10} - X_{20}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{10} - X_{20}) + X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta\,(X_{22}) = X_{22} \otimes X_{20} + X_{20} \otimes X_{22};$$

$$\Delta\,(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta\,(X_{13}) = (X_{10} - X_{20}) \otimes (X_{13} - X_{23}) + (X_{11} - X_{21}) \otimes (X_{12} - X_{22}) + (X_{12} - X_{22}) \otimes (X_{11} - X_{21}) + (X_{13} - X_{23}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta\,(X_{23}) = X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta\,(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{00} - X_{10})) + (X_{10} - X_{20}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{10} - X_{20}) + X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta\,(X_{14}) = (X_{10} - X_{20}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{10} - X_{20}) + X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta\,(X_{24}) = X_{24} \otimes X_{20} + X_{20} \otimes X_{24};$$

$$\Delta\,(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta\,(X_{15}) = (X_{10} - X_{20}) \otimes (X_{15} - X_{25}) + (X_{11} - X_{21}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{11} - X_{21}) + (X_{15} - X_{25}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta\,(X_{25}) = X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{16}) = (X_{10} - X_{20}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{26}) = X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + (X_{10} - X_{20}) \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta(X_{17}) = (X_{10} - X_{20}) \otimes (X_{17} - X_{27}) + (X_{11} - X_{21}) \otimes (X_{16} - X_{26}) + (X_{12} - X_{22}) \otimes (X_{15} - X_{25}) + (X_{13} - X_{23}) \otimes (X_{14} - X_{24}) + (X_{14} - X_{24}) \otimes (X_{13} - X_{23}) + (X_{15} - X_{25}) \otimes (X_{12} - X_{22}) + (X_{16} - X_{26}) \otimes (X_{11} - X_{21}) + (X_{17} - X_{27}) \otimes (X_{10} - X_{20}) + X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta(X_{27}) = X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\varepsilon(X_{00}) = 3,\ \varepsilon(X_{10}) = 3,\ \varepsilon(X_{20}) = 1,\ \varepsilon(X_{ij}) = 0 \ (\text{for } j \geq 1);$$

(4) $\iota(X_{07}) = 2,\ \iota(X_{17}) = 1,\ \iota(X_{27}) = \frac{1}{2},\ \iota(X_{ij}) = 0$ (for $j \leq 6$);

(5) $S = id_B$,

The $(B',\ m,\ \eta,\ \Delta,\ \varepsilon,\ \iota,\ S)$ is a GWBF-algebra. Let L' be the subspace generated by X_{00}, X_{10}, X_{20}, X_{02}, X_{12}, X_{22}, X_{04}, X_{14}, X_{24}, X_{06}, X_{16} and X_{26}. It is a subalgebra and subcoalgebra of B'.

Define $\rho \in L^*$ by

$$\rho(X_{06}) = 2,\ \rho(X_{16}) = 1,\ \rho(X_{26}) = \frac{1}{2},\ \rho(X_{ij}) = 0 \ (\text{for } j = 0,\ 2,\ 4),$$

Then $(L',\ \rho,\ S|_{L'})$ is a GWBF-subalgebra of B'.

3.4. The 40-Dimensional GWBF Algebra

Define C as the 40-dimensional associative algebra generated by a set $\{X_{ij} | i = 0,\ 1,\ 2,\ 3,\ 4,\ j = 0,\ 1,\ \cdots,\ 7\}$ with the following relations:

(1) X_{00} is the unit;

(2) $X_{ij}X_{0t} = X_{0t}X_{ij} = X_{iq}$, $X_{ij}X_{1t} = X_{1t}X_{ij} = X_{iq}$, $X_{2j}X_{2t} = X_{1q}$, $X_{2j}X_{3t} = -X_{4q}$, $X_{3j}X_{2t} = X_{4q}$, $X_{2j}X_{4t} = -X_{3q}$, $X_{4j}X_{2t} = X_{3q}$, $X_{3j}X_{3t} = X_{0q}$, $X_{3j}X_{4t} = X_{4t}X_{3j} = X_{0q}$, $X_{4j}X_{4t} = X_{0q}$, where $q = (j + t) \bmod 8$ for $i = 0,\ 1,\ 2,\ 3,\ 4$ and $j,\ t = 0,\ 1,\ \cdots,\ 7$.

(3) The coproduct and counit on A are given by:

$$\Delta(X_{00}) = (X_{00} - X_{10}) \otimes (X_{00} - X_{10}) + X_{10} \otimes X_{10};$$

$$\Delta(X_{10}) = X_{10} \otimes X_{10};$$

$$\Delta(X_{20}) = X_{20} \otimes X_{20};$$

$$\Delta(X_{30}) = X_{20} \otimes X_{30} + X_{30} \otimes X_{20};$$

$$\Delta(X_{40}) = X_{10} \otimes X_{40} + X_{40} \otimes X_{20};$$

$$\Delta(X_{01}) = (X_{00} - X_{10}) \otimes (X_{01} - X_{11}) + X_{10} \otimes X_{11} + (X_{01} - X_{11}) \otimes (X_{00} - X_{10}) + X_{11} \otimes X_{10};$$

$$\Delta(X_{11}) = X_{10} \otimes X_{11} + X_{11} \otimes X_{10};$$

$$\Delta(X_{21}) = X_{20} \otimes X_{21} + X_{21} \otimes X_{20};$$

$$\Delta(X_{31}) = X_{20} \otimes X_{31} + X_{21} \otimes X_{30} + X_{30} \otimes X_{11} + X_{31} \otimes X_{10};$$

$$\Delta(X_{41}) = X_{10} \otimes X_{41} + X_{40} \otimes X_{21} + X_{11} \otimes X_{40} + X_{41} \otimes X_{20};$$

$$\Delta(X_{02}) = (X_{00} - X_{10}) \otimes (X_{02} - X_{12}) + X_{10} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{00} - X_{10}) + X_{12} \otimes X_{10};$$

$$\Delta\,(X_{12}) = X_{10} \otimes X_{12} + X_{12} \otimes X_{10};$$

$$\Delta\,(X_{22}) = X_{20} \otimes X_{22} + X_{22} \otimes X_{20};$$

$$\Delta\,(X_{32}) = X_{20} \otimes X_{32} + X_{30} \otimes X_{12} + X_{22} \otimes X_{30} + X_{32} \otimes X_{10};$$

$$\Delta\,(X_{42}) = X_{10} \otimes X_{42} + X_{40} \otimes X_{22} + X_{12} \otimes X_{40} + X_{42} \otimes X_{20};$$

$$\Delta\,(X_{03}) = (X_{00} - X_{10}) \otimes (X_{03} - X_{13}) + X_{10} \otimes X_{13} + (X_{01} - X_{11}) \otimes (X_{02} - X_{12}) + X_{11} \otimes X_{12} + (X_{02} - X_{12}) \otimes (X_{01} - X_{11}) + X_{12} \otimes X_{11} + (X_{03} - X_{13}) \otimes (X_{00} - X_{10}) + X_{13} \otimes X_{10};$$

$$\Delta\,(X_{13}) = X_{10} \otimes X_{13} + X_{11} \otimes X_{12} + X_{12} \otimes X_{11} + X_{13} \otimes X_{10};$$

$$\Delta\,(X_{23}) = X_{20} \otimes X_{23} + X_{21} \otimes X_{22} + X_{22} \otimes X_{21} + X_{23} \otimes X_{20};$$

$$\Delta\,(X_{33}) = X_{20} \otimes X_{33} + X_{30} \otimes X_{13} + X_{21} \otimes X_{32} + X_{31} \otimes X_{12} + X_{22} \otimes X_{31} + X_{32} \otimes X_{11} + X_{23} \otimes X_{30} + X_{33} \otimes X_{10};$$

$$\Delta\,(X_{43}) = X_{10} \otimes X_{43} + X_{40} \otimes X_{23} + X_{11} \otimes X_{42} + X_{41} \otimes X_{22} + X_{12} \otimes X_{41} + X_{42} \otimes X_{21} + X_{13} \otimes X_{40} + X_{43} \otimes X_{20};$$

$$\Delta\,(X_{04}) = (X_{00} - X_{10}) \otimes (X_{04} - X_{14}) + X_{10} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{00} - X_{10}) + X_{14} \otimes X_{10};$$

$$\Delta\,(X_{14}) = X_{10} \otimes X_{14} + X_{14} \otimes X_{10};$$

$$\Delta\,(X_{24}) = X_{20} \otimes X_{24} + X_{24} \otimes X_{20};$$

$$\Delta\,(X_{34}) = X_{20} \otimes X_{34} + X_{30} \otimes X_{14} + X_{24} \otimes X_{30} + X_{34} \otimes X_{10};$$

$$\Delta\,(X_{44}) = X_{10} \otimes X_{44} + X_{40} \otimes X_{24} + X_{14} \otimes X_{40} + X_{44} \otimes X_{20};$$

$$\Delta\,(X_{05}) = (X_{00} - X_{10}) \otimes (X_{05} - X_{15}) + X_{10} \otimes X_{15} + (X_{01} - X_{11}) \otimes (X_{04} - X_{14}) + X_{11} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{01} - X_{11}) + X_{14} \otimes X_{11} + (X_{05} - X_{15}) \otimes (X_{00} - X_{10}) + X_{15} \otimes X_{10};$$

$$\Delta\,(X_{15}) = X_{10} \otimes X_{15} + X_{11} \otimes X_{14} + X_{14} \otimes X_{11} + X_{15} \otimes X_{10};$$

$$\Delta\,(X_{25}) = X_{20} \otimes X_{25} + X_{21} \otimes X_{24} + X_{24} \otimes X_{21} + X_{25} \otimes X_{20};$$

$$\Delta\,(X_{35}) = X_{20} \otimes X_{35} + X_{30} \otimes X_{15} + X_{21} \otimes X_{34} + X_{31} \otimes X_{14} + X_{24} \otimes X_{31} + X_{34} \otimes X_{11} + X_{25} \otimes X_{30} + X_{35} \otimes X_{10};$$

$$\Delta\,(X_{45}) = X_{10} \otimes X_{45} + X_{40} \otimes X_{25} + X_{11} \otimes X_{44} + X_{41} \otimes X_{24} + X_{14} \otimes X_{41} + X_{44} \otimes X_{21} + X_{15} \otimes X_{40} + X_{45} \otimes X_{20};$$

$$\Delta\,(X_{06}) = (X_{00} - X_{10}) \otimes (X_{06} - X_{16}) + X_{10} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{04} - X_{14}) + X_{12} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{02} - X_{12}) + X_{14} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{00} - X_{10}) + X_{16} \otimes X_{10};$$

$$\Delta\,(X_{16}) = X_{10} \otimes X_{16} + X_{12} \otimes X_{14} + X_{14} \otimes X_{12} + X_{16} \otimes X_{10};$$

$$\Delta\,(X_{26}) = X_{20} \otimes X_{26} + X_{22} \otimes X_{24} + X_{24} \otimes X_{22} + X_{26} \otimes X_{20};$$

$$\Delta\,(X_{36}) = X_{20} \otimes X_{36} + X_{30} \otimes X_{16} + X_{22} \otimes X_{34} + X_{32} \otimes X_{14} + X_{24} \otimes X_{32} + X_{34} \otimes X_{12} + X_{26} \otimes X_{30} + X_{36} \otimes X_{10};$$

$$\Delta\,(X_{46}) = X_{10} \otimes X_{46} + X_{40} \otimes X_{26} + X_{12} \otimes X_{44} + X_{42} \otimes X_{24} + X_{14} \otimes X_{42} + X_{44} \otimes X_{22} + X_{16} \otimes X_{40} + X_{46} \otimes X_{20};$$

$$\Delta\,(X_{07}) = (X_{00} - X_{10}) \otimes (X_{07} - X_{17}) + X_{10} \otimes X_{17} + (X_{01} - X_{11}) \otimes (X_{06} - X_{16}) + X_{11} \otimes X_{16} + (X_{02} - X_{12}) \otimes (X_{05} - X_{15}) + X_{12} \otimes X_{15} + (X_{03} - X_{13}) \otimes (X_{04} - X_{14}) + X_{13} \otimes X_{14} + (X_{04} - X_{14}) \otimes (X_{03} - X_{13}) + X_{14} \otimes X_{13} + (X_{05} - X_{15}) \otimes (X_{02} - X_{12}) + X_{15} \otimes X_{12} + (X_{06} - X_{16}) \otimes (X_{01} - X_{11}) + X_{16} \otimes X_{11} + (X_{07} - X_{17}) \otimes (X_{00} - X_{10}) + X_{17} \otimes X_{10};$$

$$\Delta\,(X_{17}) = X_{10} \otimes X_{17} + X_{11} \otimes X_{16} + X_{12} \otimes X_{15} + X_{13} \otimes X_{14} + X_{14} \otimes X_{13} + X_{15} \otimes X_{12} + X_{16} \otimes X_{11} + X_{17} \otimes X_{10};$$

$$\Delta\,(X_{27}) = X_{20} \otimes X_{27} + X_{21} \otimes X_{26} + X_{22} \otimes X_{25} + X_{23} \otimes X_{24} + X_{24} \otimes X_{23} + X_{25} \otimes X_{22} + X_{26} \otimes X_{21} + X_{27} \otimes X_{20};$$

$$\Delta\,(X_{37}) = X_{20} \otimes X_{37} + X_{30} \otimes X_{17} + X_{21} \otimes X_{36} + X_{31} \otimes X_{16} + X_{22} \otimes X_{35} + X_{32} \otimes X_{15} + X_{23} \otimes X_{34} + X_{33} \otimes X_{14} + X_{24} \otimes X_{33} + X_{34} \otimes X_{13} + X_{25} \otimes X_{32} + X_{35} \otimes X_{12} + X_{26} \otimes X_{31} + X_{36} \otimes X_{11} + X_{27} \otimes X_{30} + X_{37} \otimes X_{10};$$

$$\Delta\,(X_{47}) = X_{10} \otimes X_{47} + X_{40} \otimes X_{27} + X_{11} \otimes X_{46} + X_{41} \otimes X_{26} + X_{12} \otimes X_{45} + X_{42} \otimes X_{25} + X_{13} \otimes X_{44} + X_{43} \otimes X_{24} + X_{14} \otimes X_{43} + X_{44} \otimes X_{23} + X_{15} \otimes X_{42} + X_{45} \otimes X_{22} + X_{16} \otimes X_{41} + X_{46} \otimes X_{21} + X_{17} \otimes X_{40} + X_{47} \otimes X_{20};$$

$$\varepsilon\,(X_{00}) = 2,\ \ \varepsilon\,(X_{10}) = 1,\ \ \varepsilon\,(X_{20}) = 1,\ \ \varepsilon\,(X_{ij}) = 0\ \text{(for } i = 3,\ 4 \text{ or } j \geq 1\text{)};$$

(4) $\tau\,(X_{07}) = 1,\ \ \tau\,(X_{37}) = 1,\ \ \tau\,(X_{ij}) = 0$ (for $i \neq 0,\ 3$ or $j \leq 6$);

(5) $S = id_C$.

Then $(C,\ m,\ \eta,\ \Delta,\ \varepsilon,\ \tau,\ S)$ is a GWBF-algebra. Let M be the subspace generated by $X_{i0},\ X_{i2},\ X_{i4},\ X_{i6}$ for $i = 0,\ 1,\ 2,\ 3,\ 4$. It is a subalgebra and subcoalgebra of C. Define $\theta \in M^*$ by $\theta(X_{06}) = 1,\ \theta(X_{36}) = 1,\ \theta(X_{ij}) = 0$ (for $i \neq 0,\ 3$ or $j = 0,\ 2,\ 4$). Then $(M,\ \theta,\ S|_M)$ is a GWBF-subalgebra of C.

4. Conclusion

Poposition 4.1 Let H and K be two GWBF-algebras. Then the tensor product of H and K is also a GWBF-algebra.

Proof Deine the antipode as follows

$$S_{H\otimes \mathrm{K}} = S_H \otimes S_{\mathrm{K}},$$

And the integral as follows

$$\psi_{H\otimes \mathrm{K}} = \psi_H \otimes \psi_{\mathrm{K}},$$

It is obviously that $(H \otimes \mathrm{K},\ \mathrm{S}_{H\otimes \mathrm{K}},\ \psi_{H\otimes \mathrm{K}})$ is a GWBF-algebra under the tensor product algebra structure and tensor coproduct coalgebra structure.

Poposition 4.2 Let H be a weak Hopf algebra, K be a biFrobenius-algebra. Then the tensor product of H and K is a GWBF-algebra.

Proof The antipode is given by

$$S_{H\otimes \mathrm{K}} = S_H \otimes S_{\mathrm{K}},$$

And the integral is

$$\psi_{H\otimes \mathrm{K}} = \psi_H \otimes \psi_{\mathrm{K}},$$

Where ψ_H is an integral of H.

Poposition 4.3 The GWBF-algebra A introduced in Example 3.1 is a GWBF-subalgebra of B in Example 3.2.

Proof Obviously A is the 16-dimensional GWBF-subalgebra of B in Example 3.2 generated by $\{X_{ij} | i = 0,\ 1,\ j = 0,\ 1,\ \cdots,\ 7\}$.

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