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Review Article

Several Remarks on *q*-Binomial Inverse Formula and Examples

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Abstract: In this paper, we first give some comments on the paper [J. Goldman and G. C. Rota, on the foundations of combinatorial theory IV finite vector spaces and Eulerian generating functions, *Stud. Appl. Math.*, 49: 239--258 (1970)]. In that paper, Goldman and Rota showed two incorrect inversion formulas in Section 3. We point out the formulas and give the correct versions with the proof in this this paper first. Then we give some remarks on q -binomial inverse formula concerning its applications.

Keywords: Inverse Formula, Binomial Inverse Formula, q-Binomial Inverse Formula

1. Introduction

The basic notations of this commentary are the binomial coefficients

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

where n and k are nonnegative integers with $n \geq k$, and the quantum factorial symbol [1] defined by

$$(x;q)_0=1, \quad (x;q)_n:=\prod_{k=0}^{n-1}(1-xq^k)$$

We also need to introduce the q -analog of the binomial coefficients, which we often call them q -binomial coefficients or Gaussian polynomials. Before introducing the q -binomial coefficients, we first need to give some notations. Let x be a real number, the q -real number of x is defined as [2]

$$[x]_q=\frac{1-q^x}{1-q}$$

In particular, when k is a positive integer,

$$[k]_q=1+q+\cdots+q^k$$

is called q -positive integer. The k -th order factorial of the q -number $[x]_q$ is defined as

$$[x]_{k,q}=[x]_q[x-1]_q\cdots[x-k+1]_q=\frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q^k)}$$

In particular,

$$[k]_q!=[1]_q[2]_2\cdots[k]_q$$

is called the q -factorial. The q -binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[x]_{k,q}}{[k]_q}=\frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

In particular, for a positive integer n,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad 0 \leq k \leq n.$$

This is exactly what we need in this paper.

In Section 3 ("Eulerian generating functions") of [3], Goldman and Rota showed the following inversion formula, namely, the system of equations

$$b_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_k$$

is equivalent to the system

$$a_n = \sum_{k=0}^{n} (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k$$

The above inversion formula is exactly equation (5) in Section 3 of [3]. Then Goldman and Rota said that the above inversion formula is the q-analog (let $q \to 1$) of the classical inversion formula

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k; \quad a_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \tag{1}$$

which is equation (6) in Section 3 of [3].

Actually, these two inversion formulas are not correct. In Section 2, we will give the full reason to show why the two formulas are incorrect. Then in Section 3, we will give the correct version of the q-binomial inversion formula with the proof. In section 4, we will give an example on the applications of q-binomial inversion formula and then show some remarks on the formula.

2. Why the Formulas Are Incorrect

In fact, we just need to show that (1) is incorrect since the q-binomial inversion formula Goldman and Rota gave can reduce to (1) when $q \to 1$ and if (1) is incorrect, it is impossible for the q-binomial inversion formula Goldman and Rota gave to be true.

Now, let us show that (1) is incorrect quickly. By (1), we have

$$b_0 = a_0, \quad b_1 = a_0 + a_1$$

By which we can arrive at

$$a_0 = b_0, \quad a_1 = b_1 - a_0 = b_1 - b_0 \tag{2}$$

However, by (1), we get

$$a_0 = b_0, \quad a_1 = b_0 - b_1$$

which contradicts with (2).

So, according to the above analysis, we can conclude that the two inversion formulas appeared in (2) are not correct. Next, we would like to show the correct versions.

3. The Correct Version with the Proof

By last section, we know that the two inversion formulas showed in the paper of Goldman and Rota are incorrect. Actually, the correct version of the classical binomial inversion formula appears in many literatures. So, we just post the correct version with some references. While for the q-binomial inversion formula, it is not such easy to guess. Hence, we will revise the result given by Goldman and Rota and then prove the revised result according to the works of Carlitz.

First, let us post the correct version of the classical binomial inversion formula.

Theorem 3.1. Suppose $\{a_n\}_{n\ge 0}$ and $\{b_n\}_{n\ge 0}$ are two sequences. If

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k,$$

then we have

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k,$$

and vice versa.

Actually, this theorem can be rewritten into a more symmetric form,

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \tag{3}$$

The proof of this classic theorem can be found in many literatures (see for example [4, 5, 6, 7, 8]). Furthermore, the beginning of the two summations can be changed into any nonnegative integer which is less than n.

The formulas which directly or indirectly give the formal inverse are called inversion formulas in literature. It is a powerful tool in combinatorics when it comes to the problems of summations, series transformation and etc.. There have been many different versions of inversion formulas. For example, Lagrange inversion theorem, Möbius inversion formula, Fourier inversion theorem, Mellin inversion theorem and Post's inversion formula. All these formulas or theorems cover a wide scope of Mathematics, including differential equation, combinatorics and number theory. Here, we would like to give the q-binomial inversion theorem. Next, let us move to the correct version of the q-binomial inversion formula.

Theorem 3.2. Suppose $\{a_n\}_{n\ge 0}$ and $\{b_n\}_{n\ge 0}$ are two sequences. If

$$a_n = \sum_{k=0}^{n} (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k,$$

then we have

$$b_n = \sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q a_k,$$

and vice versa.

This q -analog of the classical binomial inversion formula is not such easy to guess, especially the q -analogs of two $(-1)^k$'s in (3) are $(-1)^k q^{\frac{k(k-1)}{2}}$ and $(-1)^k q^{\frac{k(k+1)}{2}-kn}$ respectively. This is surely not something easily guessed. Before giving the full proof of Theorem 3.2, we would like to introduce a conclusion by Carlitz. In fact, the following conclusion given by Carlitz might be the most general form of the q -analog of the classical binomial inversion formula. It has great influences on the works later. Even though in the researches today, we can still find some conclusions that were motivated by the result of Carlitz.

Lemma 3.1 (Carlitz [9]). Let $\{a_i\}$ and $\{b_i\}$ be two sequences of complex numbers, let q be an arbitrary complex number such that

$$a_i + q^{-k} b_i \neq 0, \quad i = 1,2,3,\cdots, \quad k = 0,1,2,\cdots,$$

and put

$$\psi(x,n,q) = \prod_{i=1}^{n} (a_i + q^{-k} b_i)$$

Then

$$f(n) = \sum_{k=0}^{n} (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \psi(k,n,q) g(k)$$

is equivalent to

$$g(n) = \sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q (a_{k+1} + q^{-k} b_{k+1}) \frac{f(k)}{\psi(n,k+1,q)}$$

With this lemma in hand, we can then prove Theorem 3.2 quickly.

Proof. In fact, there are two ways to show Theorem 3.2 with the aid of Lemma 3.1.

The first way is that we can take $a_i = 1$ and $b_i = 0$ for all i in Lemma 3.1. Then we can arrive at the desired result.

But if we take $a_i = 0$ and $b_i = 1$ for all i in Lemma 3.1, we can also obtain the result.

So, we can conclude that Theorem 3.2 holds true.

By now, we finished the revision of the paper by Goldman and Rota.

4. Remarks on *q*-Binomial Inverse Formula

In this section, we first show an application of q -binomial inversion formula. Then through this example, we will give some remarks on the formula.

Before introducing the example, some notations need to be given first.

For $n \geq 1$, let

$$(x)_n = x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$$

denote the falling factorial (sometimes called the descending factorial, falling sequential product or lower factorial). When $n = 0$, we adopt the convention that $(x)_n = 1$. Falling factorial are very useful in mathematics, see for example [10, 11, 12, 13, 14, 15]. Motivated by the topic of q -series, we can define the q -falling factorial as follows

$$x_q^{\underline{n}} = \prod_{r=1}^{n} (x - q^{r-1})$$

Then, according to Cauchy's identity or the explicit formula for Möbius functions of lattices of subspaces, we can get that [16]

$$\sum_{k=0}^{n} (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} = x_q^{\underline{n}}$$

Considering the identity above, we can then let

$$a_k = x^{n-k}$$

Then

$$a_n = x^0 = 1$$

So, according to Theorem 3.2, we can say that there exists a b_k that is a polynomial in terms of k such that

$$\sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k = 1$$

and

$$b_n = x_q^{\underline{n}}$$

Though this is a beautiful identity, while the obscure of b_k greatly reduced the beauty of this identity.

Also, the author has proved that [17] there also exists a $b_k^{'}$ that is a polynomial in terms of k such that

$$\sum_{k=0}^{n}(-1)^k q^{\frac{k(k+1)}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q b_k^{'} = 1$$

and $b_k^{'}$ satisfies a condition listed in [17].

Remark 3.1. The rest question is the explicit forms of b_k and $b_k^{'}$. Actually, it is an extremely hard question that we cannot solve at this time. The problem not only happens here, but also happens somewhere else concerning the applications of q -binomial inversion formula.

Remark 3.2. Pairs of inverse relations are very useful in the study of combinatorial identities. In 1968, J. Riordan published a book named "Combinatorial Identities". In that book, Riordan fully illustrated this point of view. While all he used in that book is binomial inversion formula. Of course, q -series is not such popular at that time that q -analogues are absent in his book. Now, with the rapid development of q -series, the q -analogue of the binomial inversion theorem must be introduced. Just like the role of binomial inversion formula in proving combinatorial identities, q -binomial inversion theorem will play the same role in the further study of q -series. In this paper, along with the classic idea of proving binomial inversion formula, we gave the form of q -binomial inversion theorem and then proved it. With q -binomial inversion theorem in hand, we can now do much more on the proof of q -analogues of combinatorial identities. But there these identities still have some little problems. The reasons have been shown above.

Remark 3.3. Though q -binomial inversion formula is the q -analogue of the binomial inverse formula. While the applications of binomial inverse formula is much easier than that of q -binomial inversion formula. This may be because of the two q -analogues of the two $(-1)^k$'s in binomial inverse formula.

Remark 3.4. Our future work is to find a way to show the explicit form of one sequence that appears in the above two identities. Up to now, how to find the explicit form of one sequence that appears in the above identities is still an open question.

Remark 3.5. Another inverse relation is called self-inverse. It is related to the inverse formula we showed in this paper in some sense. It is also an very interesting topic. Sun [18] studied self-inverse sequences by using their generating functions and gave many interesting examples and results of self-inverse sequences. Wang [19] explored self-inverse sequences by means of linear transformations, difference operators and the umbral calculus. Wang also obtained various characterizations of self-inverse sequences from these different approaches. For S^+ Wang showed that it is a vector space over the complex field and determine its dimension. Wang still gave simpler proofs to certain results of Sun. It is worth noting that our results can give rise to many interesting identities.

5. Conclusion

In this paper, we first pointed out why the formulas showed in the paper of Goldman and Rota is not correct. Then by applying the conclusion of Carlitz, we showed the correct version of the formulas. After which, we give some remarks on the formulas. These remarks include some applications of the formula. However, we need to say that these applications, in some senses, are incomplete since as one can see that we did not find the explicit representations of some functions in the applications. Actually, finding the explicit representations of these functions is our further work.

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References

[1] G. E. Andrews, On the difference of successive Gaussian polynomials, *J. Stat. Planning and Inf.*, 1993 (34), pp. 19-22.

[2] Ch. A. Charalambides, Non-central generalized q -factorial coefficients and q -Stirling numbers, *Discrete Math.*, 275 (2004), pp. 67-85.

[3] J. Goldman and G. C. Rota, On the foundations of combinatorial theory IV finite vector spaces and Eulerian generating functions, *Stud. Appl. Math.*, 1970 (49), pp. 239-258.

[4] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Boston, 1994.

[5] M. Aigner, *A course in enumeration*, Springer, New York, 2007.

[6] L. Carlitz, Some inversion formulas, *Rend. Circ. Mat. Palermo*, 12 (1963), pp. 1-17.

[7] H. W. Goulo and L. C. Hsu, Some new inverse relations, *Duke Math. J.*, 40 (1973), pp. 885-891.

[8] J. Roroa, *Combinatorial Identities*, New York, Wiley, 1968.

[9] L. Carlitz, Some inverse relations, *Duke Math. J.*, 1973 (40), pp. 893-901.

[10] K. N. Boyadzhiev, Lah numbers, Laguerre polynomials of order negative one, and the nth derivative of $\exp(1/x)$, *Acta Univ. Sapientiae, Mathematica*, 2016 (8), pp. 22-31.

[11] B.-N. Guo and F. Qi, Six proofs for an identity of the Lah numbers, *Online J. Anal. Comb.*, 2015 (10), pp. 1-5.

[12] J. F. Steffensen, *Interpolation (2nd ed.)*, Dover Publications. (A reprint of the 1950 edition by Chelsea Publishing Co.)

[13] B. Osgood and W. Wu, Falling Factorials, Generating Functions, and Conjoint Ranking Tables, *J. Integer Seq.*, 2009 (12), Article 09.7.8.

[14] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science, 2nd ed.*, Reading, MA: Addison-Wesley, 1994.

[15] Y.-X. Wang, A. Smola and R. J. Tibshirani, *The falling factorial basis and its statistical applications*, Proceedings of the 31st International Conference on Machine Learning.

[16] J. P. S. Kung, Curious Characterizations of Projective and Affine Geometries, *Adv. Appl. Math.*, 2002 (28), pp. 523–543.

[17] Q. Zou, The q -binomial inverse formula and a representation for the q -Catalan-Qi numbers, submitted.

[18] Z. H. Sun, Invariant Sequences Under Binomial Transformations, The Fibonacci Quarterly 2001 (39), pp. 324-333.

[19] Y. Wang, Self-inverse sequences related to a binomial inverse pair, The Fibonacci Quarterly 2005 (43), pp. 46-52.