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Some Explicit and Hybrid Strong Convergence Algorithms for Solving the Multiple-Sets Split Feasibility Problem

Peiyuan Wang[1, 2, *], Jianjun Zhou[3], Risheng Wang[3], Jie Chen[3]

[1]Postdoctoral Workstation, China Marine Development and Reserch Center (CMDRC), Beijing, China

[2]Naval Aviation Institution, Huludao, China

[3]China Marine Development and Reserch Center (CMDRC), Beijing, China

Email address:

wangpy629@163.com (Peiyuan Wang)

[*]Corresponding author

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Abstract: In this paper, we present several explicit and hybrid strong convergence algorithms for solving the multiple-sets split feasibility problem (MSSFP). Firstly, we modify the existing successive, parallel and cyclic algorithms with the hybrid steepest descent method; then two new hybrid formulas based on the Mann type method are presented; Two general hybrid algorithms which can cover the former ones are further proposed. Strong convergence properties are investigated, and numerical experiments shows the compromise is promising.

Keywords: Variational Inequalities, Multiple-Sets Split Feasibility Problem, Hybrid Steepest Descent Method, Lipschitz Continuous, Inverse Strongly Monotone

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The variational inequalities is to find $x^* \in C$ such that

$$\left\langle Fx^*, x - x^* \right\rangle \geq 0, \quad \text{for all } x \in C, \tag{1}$$

where F is k -Lipschitz and η -strongly monotone mapping on H. Since Yamada [1] introduced a hybrid steepest descent method for solving the variational inequalites (1), some improved and extended work have been done by Xu and Kim [2], Zeng [3], Liu and Cai [4] and Iemtoo and Takahashi [5]. Recently, Buong and Duong [6] introduced a new hybrid steepest descent algorithm based on the Krasnosel' ski-Mann iteration, afterward, Zhou and Wang [7] improved it and proposed a simpler one. Kim and Buong [8] also introduced another formula.

Remark that when there is $C = \bigcap_{i=1}^{t} C_i$ in (1), and $C_1, C_2, \ldots, C_t$ are t closed convex subsets of H such that $\bigcap_{i=1}^{t} C_i \neq \varnothing$, we can apply the hybrid method to improve some algorithms of the multi-sets split feasibility problem (MSSFP) [9], which is formulated as

$$\begin{aligned} &\text{finding a point } x^* \in C := \bigcap_{i=1}^{t} C_i \\ &\textit{such that } Ax \in Q := \bigcap_{j=1}^{r} Q_j \end{aligned} \tag{2}$$

where $r, t \geq 1$ are integers, $C_1, C_2, \ldots, C_t$ and $Q_1, Q_2, \ldots, Q_r$ are closed convex subsets of H_1 and H_2, respectively. $A : H_1 \to H_2$ is a bounded linear operator.

We just to consider the core iterative formula with fixed stepsize in this paper. Some algorithms have been invented to solve MSSFP (2), see [9, 10, 11, 12]. In [10], Xu proposed three weak convergence algorithms *i.e.*, the successive, parallel and cyclic iteration methods to solve a simpler minimization problem in Hilbert spaces. In [13], Xu proposed a strong convergence algorithm for the cyclic type, Guo and

Yu [14] presented strong convergence of the others. Then, He and Liu [15] presented some variable Krasnosel' ski-Mann (KM) iteration algorithms, which converge weakly to a common fixed point. Deng and Chen [16] have applied the hybrid method to solve MSSFP, but their algorithms are either implicit ones or successive type.

However, for some piratical problems, the strong convergence algorithms in [13] and [14] sometimes may not have better iterative results than the weak convergence ones in [10, 15, 21, 22]. Therefore, in order to obtain strong convergence and more effective iteration formulas, we propose several explicit and hybrid strong convergence algorithms to solve MSSFP (2), it is also a combination with applying the hybrid steepest descent methods of solving (1).

The paper is organized as follows. In Section 2, we review some facts and summarize useful results. In Section 3, several explicit and hybrid algorithms are presented orderly, strong convergences are also analyzed. Some numerical results are compared in Section 4 and Section 5 concludes and leads some further discussions.

2. Preliminaries

If the solution of MSSFP (2) $\Gamma = C\bigcap A^{-1}(Q) \neq \varnothing$, then the MSSFP is equivalent to the minimization problem

$$\min_{x\in\Gamma} q(x) := \frac{1}{2}\sum_{j=1}^{r}\beta_j \left\| P_{Q_n} Ax - Ax \right\|^2, \tag{3}$$

where $\beta_j > 0$ for each $1 \le j \le r$, and $\sum_{j=1}^{r}\beta_j = 1$. The gradient of q is

$$\nabla q(x) = \sum_{j=1}^{r}\beta_j A^*\left(I - P_{Q_n}\right)Ax. \tag{4}$$

It is easy to see that ∇q is L-Lipschitzian with $L = \|A\|^2 \sum_{j=1}^{r}\beta_j$ and $(1/L)$-ism.

Lemma 2.1 [10] *Assume that the MSSFP* (2) *is consistent. Let* $T_i := P_{C_i}(I - \gamma\nabla q)$, $i = 1,2,\ldots,t$, *where* $0 < \gamma < 2/L$. *Then the mapping* $U = T_t \ldots T_1$ *is averaged; the convex combination* $S := \sum_{i=1}^{t}\alpha_i T_i$ *is averaged, where* $\alpha_i > 0$, $\sum_{i=1}^{t}\alpha_i = 1$; $T_{[n+1]} = T_{n\,mod\,t}$ *is also averaged, where the mod function takes values* $\{1,2,\ldots,t\}$.

Lemma 2.2 [10, 13, 14, 15] *Denote a averaged operator* T *be* U, S *or* $T_{[n+1]}$, *which are defined in Lemma 2.1. For any initial points* x_0, y_0 *and* z_0 *in* H, $n \ge 0$, *the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are generated by*

$$x_{n+1} = Tx_n, \tag{5}$$

$$y_{n+1} = \left(1 - a_n\right)Ty_n \tag{6}$$

and

$$z_{n+1} = \left(1 - b_n\right)z_n + b_n Tz_n, \tag{7}$$

where $\{a_n\}$ and $\{b_n\}$ are real sequeces in $(0,1)$ satisfying the conditions in [13], [14] and [15], respectively. Then $\{x_n\}$ and $\{z_n\}$ converge weakly to a solution of MSSFP (2), when the MSSFP (2) is consistent; $\{y_n\}$ converges strongly to the minimum-norm solution of MSSFP (2).

Definition 2.1 Let averaged operators U, S and $T_{[n+1]}$ are defined as in Lemma 2.1, let $f: H \to H$ be a combination that $f\left(x_1, x_2\right) := (1-\alpha_n)x_1 + \alpha_n x_2$, $n \ge 0$, where $\alpha_n \in [0,1]$. We set the mappings $X := Uf\left(S, T_{[n+1]}\right)$, $Y := Sf\left(U, T_{[n+1]}\right)$ and $Z := T_{[n+1]}f\left(U, S\right)$ be averaged operators, then let $B: H \to H$ be a averaged operator that $B := a_n X + b_n Y + c_n Z$, $n \ge 0$, where a_n, b_n and c_n are sequences in $\mathbb{R}$, and $a_n + b_n + c_n = 1$.

Lemma 2.3 [1] Let $F: H \to H$ be a k-Lipschitz continuous and η-strongly monotone mapping. For each $\lambda \in \left(0,1\right)$ and a fixed $\mu \in (0, 2\eta/k^2)$, write

$$T^{\lambda} := (I - \lambda\mu F)$$

and

$$\tau := 1 - \sqrt{1 - \mu\left(2\eta - \mu k^2\right)} \in (0,1).$$

Then we have

$$\left\|T^{\lambda}x - T^{\lambda}y\right\| \le (1 - \lambda\tau)\|x - y\|, \tag{8}$$

for all $x, y \in H$, $T^{\lambda}: H \to H$ is a contraction on H.

Lemma 2.4 [17] Let C be a nonempty closed convex subset of real Hilbert space H and let $T: C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed on C, i.e. if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.

Lemma 2.5 [18] Let $\{x_n\}$, $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] which satisfies the following condition: $0 < \varliminf_{n\to\infty}\beta_n \le \varlimsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = \left(1 - \beta_n\right)x_n + \beta_n z_n$ for all $n \ge 0$ and $\varlimsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0$; then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.6 [19] Assume $\{s_n\}$ be a sequence of nonegative real numbers satisfying the following relation $s_{n+1} \le (1 - \sigma_n)s_n + \sigma_n\delta_n$, where $\{t_n\} \subset (0,1)$ and

$\{\sigma_n\}\subset\mathbb{R}$ satisfy the following conditions: (i) $\sum_{n=1}^{\infty}\sigma_n=\infty$; (ii) $\overline{\lim_{n\to\infty}}\,\delta_n\le 0$ or $\sum_{n=1}^{\infty}|\sigma_n\delta_n|<\infty$. Then $s_n\to 0$ as $n\to\infty$.

3. Main Results

In this section, we introduce several hybrid strong convergence algorithms to solve the multi-sets split feasibility problem (2). Namely, we want to find a solution x^* of variational inequality (1) and MSSFP (2). For the successive, parallel and cyclic algorithms, we have the following theorems:

Theorem 3.1 Let H be a real Hilbert space and $F:H\to H$ be a k-Lipschitzian and η-strongly monotone mapping. Denote a averaged operator T_a be U, S or $T_{[n+1]}$, which are defined in Lemma 2.1 respectively. *Let $\{T_i\}_{i=1}^{t}$ be t averaged mappings of H, such that $\mathfrak{F}=\bigcap_{i=1}^{t}Fix(T_i)\neq\varnothing$, take a point $u\in H_1$ and $\forall x_0\in H_1$, a sequence $\{x_n\}_{n\ge 0}$ is generated by the following recursion:*

$$x_{n+1}=(I-\lambda_n\mu F)T_a x_n,\qquad n\ge 0,\tag{9}$$

where $\lambda_n\in(0,1)$ satisfying (P_1) $\lim_{n\to\infty}\lambda_n=0$; and (P_2) $\sum_{n=0}^{\infty}\lambda_n=\infty$, $T_i=P_{C_n}(I-\gamma F)$, $i=1,2,\dots,t$ and $\gamma\in(0,2/L)$. Then the sequence $\{x_n\}$ defined by (9) converges strongly a solution of MSSFP (2), and converges in norm to the unique solution of the variational inequality

$$\langle Fx^*,x-x^*\rangle\ge 0,\quad \forall x\in\Gamma.\tag{10}$$

proof. First, we show that $\{x_n\}$ is monotone and bounded. As T_a is nonexpansive, from Lemma 2.4 and (9), take $p\in\Gamma$, we have

$$\begin{aligned}\|x_{n+1}-p\|&=\|(I-\lambda_n\mu F)(T_a x_n-p)+\lambda_n\mu Fp\|\\&\le(1-\lambda_n\tau)\|x_n-p\|+\frac{\mu}{\tau}\|Fp\|\\&\le max\left\{\|x_0-p\|,\frac{\mu}{\tau}\|Fp\|\right\}.\end{aligned}$$

It indicates that $\{x_n\}$ is bounded.

Next, take

$$\begin{aligned}&\|x_{n+1}-x_n\|\\&=\|(I-\lambda_n\mu F)T_a(x_n-x_{n-1})+(\lambda_{n-1}-\lambda_n)\mu FT_a x_{n-1}\|\\&\le(1-\lambda_n\tau)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|\mu FT_a x_{n-1}\|.\end{aligned}$$

By virtue of (P_1) and (P_2) and Lemma 2.7, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{11}$$

Let $u_n=T_a x_n$, we observe that

$$\begin{aligned}\|x_n-u_n\|&\le\|x_n-x_{n+1}\|+\|x_{n+1}-u_n\|\\&\le\|x_n-x_{n+1}\|+\lambda_n\mu\|Fu_n\|.\end{aligned}$$

from (P_1) and (11), we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{12}$$

Since $\{x_n\}$ is bounded, there is a subsequence $x_{n_i}\rightharpoonup x^*$, as $i\to\infty$. In general, we may assume that $x_n\rightharpoonup x^*$, as $i\to\infty$, combining with (12) and Lemma 2.5, we have $x_n\rightharpoonup x^*\in Fix(T_a)$.

From Lemmas 2.1 and 2.2, we know $u_n\rightharpoonup\tilde{x}$, as $n\to\infty$, and $\tilde{x}\in\Gamma$. Therefore,

$$\overline{\lim_{n\to\infty}}\langle Fx^*,u_n-x^*\rangle=\langle Fx^*,\tilde{x}-x^*\rangle\ge 0,\quad x^*\in\Gamma.\tag{13}$$

Finally, we prove that $x_n\to x^*$ in norm. We take

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\&=\|(I-\lambda_n\mu F)(u_n-x^*)-\lambda_n\mu Fx^*\|^2\\&=\|(I-\lambda_n\mu F)(u_n-x^*)\|^2+\lambda_n^2\mu^2\|Fx^*\|^2\\&\quad-2\lambda_n\mu\langle(I-\lambda_n\mu F)(u_n-x^*),Fx^*\rangle\\&\le(1-\lambda_n\tau)\|x_n-x^*\|-2\lambda_n\mu\langle u_n-x^*,Fx^*\rangle\\&\quad+\lambda_n^2\mu^2\|Fx^*\|^2+2\lambda_n^2\mu^2\|Fu_n-Fx^*\|\|Fx^*\|\\&=(1-\sigma_n)\|x_n-x^*\|+\sigma_n\delta_n,\end{aligned}$$

where $\sigma_n=\lambda_n\tau$,

$$\begin{aligned}\delta_n=&\frac{-2\mu}{\tau}\langle u_n-x^*,Fx^*\rangle\\&+\frac{\lambda_n\mu^2}{\tau}\left(\|Fx^*\|^2+2\|Fu_n-Fx^*\|\|Fx^*\|\right).\end{aligned}$$

It is clear that $\sum_{n=0}^{\infty}\sigma_n=\infty$ and $\overline{\lim_{n\to\infty}}\,\delta_n\le 0$. Hence, from Lemma 2.7 we deduce that $x_n\to x^*$ as $n\to\infty$.

Remark 3.1 When $F=I$ in Theorem 3.1, the corresponding algorithm (9) will reduce to algorithm (6), and converges strongly to the minimum norm solution of MSSFP (2).

Theorem 3.2 Let H be a real Hilbert space and $F: H \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let Ω be a nonempty closed and convex subset of H, and $\{T_i\}_{i=1}^t$ be t averaged mappings of H, such that $\mathfrak{F} = \bigcap_{i=1}^t Fix(T_i) \neq \varnothing$. Given a starting point $x_0 \in \Omega$, the iteration is generated by:

$$\begin{cases} x_0 \in \Omega, y_n^0 = x_n, n \geq 0, \\ y_n^i = T_i y_n^{i-1}, i = 1, 2, \ldots, t, \\ x_{n+1} = P_\Omega \left[(I - \lambda_n \mu F) \left(\varepsilon_n T_{[n+1]} y_n^t + (1-\varepsilon_n) \sum_{i=1}^t \alpha_i T_i y_n^t \right) \right], \end{cases} \tag{14}$$

where $\alpha_i > 0$, for all i such that $\sum_{i=1}^t \alpha_i = 1$, $\lambda_n \in (0,1]$, $\varepsilon_n \in [0,1]$,

$$T_i = P_{C_i}(I - \gamma F), \quad i = 1, 2, \ldots, t,$$

$T_{n\,mod\,t} = P_{C_{n\,mod\,t}}(I - \gamma F)$ and $\gamma \in (0, 2/L)$. Then the sequence $\{x_n\}$ defined by (14) converges strongly a solution x^* of MSSFP (2), and converges in norm to the unique solution of the variational inequality (10).

proof. We know that $P_{\mathfrak{F}} x$ is well defined for each $x \in H$, we show that there exists a unique $x^* \in \mathfrak{F}$ such that

$$x^* = P_{\mathfrak{F}}(I - \mu F)x^*. \tag{15}$$

From Lemma 2.4, we know that $I - \mu F : \Omega \to H$ is contraction, and hence $P_{\mathfrak{F}}(I - \mu F) : \Omega \to \Omega$ is also a contraction on Ω. Then the Banach contraction mapping principle to deduce (15).

Write $u_n = \varepsilon_n T_{[n+1]} y_n^t + (1-\varepsilon_n) \sum_{i=1}^t \alpha_i T_i y_n^t$, then for $\forall p \in \mathfrak{F}$ and $n \geq 0$, that

$$\left\| y_n^1 - p \right\| = \left\| T_1 y_n^0 - T_1 p \right\| \leq \left\| y_n^0 - p \right\| = \left\| x_k - p \right\|,$$

and hence

$$\begin{aligned} \left\| y_n^i - p \right\| &= \left\| T_i y_n^{i-1} - T_i p \right\| \\ &\leq \left\| y_n^{i-1} - p \right\| \\ &\leq \ldots \\ &\leq \left\| y_n^0 - p \right\| = \left\| x_k - p \right\|, \quad i = 1, 2, \ldots, t. \end{aligned} \tag{16}$$

At this point, we can estimate $\|u_n - p\|^2$, by virtue of Lemma 2.1, (14) and (16), we have that

$$\begin{aligned} & \|u_n - p\|^2 \\ &= \varepsilon_n \left\| T_{[n+1]} y_n^t - p \right\|^2 + (1-\varepsilon_n) \left\| \sum_{i=1}^t \alpha_i T_i y_n^t - p \right\|^2 \\ &\quad - \varepsilon_n (1-\varepsilon_n) \left\| T_{[n+1]} y_n^t - \sum_{i=1}^t \alpha_i T_i y_n^t \right\|^2 \\ &\leq \varepsilon_n \left\| y_n^t - p \right\|^2 + (1-\varepsilon_n) \left\| y_n^t - p \right\|^2 \leq \|x_n - p\|^2, \end{aligned}$$

for all $n \geq 0$. Therefore, we have

$$\|u_n - p\| \leq \|x_n - p\|, \quad \text{for all } n \geq 0.$$

In particular, for $x^* = P_{\mathfrak{F}}(I - \gamma F)x^* \in \mathfrak{F}$, we have

$$\left\| u_n - x^* \right\| \leq \left\| x_n - x^* \right\|, \quad \text{for all } n \geq 0. \tag{17}$$

the rest of arguments follows exactly as the corresponding part in Theorem 3.1, we omit its details.

Theorem 3.3 Let H be a real Hilbert space and $F: H \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let Ω be a nonempty closed and convex subset of H, and $\{T_i\}_{i=1}^t$ be t averaged mappings of H, such that $\mathfrak{F} = \bigcap_{i=1}^t Fix(T_i) \neq \varnothing$. Given a starting point $x_0 \in \Omega$, the iteration is generated by:

$$\begin{cases} x_0 \in \Omega \\ y_n^0 = \varepsilon_n T_{[n+1]} x_n + (1-\varepsilon_n) \sum_{i=1}^t \alpha_i T_i x_n, n \geq 0, \\ y_n^i = T_i y_n^{i-1}, i = 1, 2, \ldots, t, \\ x_{n+1} = P_\Omega \left[(I - \lambda_n \mu F) y_n^t \right], \end{cases} \tag{18}$$

where $\alpha_i > 0$, for all i such that $\sum_{i=1}^t \alpha_i = 1$, $\lambda_n \in (0,1]$, $\varepsilon_n \in [0,1]$, $T_i = P_{C_i}(I - \gamma F)$, $i = 1, 2, \ldots, t$, $T_{n\,mod\,t} = P_{C_{n\,mod\,t}}(I - \gamma F)$ and $\gamma \in (0, 2/L)$. Then the sequence $\{x_n\}$ defined by (18) converges strongly to a solution x^* of MSSFP (2), and converges in norm to the unique solution of the variational inequality (10).

The proof of Theorem 3.3 is similar with Theorem 3.2, we also omit it here.

We further introduce two general hybrid strong algorithms.

Theorem 3.4 Let H be a real Hilbert space and $F: H \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let Ω be a nonempty closed and convex subset of

H. Let $g: H \to H$ be κ-contraction, and $\kappa \in (0,k)$. For given $\forall x_0 \in H$, the sequence $\{x_n\}$ is generated by:

$$x_{n+1} = (1-\omega_n)x_n + \omega_n P_\Omega\left[\lambda_n \mu g(x_n) + (I - \lambda_n \mu F)Bx_n\right], \quad n \ge 0, \tag{19}$$

where $\{\omega_n\}$ and $\{\lambda_n\}$ are two sequences in [0,1], satisfying the following conditions: (C_1) $\lim_{n\to\infty} \lambda_n = 0$, and $\sum_{n=1}^{\infty} \lambda_n = \infty$; ($C_2$) $0 < \underline{\lim}_{n\to\infty} \omega_n$; B *is defined by Definition* 2.3. *Then the sequence* $\{x_n\}$ *converges strongly to a solution* x^* *of MSSFP* (2), *and converges in norm to the unique solution of the variational inequality*

$$\left\langle g(x^*) - Fx^*, x - x^* \right\rangle \le 0, \quad \forall x \in \Gamma. \tag{20}$$

If $g = 0$, the sequence $\{x_n\}$ converges strongly to a solution of (10).

proof. First, we prove $\{x_n\}$ is bounded, for $z \in \Gamma$, from Definition 2.3, we have

$$\|Bx_n - z\| \le \|x_n - z\|, \quad \text{for all } n \ge 0,$$

and from (19), that

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&\le (1-\omega_n)\|x_n - z\| + \omega_n \big\|\lambda_n \mu\big(g(x_n) - g(z)\big) \\
&\quad + \lambda_n \mu\big(g(z) - F(z)\big) + (I - \lambda_n \mu F)(B(x_n) - z)\big\| \\
&= \big(1 - \omega_n \lambda_n(\tau - \mu\kappa)\big)\|x_n - z\| + \frac{\mu\|g(z) - F(z)\|}{\tau - \mu\kappa} \\
&\le \max\left\{\|x_n - z\|, \frac{\mu\|g(z) - F(z)\|}{\tau - \mu\kappa}\right\}.
\end{aligned}$$

By introduction, for $\forall n \ge 0$, we have

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{\mu\|g(z) - F(z)\|}{\tau - \mu\kappa}\right\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $g(x_n)$ is also bounded.

Next, we go on to show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Set $W = 2P_\Omega - I$, we know that W is nonexpansive. From Definition 2.3 we know there exists a positive constant $t \in (0,1)$ such that $B = (1-t)I + tV$, where V is a nonexpansive mapping. We can rewrite (19) as

$$x_{n+1} = \left(1 - \frac{\omega_n(1+t)}{2}\right)x_n + \frac{\omega_n(1+t)}{2}u_n, \tag{21}$$

where

$$u_n = \frac{tVx_n + \hat{z}_n + W\tilde{z}_n}{1+t}, \quad \hat{z}_n = \lambda_n \mu g(x_n) - \lambda_n \mu FBx_n,$$

$$\tilde{z}_n = \lambda_n \mu g(x_n) + (I - \lambda_n \mu F)Bx_n.$$

Therefore, by the assumption (C_1) and $t \in (0,1)$, we deduce that

$$0 < \underline{\lim}_{n\to\infty} \frac{\omega_n(1+t)}{2} \le \overline{\lim}_{n\to\infty} \frac{\omega_n(1+t)}{2} < 1. \tag{22}$$

Then from (4) and Definition 2.3, we have

$$\begin{aligned}
\|\hat{z}_{n+1} - \hat{z}_n\| &\le |\lambda_{n+1} - \lambda_n|\,\mu\big(\|g(x_{n+1})\| + \|FBx_{n+1}\|\big) \\
&\quad + \lambda_n \mu(\kappa + k)\|x_{n+1} - x_n\|,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\|\tilde{z}_{n+1} - \tilde{z}_n\| &\le |\lambda_{n+1} - \lambda_n|\,\mu\big(\|g(x_{n+1})\| + \|FBx_{n+1}\|\big) \\
&\quad + \big(1 + \lambda_n \mu(\kappa + k)\big)\|x_{n+1} - x_n\|.
\end{aligned} \tag{24}$$

from (23) and (24) we obtain

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \left(1 + \frac{\lambda_n \mu(\kappa + k)}{1+t}\right)\|x_{n+1} - x_n\| \\
&\quad + \frac{2}{1+t}|\lambda_{n+1} - \lambda_n|\,\mu\big(\|g(x_{n+1})\| + \|FBx_{n+1}\|\big), \text{ that is}
\end{aligned}$$

$$\begin{aligned}
\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\| &\le \frac{\lambda_n \mu(\kappa + k)}{1+t}\|x_{n+1} - x_n\| \\
&\quad + \frac{2}{1+t}|\lambda_{n+1} - \lambda_n|\,\mu\big(\|g(x_{n+1})\| + \|FBx_{n+1}\|\big).
\end{aligned}$$

By virtue of assumption (C_1), it is easy to get

$$\overline{\lim}_{n\to\infty}\big(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|\big) \le 0. \tag{25}$$

In view of (C_1) and (25), $\{u_n\}$ is also bounded, therefore, by using (22), (25) and Lemma 2.6, we can obtain

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \frac{\omega_n(1+t)}{2}\|u_n - x_n\| = 0. \tag{26}$$

Let $v_n = Bx_n$, we have

$$\|x_n - v_n\| \le \|x_{n+1} - x_n\| + (1-\omega_n)\|x_n - v_n\| + \omega_n \|\lambda_n \mu g(x_n) - \lambda_n \mu F B x_n\|.$$

Thus, we have

$\|x_n - v_n\| \le \frac{1}{\omega_n}\|x_{n+1} - x_n\| + \lambda_n \mu \|g(x_n) - FBx_n\|.$ From (C_1) and (26), we can derive that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{27}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that $x_{n_i} \rightharpoonup x^*$, as $i \to \infty$. Thus, we may assume that $x_n \rightharpoonup x^*$ as $n \to \infty$. From (27) and Lemma 2.5, we have $x_n \rightharpoonup x^* \in Fix(B)$.

Next, as $x_n \rightharpoonup x^* \in Fix(B)$, we can deduce that $v_n \rightharpoonup \tilde{x} \in Fix(B)$. Therefore,

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \left\langle (g - FB)x^*, v_n - x^* \right\rangle \\ &\le \left\langle (g - FB)x^*, \tilde{x} - x^* \right\rangle \le 0. \end{aligned} \tag{28}$$

Finally, from (19), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\omega_n)\|x_n - x^*\|^2 + \omega_n \|Bx_n - x^*\|^2 \\ &+ \omega_n \lambda_n^2 \mu^2 \|g(x_n) - FBx_n\|^2 \\ &+ 2\omega_n \lambda_n \mu \left\langle g(x_n) - FBx_n, Bx_n - x^* \right\rangle \\ &\le (1-\omega_n)\|x_n - x^*\|^2 + \omega_n \|Bx_n - x^*\|^2 \\ &+ \omega_n \lambda_n^2 \mu^2 \|g(x_n) - FBx_n\|^2 \\ &+ 2\omega_n \lambda_n \mu \|g(x_n) - g(x^*)\| \|Bx_n - x^*\| \\ &- 2\omega_n \lambda_n \mu \|FBx_n - FBx^*\| \|Bx_n - x^*\| \\ &+ 2\omega_n \lambda_n \mu \left\langle g(x^*) - FBx^*, Bx_n - x^* \right\rangle \\ &= (1-\sigma_n)\|x_n - x^*\|^2 + \sigma_n \delta_n, \end{aligned}$$

where $\sigma_n = 2\omega_n \lambda_n \mu (k - \kappa)$,

$$\delta_n = \frac{\lambda_n \mu}{2(L-\kappa)} \|g(x_n) - FBx_n\|^2 + \frac{1}{k-\kappa}\left\langle (g - FB)x^*, Bx_n - x^* \right\rangle.$$

From Lemma 2.7, (C_1) and (28), it is clear that $\sum_{n=1}^{\infty} \sigma_n = \infty$ and $\overline{\lim_{n\to\infty}} \delta_n \le 0$. Hence from Lemma 2.7 we obtain that $\|x_n - x^*\| \to 0$. The proof is completed.

Similar to Theorem 3.4, another algorithm and its convergence without proof is immediately obtained.

Theorem 3.5 Let H be a real Hilbert space and $F: H \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let Ω be a nonempty closed and convex subset of H. Let $g: H \to H$ be κ-contraction and $\kappa \in (0, k)$. For given $\forall x_0 \in H$, the sequence $\{x_n\}$ is generated by:

$$\begin{aligned} x_{n+1} &= (1-\omega_n)x_n \\ &+ \omega_n P_\Omega \left[\lambda_n \mu g(x_n) + B(I - \lambda_n \mu F)x_n \right], \\ &n \ge 0, \end{aligned} \tag{29}$$

where $\{\omega_n\}$ and $\{\lambda_n\}$ are two sequences in [0,1], satisfying the following conditions: (i) $\lim_{n\to\infty} \lambda_n = 0$, and $\sum_{n=1}^{\infty} \lambda_n = \infty$; ($ii$) $0 < \underline{\lim_{n\to\infty}} \omega_n$; *B is defined by Definition* 2.3. Then the sequence $\{x_n\}$ converges strongly to *the a solution of MSSFP* (2), *and converges in norm to the unique solution of the variational inequality* (20). *If* $g = 0$, *sequence* $\{x_n\}$ *converges strongly to a solution of* (10).

4. Numerical Results

In this section. We chose the algorithms (5), (6), (7), (9) and (14) to solve a modified test problem in [20], and the numerical results were compared. For (5), (6) and (7), set $T = T_{[n+1]}$. Set $T_a = T_{[n+1]}$, $F = I$ in (9) and (14). All the codes were written in Matlab 2011 and run on a PC with Pentium (R) dual-core CPU G630 (2.69 GHz).

Example 4.1.

Denote $e_0 = (0,0,\ldots,0) \in R^N$, $e_1 = (1,1,\ldots,1) \in R^N$ and $e_2 = (1,1,\ldots,1) \in R^M$. The MSSFP with $A = (a_{ij})_{M\times N}$ and $a_{ij} \in (0,1)$ generated randomly.

$C_i = \{x \in \mathbb{R}^N \mid \|x - d_i\| \le r_i\}, \quad i = 1,2,\ldots,t;$

$Q_j = \{y \in \mathbb{R}^M \mid k_j \le y \le l_j\}, \quad j = 1,2,\ldots,r.$

where d_i is the center of the ball C_i, $e_0 \le d_i \le 10e_1$, and $r_i \in (40,50)$ is the radius, d_i and r_i are generated randomly. k_j and l_j are boundary of the box Q_j, and are also generated randomly, satisfying $20e_1 \le k_j \le 30e_1$ and $40e_2 \le l_j \le 80e_2$.

For each algorithm, set $\beta_j = 1/r$, $j = 1,2,\ldots,r$ and

$\gamma = 1.95/L$. Set $a_n = \frac{1}{n+1}$ in (6), $b_n = \frac{n}{n+1}$ in (7), $\lambda_n = \frac{1}{n+10}$, $\mu = 0.1$, $\varepsilon_n = 0.5$ and $\alpha_i = \frac{1}{t}$ in (9) and (14). The stop rule is $p(x) < 10^{-4}$ with initial point $x_0 = e_0$. The numerical results are listed in table 1, where n is the number of iterations and s is the CPU time in seconds, respectively. We see that though algorithm (6) is strong convergence, it usually takes great time. (5) and (7) can get less time by appropriate constant sequence choice, but it is weak convergence. However, (9) and (14) can split the difference.

Table 1. *Result comparison of the chose algorithms.*

M×N	Algorithms	Time	20×20	30×20	40×50	50×50	60×70
	(5)	*n*	702	6478	4032	6627	10185
		s	0.2098	2.0816	1.8030	3.3260	8.1348
	(6)	*n*	94224	111260	139593	174122	143721
t=30		*s*	26.8063	35.9333	63.0368	86.1091	114.3822
	(7)	*n*	735	7312	3599	6201	10717
r=40		*s*	0.2116	2.4229	1.6338	3.1217	8.5977
	(9)	*n*	33	14585	15208	18370	18291
		s	0.0139	4.6702	6.7923	9.2566	14.7097
	(14)	*n*	5	475	623	633	613
		s	0.0739	5.9771	12.6471	13.5881	23.5791

5. Conclusions

This paper presented several strong convergence algorithms with the hybrid steep descent method for solving MSSFP. The algorithms can obatin more effective iterative results than the strong covergence ones before. However, the main drawback of the proposed algorithms is that more complex iteration formulas bring large computational complexity. In order to have less running time and iteration steps, we may continue to choose appropriate variable parameters in the hybrid steepest descent method and use variable or adaptive stepsize in the MSSFP algorithms.

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