On Analytical Approach to Semi-Open/Semi-Closed Sets

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Abstract: The concept of open and closed sets has been extensively discussed on both metric and topological spaces. Various properties of these sets have been proved under the underlying spaces. However, scanty literature is available about semi-open /semi-closed sets on these spaces. For instance, little effort has been made in introducing these sets as clopen sets in topological spaces but no literature exists of the same under metric spaces. In this paper, with reference to the already existing definitions and properties of open and closed sets in metric spaces as well as in topological spaces we shall present definitions of semi-open/ semi-closed sets and furthermore prove basic properties of these sets on metrics spaces. The results of the study will provide a deeper understanding as well as extension knowledge for the concept of open and closed sets to their somewhat counter-intuitive terms of semi- open /semi-closed.

Keywords: Open and Closed Sets, Semi-Open /Semi-Closed Sets, Metric Spaces, Topological Spaces

1. Introduction

We shall mainly give basic definitions and notions related to open/ closed sets in both metric and topological spaces which eventually shall be utilized in the sequel.

Let X be any set. Then a function $d: X \times X \to \mathbb{R}$ is said to be a *metric* on X if it has the following properties for all $x, y, z: \in \mathbb{R}$

$$M1 \; d(x, y) \geq 0$$

$$M2 \; d(x, y) = 0 \; if \; and \; only \; if \; x = y$$

$$M3 \; d(x, y) = d(y, x)$$

$$M4 \; d(x, y) + d(y, z) \leq d(x, z)$$

The real number $d(x, y)$ is called the *distance* between x and y, and the set X together with a metric d is called a *metric space* (X, d). The space X can be generalized to $\mathbb{R}^n$ with d, the discrete metric.

We point out that given any normed vector space $(V, \| \; \|)$ we may treat V as a metric space by defining $d(x, y) = \| x - y \|$ for every $x, y \in V$

We adopt the definition by [2] of a *ball* about a in $\mathbb{R}^n$ of radius r as the set

$$B_r(a) = \{x \in \mathbb{R}^n : \| x - a \| < r\} \tag{1}$$

Given a metric space (X, d) and any real number $r > 0$, the *open ball* of radius r and center x_0 is the set $B_d(x_0, r) \subset X$ defined by

$$B_d(x_0, r) = \{ x \in X : d(x, x_0) < r\} \tag{2}$$

Since the metric d is usually understood, we will generally leave off the subscript d and simply write $B(x_0, r)$. Such a set is frequently referred to as an r -*ball*. We say that a subset U of X is *open* if, given any point $x \in U$, there exists $r > 0$ and an open ball $B(x, r)$ such that $B(x, r) \subset U$.

Probably the most common example of an open set is the open unit disk D_1 in $\mathbb{R}^2$ defined by

$$D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}. \tag{3}$$

We see that given any point $x_0 \in D_1$, we can find an open ball $(x_0, r) \subset D_1$ by choosing

$r = 1 - (x_0, 0)$. The set $D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ (4)

is not open because there is no open ball centered on any of the boundary points $x^2 + y^2 = 1$ that is contained entirely within D_1 [4].

If U is an open subset of a metric space (X, d), then its complement $U^C = X - U$ is said to be *closed*. In other words, a set is closed if and only if its complement is open. For example, a moment's thought should convince you that the subset of $\mathbb{R}^2$ defined by

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \tag{5}$$

is a closed set. The *closed ball* of radius r centered at x_0 is the set $[x_0, r]$ defined in the obvious way by

$$[x_0, r] = \{x \in X : (x_0, r) \leq r\}. \tag{6}$$

Any open set U containing a point x is said to be *a neighborhood* of x, and the set $U - \{x\}$ is called *a deleted neighborhood* of x. We say that a point $x \in (X, d)$ is *an accumulation point* (also known as a limit point) of $A \subset X$ if every deleted neighborhood of x intersects A [8]. {Alternatively, a point x of $A \subset X$ is a *limit point* if every x contains a point $y \neq x$ such that $y \in A$ [3]}.

Accumulation points are useful in determining whether or not a set is closed. The principal result relating these concepts is the following, which of course gives is an alternative definition of a closed set. This is illustrated in the following theorem.

Theorem 1.1 [10]: A subset A of a metric space (X, d) is closed if and only if A contains all of its accumulation points.

Let (X, d) be a metric space, and suppose $A \subset X$. We define

(a) The *closure* of A, denoted by $Cl\ A$, to be the intersection of all closed supersets of A;

(b) The *interior* of A, denoted by $Int\ A\ (or\ A^0)$, to be the union of all open subsets of A;

(c) The *boundary* of A, denoted by $Bd\ A\ or\ (\delta A)$, to be the set of all $x \subset X$ such that every open set containing x contains both points of A and points of $A^C = X - A$.

(d) The *exterior* of A, denoted by $Ext\ A$, to be $(Cl\ A)^C = X - Cl\ A$

(e) The *derived set* of A, denoted by A', to be the set of all accumulation points of A [4].

In [3] the following are given as facts about points in interior, exterior and closure as well as on the boundary of a set.

Fact 1: Let (X, d) be a metric space and $A \subset X$. A point is interior if and only if it has an open ball that is a subset of the set

$$X \in int\ A \Leftrightarrow \exists\ \varepsilon > 0 : B(x, \varepsilon) \subset A \tag{7}$$

Fact 2: A point is in the closure if and only if any open ball around it intersects the set

$$X \in \bar{A} \Leftrightarrow \exists\ \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset \tag{8}$$

Fact 3: A point is exterior if and only if an open ball around it is entirely outside the set

$$X \in ext\ A \Leftrightarrow \exists \varepsilon > 0 : B(x, \varepsilon) \subset X \backslash A \tag{9}$$

Fact 4: A point is on the boundary if any open ball around it intersects the set and intersects the outside of the set

$$X \in \delta A \Leftrightarrow \exists \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset \tag{10}$$

An open set can also be characterized using the concept of interior points as: A subset A of a metric space (X, d) is open if every point of A is an interior point of A.

If (X, d) is a metric space, then $A \subset X$ is said to be *somewhere dense* if $int\ (cl\ A) \neq \emptyset$. The set A is said to be *nowhere dense* if it is not somewhere dense. If $cl\ A = X$, then A is said to be *dense* in X [4].

Theorem 1.2 [4]: A subset A of a metric space (X, d) is dense if and only if every open subset U of X contains some point of A.

Shifting our focus to topological spaces, it is worth noting that the theory of topological spaces provides a setting for the notions of continuity and convergence which is more general than that provided by the theory of metric spaces.

A *topological space* denoted by (X, τ) is a non-empty set X together with a collection of τ subsets, (referred to as open sets), that satisfies the following conditions:

(i) The empty set; and the whole set X are open sets,

(ii) The union of any collection of open sets is itself an open set,

(iii) The intersection of any finite collection of open sets is itself an open set.

However it is customary to denote this topological space simply by X if no confusion will arise. Note that any metric space may be regarded as a topological space since all the topological space axioms are satisfied by the collection of open sets in any metric space. Also, any subset X of n dimensional Euclidean space $\mathbb{R}^n$ is a topological space. For instance, a subset V of X is open in X if and only if, given any point v of V, there exists some $\delta > 0$ such that

$$x \in X : |x - v| < \delta \subset V. \tag{11}$$

In particular $\mathbb{R}^n$ is itself a topological space whose topology is generated by the Euclidean distance function on $\mathbb{R}^n$. This topology on $\mathbb{R}^n$ is referred to as the usual topology on $\mathbb{R}^n$ [1].

2. Related Literature Review

This section presents a review of related literature on basic properties of open and closed sets as applied both in metric and topological spaces. We also present some results which so far exist about semi-open sets in topological spaces.

2.1. Properties of Open Sets in Metric Spaces

The fundamental characterizations of open sets are contained in the following three theorems.

Theorem 2.1.1 [10]: Let (X, d) be a metric space. Then any open ball is an open set.

Theorem 2.1.2 [6]: Let (X, d) be a metric space. Then

(a) Both X and $\emptyset$ are open sets.

(b) The intersection of a finite number of open sets is open.

(c) The union of an arbitrary number of open sets is open.

Remark 2.1.3: It is remarkable that in an arbitrary metric space the structure of the open sets can be very complicated. However, the most general description of an open set is contained in the following.

Theorem 2.1.4 [4]**:** A subset U of a metric space (X, d) is open if and only if it is the union of open balls.

Remark 2.1.5: We note that a set is never open in and of itself. Rather, a set is open only with respect to a specific metric space containing it. For example, the set of numbers $[0, 1)$ is not open when considered as a subset of the real line because any open interval about the point 0 contains points not in $[0, 1)$. However, if $[0, 1)$ is considered to be the entire space X, then it is open by Theorem 2.1.2 (a).

2.2. Properties of Closed Sets in Metric Spaces

Theorem 2.2.1 [6]: Let (X, d) be a metric space. Then any closed ball is a closed set.

The proof of the closed set analogue of Theorem 2.1.2 is discussed by several authors. For instance refer to [8, 6]. This is illustrated as in the theorem that follows.

Theorem 2.2.2 [6]: Let (X, d) be a metric space. Then

(a) Both X and $\emptyset$ are closed sets.

(b) The intersection of an arbitrary number of closed sets is closed.

(c) The union of a finite number of closed sets is closed.

The important difference to realize is that the intersection of an arbitrary number of closed sets is closed, while only the union of a finite number of closed sets is closed.

WLOG, it thus follows the theorem:

Theorem 2.2.3 [6]:

(a) Let A be any set in a metric space S. Then A is $c\,l\,o\,s\,e\,d \Leftrightarrow A$ is open.

(b)The space S is both open and closed.

(c)The null set is both open and closed.

2.3. Open/Closed and Semi-Open Sets in Topological Spaces

We present some existing literature about open/closed and semi-open sets in topological spaces.

[1] gives two examples of cases for a topology as:

Example 1: Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example 2: Given any set X, one can define a topology on X in which the only open sets are the empty set; and the whole set X.

Further, by defining a subset F of as a closed set if and only if its complement $X \backslash F$ is an open set, [1] provides the following result from the definition of a topological space.

Proposition 2.3.1 [1]: Let X be a topological space. Then the collection of closed sets of X has the following properties:

(i) The empty set; and the whole set X are closed sets,

(ii) The intersection of any collection of closed sets is itself a closed set,

(iii) The union of any finite collection of closed sets is itself a closed set.

Similarly, by giving the following definition of a neighborhood, [1] claims that one can readily verify that this definition of neighborhoods in topological spaces is consistent with that for neighborhoods in metric spaces. This notion is presented in Lemma 2.3.2. We first give the definition of a neighborhood: - Let X be a topological space, and let x be a point of X. Let N be a subset of X which contains the point x. Then N is said to be a *neighborhood* of the point x if and only if there exists an open set U for which $x \in U$ and $U \subset N$.

Lemma 2.3.2 [1]: Let X be a topological space. A subset V of X is open in X if and only if V is a neighborhood of each point belonging to V.

According to the Wikipedia, the free encyclopedia, a *clopen set* (representing a closed-open set) in a topological space is a set which is both open and closed. This set shall refer to a semi open/semi-closed set in our study.

However, we note that the concept of semi-open sets in topological spaces was introduced and discussed by [7]. He defines a subset A of a topological space X as semi-open (written $s\,.\,o$.) if and only if there exists an open set O such that $O \subset A \subset cO$ where cO denotes the closure operator in X. Under this context, [7] presents some properties of semi-open sets in the following theorems:

Theorem 2.3.3 [7]: A subset A in a topological space X is $s.o.$ if and only if $A \subset cInt\,A, Int$ denoting the interior operator.

Theorem 2.3.4 [7]: Let $\{A_\alpha\}, \alpha \in \wedge$ be a collection of $s.o.$ sets in a topological space X. Then $\cup A_\alpha, \alpha \in \wedge$ is $s.o.$

Theorem 2.3.5 [7]: Let A be $s.o.$ in the topological space X and suppose $\subset B \subset c\,A$. Then B is $s.o.$

[7] also gives the following concluding remark.

Remark 2.3.6: If O is open in X, then O is semi-open in X. The converse is clearly false.

We shall therefore, extent the known results of open/closed sets on metric and topological spaces to semi-open/ semi-closed sets in metric spaces.

3. Main Results

3.1. Precise Analytical Definitions of Semi-Open/ Semi-Closed Sets in Metric Spaces

We therefore give our precise definitions of semi-open/ semi-closed sets in metric spaces.

Definition: Semi-open/ semi-closed set

A subset S of a metric space M is said to be a semi-open/semi-closed if it contains some of its limit points. Alternatively, a subset S of a metric space M is said to be semi-open/semi-closed if the neighborhood of some elements is contained in S.

Example

$$S = \{\frac{1}{n} : n = 1, 2, 3 \dots \dots\}$$

Implies that $S = [1, \frac{1}{2}, \frac{1}{3}, \dots 0)$

We note that 0 is a limit point of S but it is not an element of S implying that the set S is not closed. Also the set S is not open since the neighborhood of 0 is not properly contained in S. This then implies that the set S is semi-open/ semi-closed set.

We recall the definitions of a limiting point and neighborhood of a point x of a subset $S \subset M$ as:

Definition: A Limiting point

We say that $x \in S \subset M$ is called *a limiting point/cluster point/accumulation point* of subset S if the neighborhood of a point x contains other points other than x, i.e.

$$N(x,r)\backslash\{x\} \cap S \neq \emptyset. \tag{12}$$

Definition: A Neighborhood

A neighborhood of the point $x \in M$ is defined as $\forall\, \varepsilon > 0$ we have

$$N(x, \varepsilon) = |x - y| < \varepsilon \,\forall\, y \in M. \tag{13}$$

Remark 3.1.1: Semi-open/semi-closed sets can be categorized into two main categories on the basis of the position of openness and closedness. The categories are:

(i) Lower open-upper closed set denoted by $S\,(a, b]$ where $S \in M \,\forall\, a, b \in S$.

(ii) Lower closed-upper open set denoted by $S\,[a, b)$ where $S \in M \,\forall\, a, b \in S$.

Definition: Semi-open/Semi-closed balls (spheres). The set

$$S(a, r] = \{x \in M : |x - a| \leq r\} \tag{14}$$

is called *lower open-upper closed ball/sphere* whereas the set

$$S\,[a, r) = \{x \in M : |x - a| < r\} \tag{15}$$

is called *lower closed-upper open ball/sphere.*

Remark 3.1.2: The lower open-upper closed sphere shall refer to a semi-closed sphere whereas the lower closed-upper open sphere shall be a semi-open sphere.

Alternative definition of semi-open/semi-closed sets

Definition: We say that a subset $G \subset M$ is a semi-open set if

$$\forall\, x \in G, \exists\, r > 0 : S\,[x, r) \subset G, \tag{16}$$

while a subset $\mathfrak{H} \subset M$ is a semi-closed set if

$$\forall x \in \mathfrak{H}, \exists\, r > 0 : S\,(x, r] \subset \mathfrak{H}. \tag{17}$$

Example.

Show that $[-1, 0)$ is semi-open.

Solution.

By definition we need to show that

$$\exists\, S\,[x, r) \subset [-1, 0) = G \; i.e. [-1, 0) = G \tag{18}$$

is a semi-open set.

Taking any arbitrary element $x \in [-1, 0)$ and let $r = min\,\{-1 - x, x\}$

Clearly, $r > 0$ and $S[x, r) \subset [-1, 0)$.

3.2. Analytical Properties of Semi-Open/Semi-Closed Set in Metric Spaces

Theorem 3.2.1:

(a) An arbitrary union of semi-open (resp. semi-closed) sets is semi-open (resp. semi-closed).

(b) Finite intersection of semi-open (resp. semi-closed) sets is semi-open (resp. semi-closed).

Proof of (a):

Case 1: Semi-open sets

We let $G_\lambda : \lambda \in \Lambda$ be a family of semi-open sets.

We need to show that $G = \cup G_\lambda : \lambda \in \Lambda$ is semi-open i.e.

$$\exists\, S[x, r) \subset G = \cup G_\lambda : \lambda \in \Lambda \tag{19}$$

Let $x \in \cup G_\lambda : \lambda \in \Lambda$ then it implies that $x \in G_\lambda : \lambda \in \Lambda$.

Since $G_\lambda : \lambda \in \Lambda$ is semi-open, by definition

$$\exists\, r > 0 : S[x, r) \subset \cup G_\lambda : \lambda \in \Lambda \tag{20}$$

So $S[x, r) \subset \cup G_\lambda : \lambda \in \Lambda = G$ implying that G is semi-open.

Case 2: Semi-closed sets

To show that arbitrary union of semi-closed set is also semi-closed, we shall let $\mathfrak{H}_\lambda : \lambda \in \Lambda$ be a family of semi-closed sets. We then show that

$$\mathfrak{H} = \cup \mathfrak{H}_\lambda : \lambda \in \Lambda \; i.e.\, \exists\, S(x, r] \subset \mathfrak{H} = \cup \mathfrak{H}_\lambda : \lambda \in \Lambda \tag{21}$$

Let $x \in \cup \mathfrak{H}_\lambda : \lambda \in \Lambda$, this implies that $x \in \mathfrak{H}_\lambda : \lambda \in \Lambda$. Since $\mathfrak{H}_\lambda : \lambda \in \Lambda$ is semi-closed, by definition

$$\exists\, r > 0 : s(x, r] \subseteq \mathfrak{H}_\lambda \subset \cup \mathfrak{H}_\lambda : \lambda \in \Lambda \tag{22}$$

So $S(x, r] \subset \cup \mathfrak{H}_\lambda : \lambda \in \Lambda = \mathfrak{H}$, implying that $\mathfrak{H}$ is semi-closed.

Proof of (b):

We prove for the case of semi-open sets (the case of semi-closed is an analogous)

Let $G_1, G_2, \ldots . G_n$ be a family of semi-open sets. We need to show that $G = \bigcap\limits_{i=1}^{n} G_i$ is also semi-open. Let

$$x \in \bigcap\limits_{i=1}^{n} G_i \Rightarrow x \in G_i \,\forall i = 1, 2, 3, \ldots, n \tag{23}$$

Since $G_i \,\forall i = 1, 2, \ldots \ldots, n$ are semi-open

$$\exists\, r_i > 0 : S(x_i, r_i] \subset G_i \,\forall i = 1, 2, \ldots . n. \tag{24}$$

Taking $r = \min\{r_i\}$ then clearly $r > 0$ and

$$S(x, r] \subseteq S(x_i, r_i] \subseteq G_i \subset \bigcap\limits_{i=1}^{n} G_i \tag{25}$$

$$\Rightarrow S(x, r] \subset G$$

Therefore, $S(x, r] \subset \bigcap\limits_{i=1}^{n} G_i = G \Longrightarrow$ finite intersection of

semi-open sets is also semi-open.

Remark 3.2.2:

Since in $\mathbb{R}^1$ an open ball is an open interval and also a closed ball is a closed interval, it follows that in $\mathbb{R}^1$ a semi-open/semi-closed ball is a semi-open/semi-closed interval respectively.

3.3. Relative Semi-Closed and Semi-Open Subsets

Recall: We say that a subset F of any metric space M is open if its complement, F^C is closed, it follows then that a subset $\mathcal{F}$ of any metric space M is semi-closed/lower open-upper closed if its complement is semi-open/lower closed-upper open.

Note: The sets F and $\mathcal{F}$ are disjoint.

Theorem 3.3.1:

The complement of a semi-closed set is semi-open.

Proof:

Equivalently, we show that the complement of a lower open-upper closed set is the lower closed-upper open. In case 1, we consider the lower open end. Now since the lower part is open, then its complement must be closed. Suppose $S \subset M$ is the lower open-upper closed set, then we prove that S^C is lower closed- upper open set.

If the lower part of S is open, then we prove that the complement of the lower part is closed.

Let $X_n \to x$ in M of the lower part of S and $X_n \in S^c_{(lowerpart)} \, \forall n$, we need to show that $x \in S^c_{(lowerpart)}$. Since the lower part of S is open, then

$$\exists r > o: |x - y| < r \; \forall y \in S_{(lower\;part)}. \tag{26}$$

Now if

$$X_n \to x \; \exists N: |X_n - x| < r \; \forall \, n \geq N \tag{27}$$

This implies that $X_n \in S_{(lowerpart)}$. $\implies X_n \in S_{(lowerpart)}$ and $X_n \in S^c_{(lowerpart)}$ is a contradiction, since no element could be in $S_{(lowerpart)}$ and $S^c_{(lowerpart)}$ at the same time.

In case 2, we consider the upper closed end. Since the upper part of S is closed, then its complement is open. Now we suppose that $S \subset M$ is the lower open-upper closed set then we prove that S^c is the lower closed- upper open set.

This is proved by contradiction. Suppose $S^c_{(upperpart)}$ is not open, then it implies that $S^c_{(upperpart)}$ is closed.

Now since $S^c_{(upperpart)}$ is closed, $\forall \, r_n > 0$ and $x \in S^c_{(upper\;part)}$

$$\exists X_n: |X_n \to x| < r_n \tag{28}$$

But $X_n \in S$ since the upper part of S is open.

Let $\quad r_n = \frac{1}{n} \, \forall \, n \in N, \Rightarrow X_n \to x,$ meaning that $X_n \in S_{(upperpart)} \, \forall \, n$ and $X_n \to x \in S^c_{(upperpart)}$

$\implies$ the upper part of s is not closed since it fails to contain all its limit points $e.g. x$ and therefore, our initial assumption

that the upper part of the complement of S is not open is invalid implying that the upper part of the complement of S must be open.

Corollary 3.3.2: The complement of the lower closed-upper open set/semi-open set is lower open-upper closed/semi-closed set.

4. Conclusion and Suggestions for Future Research

In this paper, with reference to the already existing definitions and properties of open and closed sets in metric spaces as well as in topological spaces precise definitions of semi-open/ semi-closed sets were given and furthermore basic analytical properties of these sets on metrics spaces were proved. A version of the definition of semi-closed and semi-open sets in terms of relative complementation was also introduced. However, it is not certain that the analytical properties discussed in section 3.2 would also hold under this context of relative complementation. This could thus form a basis for a venture in the future research as well as extending these properties to topological spaces.

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