Asymptotic Properties of Optimized Type CVaR Estimator for NA Random Variables

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Abstract: VaR and CVaR are important risk measures, which are widely used in finance, economy, insurance and other fields. However, VaR is not a coherent risk quantity, and it is not sufficient to measure tail risk. CVaR (also known as expected shortfall, ES) is a coherent risk measure, and it makes up for the defect that VaR is not enough to measure tail risk. Therefore, CVaR has been paid more and more attention in both application and theory fields. Rockafellar and Uryasev (2000) and Trindade et al (2007) proposed an optimized type CVaR estimator and studied some asymptotic properties of the estimator. Since then, some scholars have discussed the properties of the estimator in the cases of ρ-mixing, φ-mixing and α-mixing. In this paper, we shall study the asymptotic properties of the optimized type CVaR estimator in the case where the samples are NA random variables. The consistency and the asymptotic normality of the optimized type CVaR estimator and their corresponding convergence rates are obtained. The convergence rates of estimation are $n^{-1/2}$ or near to $n^{-1/2}$. These results also establish the asymptotic relations of the optimized type CVaR estimator and the common CVaR estimator. And their deviation converges almost surely to 0 at the rate of $n^{-1/2}$.

Keywords: CVaR Estimator, Consistency, Asymptotic Normality, Convergence Rate, NA Sequence

1. Introduction

In the fields of finance, economy and insurance, it is very important to measure risk accurately. In financial filed, VaR is widely used to measure risk, which refers to the maximum possible loss under a certain confidence level for financial assets or portfolios in the holding period. Let X be a loss variable of financial asset or portfolio, for a given small probability $\alpha \in (0,1)$, in the confidence level of $1 - \alpha$, VaR is defined as

$$VaR_\alpha(X) = \inf\{u: F(u) \geq 1 - \alpha\}, \quad (1)$$

where $F(u)$ is distribution function of loss variable.

However, Artzner et al. [2-3] found that the VaR doesn't satisfy subadditivity, namely, it hasn't convexity. So VaR is not a coherent risk measure, and it is insufficient to measure the tail risk. As an alternative measure of risk, CVaR (also known as expected shortfall, ES) is proposed, also known as the average excess loss, which denotes the expected loss which greater than the given VaR value for a portfolio, and its formula as

$$CVaR_\alpha(X) = E\big(X|X \geq VaR_\alpha(X)\big). \quad (2)$$

Rockafellar and Uryasev [15] proved that the definition of CVaR is equivalent to the solution of an optimization problem, namely

$$CVaR_\alpha(X) = \inf_{x \in R}\{x + \alpha^{-1}E[X - x]^+\}, \quad (3)$$

where $[a]^+ = \max\{0, a\}$. Pflug [14] proved the CVaR satisfies subadditivity, also is a coherent risk measurement. More researches are available in Rockafellar and Uryasev [16], Föllmer and Schied [6].

CVaR has better properties than VaR. Therefore, it is suitable as the optimization tool of investment portfolio,

which attracts scholar's widely attention. For example, Andrew et al. [1], Mansini et al. [12], Noyan and Rodulf [13], Sun et al. [22]. Noyan and Rodulf [13] extended to multivariate CVaR, expand its application range. Also in other fields it is widely used, such as risk issues of books and newspapers selling (Gotoh and Takano, [7]), optimal placement of the gas detector in the petrochemical plant (Legg et al., [8]), wind power generation system (Chen et al., [4]), and so on.

Scaillet [18], Luo and Yang [10], Luo and Ou [11] and references therein studied the CVaR kernel estimator. Escanciano and Mayoral [5], Wang and Zhao [24] studied the CVaR semi-parametric estimation. On the basis of the equivalence definition of formula (3), Trindade et al. [23] proposed a kind of optimization CVaR estimation, and studied the consistency and the asymptotic normality of the optimization CVaR estimation. For the optimization CVaR estimation, Luo and Yang [10] studied the strong consistency and asymptotic normality under the ρ-mixing samples, and its convergence rate. Xing et al. [25] studied the strong consistency under the ϕ-mixing sample, and gave the rate of convergence. However, negatively associated (NA) sequence is also common in dependence. It was proposed by Joag-Dev in 1983, and then attracts many scholars, such as Su Chun [20], Roussas [17], Yang [27], Zhou et al. [28] and so on. However, as we know, the asymptotic normality on the optimal CVaR estimation under NA sequence hasn't been studied, so this article will focus on the consistence and asymptotic normality of optimization CVaR estimation under NA sequence.

In the end of this section, we give some relevant definitions as follows.

The random variables $X_1, X_2, \cdots, X_n, n \geq 2$ are said to be negatively associated (NA), if, for two arbitrary disjoint subsets A_1 and A_2 of $\{1, 2, \cdots, n\}$,

$$\text{Cov}\left(f_1(x_i, i \in A_1), f_2(x_j, j \in A_2)\right) \leq 0, \tag{4}$$

where f_1 and f_2 are non-increasing or nondecreasing functions. Random sequence $\{X_n, n \geq 1\}$ is said to be NA if

$X_1, X_2, \cdots, X_n$ are NA for any $n \geq 2$.

Let $h_\alpha(X, x) = x + \alpha^{-1}[X - x]^+$, $\theta^* = CVaR_\alpha(X)$, from the (3) we have

$$\theta^* = \inf_{x \in \mathbb{R}} E h_\alpha(X, x). \tag{5}$$

Assume $X_1, X_2, \cdots, X_n$ is a sample coming from the population X, Rockafellar and Uryasev [15] and Trindade et al [23] define the optimization estimation for θ^*, namely

$$\hat{\theta}_n = \inf_{x \in \mathbb{R}} n^{-1} \sum_{i=1}^{n} h_\alpha(X_i, x). \tag{6}$$

2. Main Result

To describe our main results, we firs introduce some denotations. For any given $x \in \mathbb{R}$, let

$$Z_i(x) = h_\alpha(X_i, x), g_n(x) = \frac{1}{n}\sum_{i=1}^{n} h_\alpha(X_i, x) = \frac{1}{n}\sum_{i=1}^{n} Z_i(x), \tag{7}$$

$$S_n(x) = \sigma_n^{-1}(x)\left(g_n(x) - E g_n(x)\right), F_n^x(u) = P(S_n(x) < u), \tag{8}$$

where $\sigma_n^2(x) = \text{Var}(g_n(x))$.

We need the following basic assumptions.

(A1) $\{X_i : i \geq 1\}$ is a sequence of identically distributed NA random variables with $EX_1 = 0$ and $E(|X_1|^{2+\delta}) < \infty$ for some $\delta > 0$.

(A2)　　　　　$u(1) < \infty$　　　　　, 　　　　where $u(n) = \sup_{j \geq 1} \sum_{i:|i-j| \geq n} |\text{Cov}(X_i, X_j)|$.

(A3) $\sigma^2(x) > 0$ for any $x \in \mathbb{R}$, where $\sigma^2(x) = \lim_{n \to \infty} n\sigma_n^2(x)$.

(A4) There exist positive integers p_n and q_n such that

$$p_n + q_n \leq n, q_n p_n^{-1} \leq C < \infty, \tag{9}$$

and, as $n \to \infty$,

$$\gamma_{1n} \to 0, \gamma_{2n} \to 0, u(q_n) \to 0. \tag{10}$$

where $\gamma_{1n} = q_n p_n^{-1}, \gamma_{2n} = p_n n^{-1}$.

Because $h_\alpha(X, x)$ is monotone increasing about X, $\{Z_i(x) : i \geq 1\}$ is also a NA sequence. Under the assumptions (A1) and (A2), we have that

$$n\sigma_n^2(x) = n^{-1}\sum_{i=1}^{n} \text{Var}(Z_i(x)) + 2n^{-1}\sum_{1 \leq i < j \leq n} \text{Cov}\left(Z_i(x), Z_j(x)\right)$$
$$= \text{Var}(Z_1(x)) + 2n^{-1}\sum_{1 \leq i < j \leq n} \text{Cov}\left(Z_i(x), Z_j(x)\right). \tag{11}$$

Let $\Gamma_n = n^{-1}\sum_{1 \leq i < j \leq n} \text{Cov}\left(Z_i(x), Z_j(x)\right)$. By Lemma 4.1 in section 4,

$$|\Gamma_n| \leq Cn^{-1}\sum_{1 \leq i < j \leq n} |\text{Cov}(X_i, X_j)|$$

$$= Cn^{-1}\sum_{j=2}^{n}\sum_{i=1}^{j-1} |\text{Cov}(X_i, X_j)|$$

$$\leq Cn^{-1}\sum_{j=2}^{n}\sum_{i:|i-j| \geq 1} |\text{Cov}(X_i, X_j)|$$

$$\leq Cn^{-1}\sum_{j=2}^{n}\sup_{j \geq 1}\sum_{i:|i-j| \geq 1} |\text{Cov}(X_i, X_j)|$$

$$\leq C\sup_{j \geq 1}\sum_{i:|i-j| \geq 1} |\text{Cov}(X_i, X_j)| = Cu(1) < \infty. \tag{12}$$

Hence, Γ_n is monotonous and bounded about n. Therefore, Γ_n has finite limit as $n \to \infty$, so that $\lim_{n \to \infty} n\sigma_n^2(x) = \sigma^2(x) < \infty$, and the assumption (A3) implies that

$$\sigma_n^2(x) = O(n^{-1}), \sigma_n^{-2}(x) = O(n). \tag{13}$$

Theorem 2.1. Assume that $\{X_i : i \geq 1\}$ is a sequence of identically distributed NA random variables with $EX_i = 0$ and $E|X_i|^r < \infty$ for some $r \geq 1$. Then we have

$$\hat{\theta}_n - \theta^* = o(n^{-\tau}), a.s., \tag{14}$$

where (1) when $1 \leq r < 2$, $\tau = 1 - \frac{1}{r}$; (2) when $r \geq 2$, we can take any $\tau \in [0, 1/2)$.

Theorem 2.2. Suppose that the assumptions (A1)-(A4) are

satisfied. Then for any given $x \in \mathbb{R}$, we have

$$\sup_u |F_n^x(u) - \Phi(u)| \leq C\left(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + \gamma_{2n}^{\delta/2} + u^{1/3}(q_n)\right), \quad (15)$$

where $\Phi(u)$ is the distribution function of $N(0,1)$.

Let λ_1 and λ_2 satisfy $0 < \lambda_1 < \lambda_2 < 1$. Take $q_n = [n^{\lambda_1}]$, $p_n = [n^{\lambda_2}]$ where $[x]$ denotes the integer part of x. In this case, the assumption (A4) is satisfied. Therefore, we have the following corollary.

Corollary 2.1. If the assumptions (A1)-(A3) are satisfied, then for any given $x \in \mathbb{R}$, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (h_\alpha(X_i, x) - Eh_\alpha(X_i, x)) \xrightarrow{d} N(0, \sigma^2(x)). \quad (16)$$

Theorem 2.3. Let $q = VaR_\alpha(X), q^- = \sup\{x : F(x) \leq 1 - \alpha\}$. If (A1)-(A3) are satisfied, we have

$$\hat{\theta}_n = \inf_{x \in [q^-, q]} n^{-1} \sum_{i=1}^{n} h_\alpha(X_i, x) + o_p(n^{-1/2}). \quad (17)$$

There are two cases to consider:
(i) If $q = q^-$, we have

$$\hat{\theta}_n = n^{-1} \sum_{i=1}^{n} h_\alpha(X_i, q) + o_p(n^{-1/2}), \quad (18)$$

and

$$n^{1/2}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, \sigma^2(q)), \quad (19)$$

where

$$\sigma^2(q) = \lim_{n \to \infty} n\sigma_n^2(q) = \lim_{n \to \infty} (n\alpha^2)^{-1} Var(\sum_{i=1}^{n} [X_i - q]^+). \quad (20)$$

Moreover,

$$\hat{\theta}_n = n^{-1} \sum_{i=1}^{n} h_\alpha(X_i, q) + o(n^{-\tau}), a.s. \quad (21)$$

for any $\tau \in (0, 1/2)$.

(ii) If $q > q^-$, then the limit distribution of $n^{1/2}(\hat{\theta}_n - \theta^*)$ is not a normal distribution.

Remark 2.1. Recall that X is a loss variable of financial asset or portfolio. In practice, the loss variable is usually continuous, so that $q = q^-$ in Theorem 2.3.

Remark 2.2. Let $\tilde{\theta}_n = n^{-1} \sum_{i=1}^{n} h_\alpha(X_i, q)$. Note that $q = VaR_\alpha(X)$. Then

$$\begin{aligned}\tilde{\theta}_n &= n^{-1} \sum_{i=1}^{n} (q + \alpha^{-1}[X_i - q]^+) \\ &= VaR_\alpha(X) + \frac{1}{n\alpha} \sum_{i=1}^{n} [X_i - VaR_\alpha(X)]^+, \end{aligned} \quad (22)$$

it is a common estimator of CVaR. Therefore, the equations (17), (18) and (21) establish the asymptotic relations of the optimized type CVaR estimator $\tilde{\theta}_n$ and the common CVaR estimator $\tilde{\theta}_n$.

Remark 2.3. To give the confidence interval of CVaR θ^* by (19), we need to estimate the variance $\sigma^2(q)$. Let $\hat{q}_n = \hat{F}_n^{-1}(1 - \alpha)$ where $\hat{F}_n^{-1}(x)$ is empirical distribution function of sample $X_1, X_2, \cdots, X_n$. Denote $V_i = [X_i - \hat{q}_n]^+$, $i = 1, 2, \cdots, n$. From (20), it is easy to get the estimation of $\sigma^2(q)$ as following

$$\hat{\sigma}^2(q) = \frac{1}{(n\alpha)^2} \sum_{i=1}^{n} (V_i - \overline{V})^2, \quad (23)$$

where $\overline{V} = \frac{1}{n} \sum_{i=1}^{n} V_i$.

3. Proof of Theorem 2.1

From the definition of $\hat{\theta}_n$ and θ^*, we have

$$n^\tau |\hat{\theta}_n - \theta^*|$$

$$= n^\tau \left| (\hat{\theta}_n - \theta^*) I_{(\hat{\theta}_n - \theta^* \geq 0)} + (\hat{\theta}_n - \theta^*) I_{(\hat{\theta}_n - \theta^* < 0)} \right|$$

$$\leq n^\tau |\hat{\theta}_n - \theta^*| I_{(\hat{\theta}_n - \theta^* \geq 0)} + n^\tau |\hat{\theta}_n - \theta^*| I_{(\hat{\theta}_n - \theta^* < 0)}$$

$$= n^\tau \left| \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) - \inf_{y \in \mathbb{R}} Eh_\alpha(X_1, y) \right| I_{(\hat{\theta}_n - \theta^* \geq 0)}$$

$$+ n^\tau \left| \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) - \inf_{y \in \mathbb{R}} Eh_\alpha(X_1, y) \right| I_{(\hat{\theta}_n - \theta^* < 0)}$$

$$= n^\tau \inf_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) - \inf_{y \in \mathbb{R}} Eh_\alpha(X_1, y) \right| I_{(\hat{\theta}_n - \theta^* \geq 0)}$$

$$+ n^\tau \inf_{y \in \mathbb{R}} \left| Eh_\alpha(X_1, y) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) \right| I_{(\hat{\theta}_n - \theta^* < 0)}$$

$$\leq n^\tau \inf_{x \in \mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) - Eh_\alpha(X_1, x) + Eh_\alpha(X_1, x) - \inf_{y \in \mathbb{R}} Eh_\alpha(X_1, y) \right|$$

$$+ n^\tau \inf_{y \in \mathbb{R}} \left| Eh_\alpha(X_1, y) - \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, y) + \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, y) - \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) \right|$$

$$\leq \inf_{x \in \mathbb{R}} \left\{ n^\tau \left| \frac{1}{n} \sum_{i=1}^{n} h_\alpha(X_i, x) - Eh_\alpha(X_1, x) \right| + n^\tau |Eh_\alpha(X_1, x) - \inf_{y \in \mathbb{R}} Eh_\alpha(X_1, y)| \right\}$$

$$+\inf_{y\in\mathbb{R}}\left\{n^{\tau}\left|Eh_{\alpha}(X_1,y)-\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,y)\right|+n^{\tau}\left|\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,y)-\inf_{x\in\mathbb{R}}\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,x)\right|\right\} \quad (24)$$

From Theorem 1 and related contents of Rockafellar and Uryasev [15], we know that $Eh_{\alpha}(X_1,y)$ and $\frac{1}{n}\sum_{i=1}^{n}h_{\alpha}(X_i,y)$ are convex function about y. Hence, there exist $y_1\in R$ and $y_2\in R$ such that

$$Eh_{\alpha}(X_1,y)=\inf_{y\in\mathbb{R}}Eh_{\alpha}(X_1,y_1),\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,y_2)=\inf_{x\in\mathbb{R}}\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,x). \quad (25)$$

Combining (24) and (25), we have

$$n^{\tau}\left|\hat{\theta}_n-\theta^*\right|$$

$$\leq n^{\tau}\left|\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,y_1)-Eh_{\alpha}(X_1,y_1)\right|+n^{\tau}\left|Eh_{\alpha}(X_1,y_2)-\tfrac{1}{n}\textstyle\sum_{i=1}^{n}h_{\alpha}(X_i,y_2)\right|$$

$$\leq n^{\tau-1}\left|\textstyle\sum_{i=1}^{n}[h_{\alpha}(X_i,y_1)-Eh_{\alpha}(X_1,y_1)]\right|+n^{\tau-1}\left|\textstyle\sum_{i=1}^{n}[h_{\alpha}(X_i,y_2)-Eh_{\alpha}(X_1,y_2)]\right| \quad (26)$$

Therefore, to prove the Theorem we only need to prove that, for any given $x\in\mathbb{R}$,

$$\tfrac{1}{n^{1-\tau}}\textstyle\sum_{i=1}^{n}[\,h_{\alpha}(X_i,x)-Eh_{\alpha}(X_i,x)]\overset{a.s.}{\to}0. \quad (27)$$

(1) For $1\leq r<2$, by Theorem 2 in Su and Wang [21], we know (27) is true, and $\tau=1-1/r$.
(2) For $r\geq 2$, take $p\in[1,2)$ and let $\tau=1-1/p$. Then $E|X_1|^p\leq(E|X_1|^r)^{p/r}<\infty$ and $0\leq\tau<1/2$. Hence

we obtain (27) from the result of case (1).

4. Proof of Theorem 2.2

The following relevant lemmas will be used in the proof of theorem.

Lemma 4.1. Assume that X_1 and X_2 are NA. Then

$$\left|\mathrm{Cov}\big(h_{\alpha}(X_1,x),h_{\alpha}(X_2,x)\big)\right|\leq\alpha^{-2}|\mathrm{Cov}(X_1,X_2)|. \quad (28)$$

Proof Recall that $h_{\alpha}(X_i,x)=x+\alpha^{-1}[X_i-x]^+$, have

$$\mathrm{Cov}\big(h_{\alpha}(X_1,x),h_{\alpha}(X_2,x)\big)=\alpha^{-2}\mathrm{Cov}([X_1-x]^+,[X_2-x]^+). \quad (29)$$

From Hoeffding Lemma in Lehmamn [9],

$$\mathrm{Cov}([X_1-x]^+,[X_2-x]^+)$$
$$=\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{P([X_1-x]^+\leq u,[X_2-x]^+\leq v)-P([X_1-x]^+\leq u)P([X_2-x]^+\leq v)\}\,dudv$$
$$=\textstyle\int_{0}^{\infty}\int_{0}^{\infty}\{P([X_1-x]^+\leq u,[X_2-x]^+\leq v)-P([X_1-x]^+\leq u)P([X_2-x]^+\leq v)\}\,dudv. \quad (30)$$

When $u\geq 0$ and $v\geq 0$, it is easy to know the following event equations

$$\{[X_1-x]^+\leq u\}=\{X_1-x\leq u\},\{[X_2-x]^+\leq v\}=\{X_2-x\leq v\},$$
$$\{[X_1-x]^+\leq u,[X_2-x]^+\leq v\}=\{X_1-x\leq u,X_2-x\leq v\}.$$

Hence

$$\mathrm{Cov}([X_1-x]^+,[X_2-x]^+)$$
$$=\textstyle\int_{0}^{\infty}\int_{0}^{\infty}\{P(X_1-x\leq u,X_2-x\leq v)-P(X_1-x\leq u)P(X_2-x\leq v)\}\,dudv$$
$$=\textstyle\int_{0}^{\infty}\int_{0}^{\infty}\{P(X_1\leq u+x,X_2\leq v+x)-P(X_1\leq u+x)P(X_2\leq v+x)\}\,dudv \quad (31)$$
$$=\textstyle\int_{x}^{\infty}\int_{x}^{\infty}\{P(X_1\leq u,X_2\leq v)-P(X_1\leq u)P(X_2\leq v)\}\,dudv$$

Due to NA, $P(X_1\leq u,X_2\leq v)-P(X_1\leq u)P(X_2\leq v)\leq 0$ for any $u,v\in\mathbb{R}$, so we get

$$|\mathrm{Cov}([X_1-x]^+,[X_2-x]^+)|$$
$$=\left|\textstyle\int_{x}^{\infty}\int_{x}^{\infty}\{P(X_1\leq u,X_2\leq v)-P(X_1\leq u)P(X_2\leq v)\}\,dudv\right|$$
$$\leq\left|\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{P(X_1\leq u,X_2\leq v)-P(X_1\leq u)P(X_2\leq v)\}\,dudv\right| \quad (32)$$
$$=|\mathrm{Cov}(X_1,X_2)|.$$

Combine (29) with (32) yields (28). Complete the proof.

Lemma 4.2. (Luo, [10]) Assume that $E|X|^r<\infty$ for some $r>0$. Then there exists a positive constant C no depending

on x such that

$$E|h_{\alpha}(X,x)-Eh_{\alpha}(X,x)|^r\leq CE|X|^r.$$

Lemma 4.3. (Yang, [26]) Suppose that $\{X_i : i \geq 1\}$ is a NA random sequence which satisfies $EX_i = 0$ and $E|X_i|^r < \infty$ for some $r > 1$. Then there exists a positive constant C no depending on n such that

(1) for $1 < r \leq 2$,

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^r \leq C \sum_{i=1}^{n} E|X_i|^r; \qquad (33)$$

(2) for $r > 2$,

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^r \leq C \left(\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} E X_i^2 \right)^{r/2} \right). \quad (34)$$

Lemma 4.4. (Yang, [27]) Let $\{X_j : j \geq 1\}$ be a NA random sequence, $\{a_j : j \geq 1\}$ be a real-value constant sequence, $1 = m_0 < m_1 < m_2 < \cdots < m_\tau = n$. Denote $Y_l = \sum_{j=m_{l-1}+1}^{m_l} a_j X_j$ for $1 \leq l \leq k$. Then

$$\left| E\exp\left(it \sum_{l=1}^{k} Y_l\right) - \prod_{l=1}^{k} E \exp(itY_l) \right| \leq$$
$$4t^2 \sum_{1 \leq s < j \leq n} |a_s a_j| \left| \mathrm{Cov}(X_s, X_j) \right|.$$

For any given $x \in \mathbb{R}$, let $Y_i = \sigma_n^{-1}(x)n^{-1}\big(Z_i(x) - EZ_i(x)\big)$, $i = 1, 2, \cdots, n$, so that $S_n(x) = \sum_{i=1}^{n} Y_i$. Let $k_n = [n/(p_n + q_n)]$, where $[x]$ denotes the integer part of x as mentioned before. Then S_n may be split as $S_n = S_n' + S_n'' + S_n'''$, where

$$S_n' = \sum_{m=1}^{k_n} y_{n,m}, \quad S_n'' = \sum_{m=1}^{k_n} y'_{n,m}, \quad S_n''' = y'_{n,k_n+1},$$

$$y_{n,m} = \sum_{i=t_m}^{t_m+p_n-1} Y_i, \quad y'_{n,m} = \sum_{i=l_m}^{l_m+q_n-1} Y_i, \quad y'_{n,k_n+1} = \sum_{i=k_n(p_n+q_n)+1}^{n} Y_i,$$

$$t_m = (m-1)(p_n+q_n)+1, \quad l_m = (m-1)(p_n+q_n)+p_n+1, \quad m = 1, 2, \cdots, k_n.$$

Lemma 4.5. Assume that the assumptions (A1)-(A4) are satisfied. Then

$$E(S_n'')^2 \leq C\gamma_{1n}, \quad E(S_n''')^2 \leq C\gamma_{2n}, \qquad (35)$$

$$P\big(|S_n''| \geq \gamma_{1n}^{1/3}\big) \leq C\gamma_{1n}^{1/3}, \quad P\big(|S_n'''| \geq \gamma_{2n}^{1/3}\big) \leq C\gamma_{2n}^{1/3}. \quad (36)$$

Proof From (13), it is easy to get that

$$E(Y_i^2) = E\big\{\sigma_n^{-1}(x)n^{-1}\big(Z_i(x) - EZ_i(x)\big)\big\}^2$$
$$= \sigma_n^{-2}n^{-2} E\big(Z_i(x) - EZ_i(x)\big)^2 \qquad (37)$$
$$\leq C\sigma_n^{-2}(x)n^{-2}$$
$$\leq Cn^{-1}.$$

By Lemma 4.3, we can get that

$$E(S_n'')^2 = E\left(\sum_{m=1}^{k_n} y'_{n,m}\right)^2 \leq C \sum_{m=1}^{k_n} E\left|y'_{n,m}\right|^2 \leq C \sum_{m=1}^{k_n} \sum_{i=l_m}^{l_m+q_n-1} E|Y_i|^2. \qquad (38)$$

Using (37), (38) and (9) yields that

$$E(S_n'')^2 \leq Ck_n q_n n^{-1} \leq \frac{Cq_n}{p_n+q_n} \leq \frac{Cq_n p_n^{-1}}{1+q_n p_n^{-1}} \leq Cq_n p_n^{-1} = C\gamma_{1n}.$$

Note that $n - k_n(p_n + q_n) < p_n + q_n$, we have

$$E(S_n''')^2 \leq C \sum_{i=k_n(p_n+q_n)+1}^{n} E(Y_i^2)$$
$$\leq C(n - k_n(p_n+q_n))n^{-1}$$
$$\leq C(p_n+q_n)n^{-1}$$
$$\leq C(1 + q_n p_n^{-1})p_n n^{-1}$$
$$\leq C\gamma_{2n}.$$

Therefore, (35) holds. By the Markov inequality, (36) is immediately yielded form (35). Complete the proof.

Let $s_n^2 = \sum_{m=1}^{k_n} Var(y_{n,m})$. We have the following lemma.

Lemma 4.6. Under (A1)-(A4), have

$$|s_n^2 - 1| \leq C\big\{\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + u(q_n)\big\}. \qquad (39)$$

Proof Because that $E(S_n')^2 = \sum_{i=1}^{k_n} Var(y_{n,m}) + 2\sum_{1 \leq i < j \leq k_n} Cov(y_{n,i}, y_{n,j})$, we have that

$$s_n^2 - 1 = (E(S_n')^2 - 1) - 2\sum_{1 \leq i < j \leq k_n} Cov(y_{n,i}, y_{n,j}). \quad (40)$$

Note that $E(S_n) = 0$ and $E(S_n)^2 = Var(S_n) = 1$, we know that

$$E(S_n')^2 = E[S_n - (S_n'' + S_n''')]^2 = 1 + E(S_n'' + S_n''')^2 - 2E[S_n(S_n'' + S_n''')],$$

hence, from Lemma 4.5,

$$|E(S_n')^2 - 1| = |E(S_n'' + S_n''')^2 - 2E[S_n(S_n'' + S_n''')]|$$
$$\leq 2(E|S_n''|^2 + E|S_n'''|^2) + 2(E|S_n|^2)^{1/2}(E|S_n'' + S_n'''|^2)^{1/2}$$
$$\leq 2(E|S_n''|^2 + E|S_n'''|^2) + 2(E^{1/2}|S_n''|^2 + E^{1/2}|S_n'''|^2) \qquad (41)$$
$$\leq C(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2}).$$

And

$$
\begin{aligned}
\left|\sum_{1\le i<j\le k_n} \mathrm{Cov}(y_{n,i}, y_{n,j})\right| &\le \sum_{1\le i<j\le k_n} \sum_{s=t_i}^{t_i+p_n-1} \sum_{t=t_j}^{t_j+p_n-1} |\mathrm{Cov}(Y_s, Y_t)| \\
&\le C \sum_{i=1}^{k_n-1} \sum_{j=i+1}^{k_n} \sum_{s=t_i}^{t_i+p_n-1} \sum_{t=t_j}^{t_j+p_n-1} n^{-1} |\mathrm{Cov}(X_s, X_t)| \\
&\le C n^{-1} \sum_{i=1}^{k_n-1} \sum_{s=t_i}^{t_i+p_n-1} \sup_{j\ge 1} \sum_{t:|t-j|\ge q_n} |\mathrm{Cov}(X_j, X_t)| \\
&\le C u(q_n).
\end{aligned}
\tag{42}
$$

Therefore, (39) holds from (40)-(42). Complete the proof.

Lemma 4.7. (Yang, [27]) Suppose that $\{\zeta_n : n \le 1\}$ and $\{\eta_n : n \le 1\}$ are two random sequences, $\{\gamma_n : n \le 1\}$ is a positive constant sequence, and $\gamma_n \to 0$. If

$$
\sup_u |F_{\zeta_n}(u) - \Phi(u)| \le C\gamma_n,
$$

then for any $\epsilon > 0$, have

$$
\sup_u |F_{\zeta_n+\eta_n}(u) - \Phi(u)| \le C(\gamma_n + \epsilon + P(|\eta_n| \ge \epsilon)).
$$

Proof of Theorem 2.2 Assume that $\{\eta_{n,m} : m = 1,2,\cdots, k_n\}$ be the independent random variables, and the distribution of $\eta_{n,m}$ is the same with $y_{n,m}$ for $m = 1,2,\cdots, k_n$. Let $T_n = \sum_{m=1}^{k_n} \eta_{n,m}$, $B_n^2 = \sum_{m=1}^{k_n} \mathrm{Var}(\eta_{n,m})$, $\hat{F}_n^x(u), G_n^x(u)$ and $\tilde{G}_n^x(u)$ are the distribution functions of $S_n', T_n/B_n$ and T_n respectively. Recall $s_n^2 = \sum_{m=1}^{k_n} \mathrm{Var}(y_{n,m})$, it is obvious that $B_n^2 = s_n^2$, $\tilde{G}_n^x(u) = G_n^x(u/s_n)$, and

$$
\begin{aligned}
&\sup_u |\hat{F}_n^x(u) - \Phi(u)| \\
&\le \sup_u |\hat{F}_n^x(u) - \hat{G}_n^x(u)| + \sup_u |\hat{G}_n^x(u) - \Phi(u/s_n)| + \sup_u |\Phi(u/s_n) - \Phi(u)| \\
&= \sup_u |\hat{F}_n^x(u) - \hat{G}_n^x(u)| + \sup_u |\hat{G}_n^x(u/s_n) - \Phi(u/s_n)| + \sup_u |\Phi(u/s_n) - \Phi(u)| \\
&= J_{1n} + J_{2n} + J_{3n}.
\end{aligned}
\tag{43}
$$

First to estimate J_{2n}. By Lemma 4.3 we know

$$
\begin{aligned}
\sum_{m=1}^{k_n} E |\eta_{n,m}|^{2+\delta} &\le C \sum_{m=1}^{k_n} \left(\sum_{i=t_m}^{t_m+p_n-1} E |Y_i|^{2+\delta} + \left(\sum_{i=t_m}^{t_m+p_n-1} E Y_i^2 \right)^{(2+\delta)/2} \right) \\
&\le C \sum_{m=1}^{k_n} \left(\sum_{i=t_m}^{t_m+p_n-1} \sigma_n^{-(2+\delta)} n^{-(2+\delta)} + \left(\sum_{i=t_m}^{t_m+p_n-1} n^{-1} \right)^{(2+\delta)/2} \right) \\
&\le C \left(\sum_{i=1}^{n} n^{-1-\delta/2} + (p_n n^{-1})^{(1+\delta/2)} \right) \\
&\le C \left(n^{-\delta/2} + k_n (p_n n^{-1})^{(1+\delta/2)} \right) \\
&\le C \left(n^{-\delta/2} + (p_n n^{-1})^{\delta/2} \right) \\
&\le C (p_n n^{-1})^{\delta/2} \\
&= C \gamma_{2n}^{\delta/2}.
\end{aligned}
\tag{44}
$$

Note that $B_n^2 = s_n^2 \to 1$ by Lemma 4.6. From the Berry-Esseen theorem, we get that

$$
J_{2n} \le C B_n^{-(2+\delta)} \sum_m^{k_n} E |y_{n,m}|^{2+\delta} \le C \gamma_{2n}^{\delta/2}. \tag{45}
$$

Next to estimate J_{1n}. Assume $\varphi(t)$ and $\psi(t)$ are characteristic functions of S_n' and T_n respectively. By Berry-Esseen Inequality, we have

$$
J_{1n} \le \int_{-T}^{T} \left| \frac{\varphi(t)-\psi(t)}{t} \right| dy + T \sup_u \int_{|y|\le c/T} |\tilde{G}_n^x(u+y) - \tilde{G}_n^x(u)| dy. \tag{46}
$$

Note that

$$
\psi(t) = E\exp(itT_n) = E\exp\left(it \sum_{m=1}^{k_n} \eta_{n,m}\right) = \prod_{m=1}^{k_n} E\exp(it\eta_{n,m}) = \prod_{m=1}^{k_n} E\exp(ity_{n,m}), \tag{47}
$$

and utilize Lemma 4.4

$$
\begin{aligned}
|\varphi(t) - \psi(t)| &= \left| E\exp\left(it \sum_{m=1}^{k_n} y_{n,m}\right) - \prod_{m=1}^{k_n} E\exp(ity_{n,m}) \right| \\
&\le 4t^2 \sum_{1\le i<j\le k_n} \sum_{u=t_i}^{t_i+p_n-1} \sum_{v=t_j}^{t_j+p_n-1} |\mathrm{Cov}(Y_u, Y_v)| \\
&\le 4t^2 u(q_n).
\end{aligned}
\tag{48}
$$

On the other hand,

$$
\begin{aligned}
\sup_u \left| \tilde{G}_n^x(u+y) - \tilde{G}_n^x(u) \right| \quad &= \sup_u |G_n^x((u+y)/s_n) - G_n^x(u/s_n)| \\
&\leq \sup_u \left| G_n^x((u+y)/s_n) - \Phi((u+y)/s_n) \right| \\
&\quad + \sup_u |\Phi((u+y)/s_n) - \Phi(u/s_n)| + \sup_u |G_n^x(u/s_n) - \Phi(u/s_n)| \\
&\leq C(\gamma_{2n}^{\delta/2} + |y|/s_n) \leq C(\gamma_{2n}^{\delta/2} + |y|).
\end{aligned}
\tag{49}
$$

Let $T = u^{-1/3}(q_n)$ then have

$$
\begin{aligned}
J_{1n} \quad &\leq C \left(\int_{-T}^T |4tu(q)| \, dt + T \int_{|y| \leq C/T} (\gamma_{2n}^{\delta/2} + |y|) dy \right) \\
&\leq C(u(q_n)T^2 + \gamma_{2n}^{\delta/2} + 1/T) \\
&\leq C(u^{1/3}(q_n) + \gamma_{2n}^{\delta/2}).
\end{aligned}
\tag{50}
$$

Finally to estimate J_{3n}. By mean value theorem, there exists $\xi_u \in [u, u/s_n] \cup [u/s_n, u]$ such that

$$
\begin{aligned}
J_{3n} \quad &= \sup_u |\Phi(u/s_n) - \Phi(u)| \\
&= \sup_u \frac{1}{\sqrt{2\pi}} e^{-\xi_u^2/2} |u/s_n - u| \\
&\leq \frac{|s_n - 1|}{\sqrt{2\pi}s_n} \sup_u |u| \, e^{-u^2 \min\{1, 1/s_n^2\}/2} \\
&= \frac{|s_n - 1|}{\sqrt{2\pi}s_n} \frac{1}{\sqrt{\min\{1, 1/s_n^2\}}}.
\end{aligned}
\tag{51}
$$

From Lemma 4.6,

$$
J_{3n} \leq C\{\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + u(q_n)\}. \tag{52}
$$

Combining (43), (45), (50) and (52) yields that

$$
\sup_u \left| \tilde{F}_n^x(u) - \Phi(u) \right| \quad \leq C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{2n}^{\delta/2} + u^{1/3}(q_n) \right). \tag{53}
$$

By Lemma 4.5 and Lemma 4.7, we have

$$
\begin{aligned}
\sup_u |F_n^x(u) - \Phi(u)| \quad &\leq C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + \gamma_{2n}^{\delta/2} + u^{1/3}(q_n) \right) \\
&\leq C \left(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + \gamma_{2n}^{\delta/2} + u^{1/3}(q_n) \right).
\end{aligned}
\tag{54}
$$

Complete the proof.

5. Proof of Theorem 2.3

Lemma 5.1. (Shapiro, [19]) Suppose that, for a sequence $\{\tau_n\}$ of positive number converging to infinity, the sequence $\tau_n\{\hat{f}_n(x) - f(x)\}$, of random elements of $C(S)$, converges in distribution to a random element $Y(x)$ of $C(S)$. Then

$$
\tau_n \left(\inf_{x \in S} \hat{f}_n(x) - \inf_{x \in S} f(x) \right) \xrightarrow{d} \inf_{x \in S^*(f)} Y(x),
$$

where $S \subseteq \mathbb{R}$ and $f(x): S \to \mathbb{R}$, $\hat{f}_n(x)$ is sample average of $f(x)$, $S^*(f)$ is a set of points that $f(x)$ reaches minimum values in these points.

Lemma 5.2. (Pflug, [14]) Suppose that $F(x)$ is distribution function of random variable X. For $\alpha \in (0,1)$, let $a = \sup\{u: F(u) \leq 1 - \alpha\}$ and $b = \inf\{u: F(u) \geq 1 - \alpha\}$. Then

$$
[a, b] = \text{argmin}\left\{ x + \frac{1}{\alpha}E[X - x]^+, x \in \mathbb{R} \right\}.
$$

particularly, $\quad F^{-1}(1 - \alpha) \in \text{argmin}\left\{ x + \frac{1}{\alpha}E[X - x]^+: x \in \mathbb{R} \right\}$.

Proof of Theorem 2.3 From Corollary 2.1, we know that, for any $x \in \mathbb{R}$,

$$
n^{1/2}[n^{-1}\sum_{i=1}^n h_\alpha(X_i, x) - Eh_\alpha(X, x)] \xrightarrow{d} N(0, \sigma^2(x)). \tag{55}
$$

Note that $[q^-, q] = \text{argmin}\{Eh_\alpha(X, x), x \in \mathbb{R}\}$ from Lemma 5.2. So, by Lemma 5.1, we have

$$
n^{1/2}(\hat{\theta}_n - \theta^*) \xrightarrow{d} \inf_{x \in [q^-, q]} N(0, \sigma^2(x)), \tag{56}
$$

and

$$
\begin{aligned}
&\inf_{x \in [q^-, q]}\{n^{1/2}[n^{-1}\sum_{i=1}^n h_\alpha(X_i, x) - Eh_\alpha(X, x)]\} \\
&\xrightarrow{d} \inf_{x \in [q^-, q]} N(0, \sigma^2(x)).
\end{aligned}
\tag{57}
$$

Thus, (56) and (57) imply that

$$
\begin{aligned}
\hat{\theta}_n - \theta^* &= \inf_{x \in [q^-, q]}\{n^{-1}\sum_{i=1}^n h_\alpha(X_i, x) - Eh_\alpha(X, x)\} + \\
&\quad o_p(n^{-1/2}).
\end{aligned}
\tag{58}
$$

Lemma 5.2 tells us that $Eh_\alpha(X, x) = \inf_{x \in [q^-, q]} Eh_\alpha(X, x) = \theta^*$ for any $x \in [q^-, q]$. Therefore, (58) is equivalent to

$$
\hat{\theta}_n = \inf_{x \in [q^-, q]} n^{-1}\sum_{i=1}^n h_\alpha(X_i, x) + o_p(n^{-1/2}), \tag{59}
$$

that is (17).

When $q = q^-$, it is obvious that (18) and (19) hold from (59) and (56), respectively. Let $\eta_n = \hat{\theta}_n - n^{-1}\sum_{i=1}^n h_\alpha(X_i, q)$, then

$$
\eta_n = (\hat{\theta}_n - \theta^*) - (n^{-1}\sum_{i=1}^n [h_\alpha(X_i, q) - Eh_\alpha(X_i, q)]). \tag{60}
$$

From Theorem 2.1 and (27), we know that $\eta_n = o(n^{-\tau}), a.s.$, so obtain (21).

When $q > q^-$, from (56), the limiting distribution of $n^{1/2}(\hat\theta_n - \theta^*)$ is a minimum value of a family of normal distributions, so it isn't a normal distribution.

6. Conclusion

We know that ARMA models with negative autocorrelation coefficients are NA models, so NA dependent samples are widely existed in practice. Under NA variables, therefore, it is meaningful to discuss the asymptotic properties of the optimized type CVaR estimator proposed by Rockafellar and Uryasev [15] and Trindade et al [23]. We derive the consistency and the asymptotic normality of the estimator, and the consistency rates are $n^{-1/2}$ or near to $n^{-1/2}$. These results show that the estimatior has good theoretical properties in NA dependent environment. And the confidence interval of CVaR can be given by using the property of asymptotic normality.

Fund

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